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POSITIVE DEFINITENESS IN QUASI-UNIFORM SPACES

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The notion of positive definiteness has been introduced in quasi-uniform spaces and some results of Dugundji (1976) and Wong (1974) have been extended to quasi-uniform spaces.

1. INTRODUCTION

Dugundji (1976) has introduced the concept of positive definiteness and used it to derive certain fixed point theorems for continuous maps in metric spaces. Wong (1974) has extended the idea to uniform spaces. In this paper the concept of positive definiteness has been introduced in quasi-uniform spaces and the related results have been established therein.

2. DEFINITIONS

We recall the following definitions due to Dugundji (1976) and Wong (1974). In what follows R_0 will denote the subspace $[0, \infty)$ of the real line with usual topology.

Definition 1 (Dugundji 1976) — Let (X, d) be a metric space and $A \subset X$. A map $P : X \to R_0$ is said to be positive definite mod A if

$$P_A(U) = \inf \{P(x) \mid d(x, A) > \epsilon\}$$

is positive for each $\epsilon > 0$.

Definition 2 (Wong 1974) — Let (X, u) be a uniform space and $A \subset X$. A function $P : X \to R_0$ is called positive definite mod A if

$$P_A(U) = \inf \{P(x) \mid x \in X - U[A]\}$$

is positive for each $U \in u$.

We now introduce the concept of positive definiteness in quasi-uniform spaces.

Definition 3 — Let (X, h) be a quasi-uniform space and $A \subset X$. A function $P : X \to R_0$ is called positive definite mod A if $P_A(U) = \inf \{P(x) \mid x \in X - U^{-1}[A]\}$ is positive for each $U \in h$.

3. RESULTS

It is easy to see that our definition is in agreement with Wong's definition of positive definiteness in uniform spaces since $U \in u \Leftrightarrow U^{-1} \in u$, where u is the uniformity for X. We now state some properties of P_A and P.

If $U = X \times X$, it is easy to see that $P_A(U) = \infty$. It can be easily verified that P_A is monotone, i.e., $U, V \in h$ and $U \subset V \Rightarrow P_A(U) \leqslant P_A(V)$. If $r > 0$ and P is positive definite mod A, then rP will also be positive definite mod A. If P and S are both positive definite mod A, then $P \cdot S$ and $P \wedge S$ are positive definite mod A. If P is positive definite mod A and $P \leqslant S$, then S will also be positive definite mod A. Hence $P \vee S$ and $P + S$ are positive definite mod A if at least one of P and S is positive definite mod A. As remarked by Dugundji, $P + S$ may be positive definite mod A even though neither is. Since a metric space is also a quasi-uniform space, the example given by Dugundji will work here as well. It is easy to see that if P is positive definite mod A, then P is positive definite mod B for any $B \supset A$. Also any map $S : X \to R_0$ satisfying $S(x) \geqslant P(x)$ on $X - A$ is positive definite mod A if P is positive definite mod A.

Lemma 1 — If A is compact in the conjugate quasi-uniformity h^{-1} and g is any other quasi-uniformity on X such that its conjugate is compatible with h^{-1}, then P will remain positive definite mod A in (X, g).

PROOF : If $V \in g$, then $V^{-1}[A]$ will be a neighbourhood of A in the topology of the conjugate quasi-uniformity g^{-1} and, since g^{-1} and h^{-1} are compatible, $V^{-1}[A]$ is a neighbourhood of A in h^{-1}. Moreover, the compactness of A in h^{-1} implies that there exists a $U \in h$ such that $U^{-1}[A] \subset V^{-1}[A]$ (see Murdheshwar and Naimpally 1966, p. 55). Hence

$$\inf \{P(x) \mid x \in X - V^{-1}[A]\} \geqslant \inf \{P(x) \mid x \in X - U^{-1}[A]\}$$

which is positive since P is positive definite mod A in (X, h).

Lemma 2 — If P is positive definite mod A in (X, h), then $P^{-1}(0) \subset \bar{A}$.

PROOF : Let $x \in P^{-1}(0)$. Then $P(x) = 0$. Since P is positive definite mod A, $P_A(U) > 0$ for all $U \in h \Rightarrow x \in U^{-1}[A] \; \forall \; U \in h \Rightarrow x \in \bigcap_{U \in h} U^{-1}[A] = \bar{A}$. It may, however, be noted that the possibility $P^{-1}(0) = \phi$ is not excluded.

Lemma 3 — If P is a closed map on X with the topology of the conjugate h^{-1} of the quasi-uniformity h and A is closed in the topology of h, then P is positive definite mod A in (X, h) iff $P \mid X - A$ has no zero.

PROOF : Let P be positive definite mod A in (X, h). Then

$$P(z) = 0 \Rightarrow z \notin X - Q^{-1}[A]$$

for each $Q \in h \Rightarrow z \in Q^{-1}[A] \Rightarrow z \in \bigcap_{Q \in h} Q^{-1}[A] = \bar{A} = A$, since A is closed in (X, h). This shows that $P \mid_{X-A}$ has no zero.

Conversely, if $P \mid_{X-A}$ has no zero and $\inf \{P(x) \mid x \in X - Q^{-1}[A]\} = 0$ for some Q in h, then there exists a sequence $\{x_n\}$ of points in $X - Q^{-1}[A]$ such that

$P(x_n) \to 0$. Since $Q^{-1}[A]$ is a neighbourhood of A in the conjugate quasi-uniformity of h, there exists a set B, open in the topology of h^{-1}, such that $A \subset B \subset Q^{-1}[A]$, i.e. $X - A \supset X - B \supset X - Q^{-1}[A]$. But P is a closed map on (X, h^{-1}). Therefore, $P(X - B)$ is a closed set in R_0 and since

$$\{P(x_n)\} \subset P\{X - Q^{-1}[A]\} \subset P(X - B),$$

the limit point 0 must belong to $P(X - B)$, i.e. there exists a point $x \in X - B \subset X - A$ such that $P(x) = 0$, which is a contradiction.

Following Kelley (1955) we use the following notation. If (X, u), (Y, v) are two (quasi) uniform spaces, then

$$W(U, V) = \{((x, y), (p, q)) \mid (x, p) \in U, (y, q) \in V\}$$

where U, V are members of the (quasi) uniformities u and v respectively.

Lemma 4 — Let (X, h) be a quasi-uniform space. Let $X \times X$ be endowed with the product quasi-uniformity $h \times h^{-1}$ $(h^{-1} \times h)$. Let P be a function : $X \times X \to R_0$. Then the following conditions are equivalent.

(a) P is positive definite mod $\triangle(X)$, where $\triangle(X)$ is the diagonal of $X \times X$.

(b) For any $V \in h(h^{-1})$ there exists a $\delta(V) > 0$ such that

$$P^{-1}\Big([0, \delta(V))\Big) \subset V.$$

PROOF : Let P be positive definite mod $\triangle(X)$ and $V \in h$. Then there exists a $U \in h$ such that $U \circ U \subset V$. Let $P_{\triangle(X)}(W(U, U^{-1})) = k > 0$ implying

$$\inf\{P(x, y) \mid (x, y) \in X \times X - (W(U, U^{-1}))^{-1}[\triangle(X)]\} = k > 0,$$

i.e. $\inf\{P(x, y) \mid (x, y) \notin U^{-1}[p] \times U[p] \text{ for any } p \in X\} = k > 0$,

i.e. if $(x, y) \notin U^{-1}[p] \times U[p]$ for any $p \in X$, then $P(x, y) \geqslant k > 0$.

Its contrapositive statement is : if $P(x, y) < k$, then $(x, y) \in U^{-1}[p] \times U[p]$ for some $p \in X$ or, $(p, x) \in U^{-1}$ and $(p, y) \in U$, or $(x, p) \in U$ and $(p, y) \in U$, or $(x, y) \in U \circ U \subset V$. Taking $k = \delta(V)$, we have $P^{-1}[0, \delta(V)) \subset V$.

(b) $\Rightarrow$ (a). It is sufficient to prove the stated property for basic members of $h \times h^{-1}$. It can be shown that basic members of $h \times h^{-1}$ are of the type $W(U, U^{-1})$, $U \in h$. Let $V \in h$. By hypothesis there exists a $k > 0$ such that $P^{-1}([0, k)) \subset V$. Now, $P(x, y) < k \Rightarrow (x, y) \in V$

$$\begin{aligned} &\Rightarrow (x, y) \in V, (y, y) \in V \\ &\Rightarrow (x, y) \in V, (y, y) \in V^{-1} \\ &\Rightarrow (x, y), (y, y) \in W(V, V^{-1}) \\ &\Rightarrow (x, y) \in (W(V, V^{-1}))^{-1}[\triangle(X)]. \end{aligned}$$

Hence, $\inf \{P(x, y) \mid (x, y) \notin (W(V, V^{-1}))^{-1} [\triangle(X)]\} \geqslant k > 0$.

Therefore P is positive definite mod $\triangle(X)$.

Theorem 1 —Let (X, h) be a complete quasi-uniform space and V, a function : $X \to R_0$ such that (i) V is lower semi-continuous in the induced topology of h, (ii) $\inf V(X) = 0$, (iii) $P_V : X \times X \to R_0$ defined by $(x, y) \to V(x) + V(y)$ is positive definite mod $\triangle(X)$, where $X \times X$ is endowed with the topology of $h \times h^{-1}$; then $V(p) = 0$ for some $p \in X$. If the space is T_1, then the point p is unique. Moreover, if $\{x_n\}$ is a sequence in X such that $\{V(x_n)\}$ converges to 0, then $\{x_n\}$ will converge to p.

PROOF : Consider $F_n = \{x \in X \mid V(x) \leqslant (1/n)\}$, $n = 1, 2, 3, \ldots$. Since V is lower semicontinuous, F_n is closed in the topology induced by h, Since $\inf V(X) = 0$, each F_n is nonempty. Clearly $\{F_n\}$ forms a filter basis on X. Let F be the generated filter. Consider any $Q \in h$. By Lemma 4 there exists a number $r > 0$ such that $P_V^{-1}([0, r)) \subset Q$. Now, for $x, y \in F_n$, $P_V(x, y) = V(x) + V(y) \leqslant (2/n)$. If we choose n such that $n > (2/r)$ then $F_n \times F_n \subset P_V^{-1}([0, r)) \subset Q$. Choose $z \in F_n$. Then $Q[z] \supset F_n \times F_n[z] \supset F_n$ and so $Q[z]$ belongs to the filter $\mathcal{F}$ generated by $\{F_n\}$. Therefore $\mathcal{F}$ is a h-Cauchy filter. By completeness of X there exists a cluster point p of $\mathcal{F}$. Hence $p \in \bar{F}_n$ for all n. Since F_n is closed, $p \in F_n$ and so $V(p) \leqslant (1/n)$ for each n. Hence $V(p) = 0$.

To prove the uniqueness of p, we see that, if $q \neq p$ is another point of X such that $V(q) = 0$, then $P_V(p, q) = V(p) + V(q) = 0$. Hence for each $Q \in h$, $(p, q) \in Q$. Thus $(p, q) \in \bigcap_{Q \in h} Q = \triangle(X)$ if the space is T_1, (see Murdheshwar and Naimpally 1966, p. 36). Therefore $p = q$.

To prove the second part, let $\{x_n\}$ be a sequence in X such that $V(x_n) \to 0$. Without loss of generality we may assume that $V(x_n)$ monotonically tends to zero. Hence, for each n, there exists an integer n_i such that $V(x_{n_i}) < (1/n)$. Consider $G_n = \{x_n, x_{n+1}, \ldots\}$. Obviously $\{G_n\}$ is a filter basis and $G_{n_i} \subset F_n$. Therefore the filter $\mathcal{G}$ generated by $\{G_n\}$ is finer than the filter $\mathcal{F}$ generated by $\{F_n\}$. Since $\mathcal{F}$ is h-Cauchy, $\mathcal{G}$ will also be h-Cauchy. Therefore $\mathcal{G}$ will have a cluster point q, say. Hence $q \in \bar{G}_{n_i} \subset F_n$, since F_n is closed. This shows that $V(q) \leqslant (1/n)$ for each n. Therefore $V(q) = 0$. By the uniqueness of p, $p = q$.

Remark 1 : Our theorem shows that the uniqueness of the point p can be established by means of the T_1-axiom only in the case of uniform spaces, whereas Wong (1974) assumes the space to be T_2 to derive the same result. However, this is not so surprising since a uniform space is T_2 whenever it is T_1.

Remark 2 : A fixed point theorem for complete quasi-metric spaces can be derived using the above theorem with $V(x) = d(x, Tx)$. If the function V satisfies the conditions of Theorem 1, then there will exist a point p such that $V(p) = 0$ implying p is a fixed point of T. This point is unique since a quasi-metric space is T_1.

Theorem 2 — Let A be a closed subset of a quasi-uniform space (X, h). Then A is compact implies that for every map $V : X \to R_0$ which is lower semicontinuous with $\inf V(X) = 0$ and is positive definite mod A in (X, h^{-1}), $V(a) = 0$ for some $a \in A$.

PROOF : Let, if possible $\inf\limits_{x \in A} V(x) = \beta > 0$. Then $\{x \mid V(x) > \frac{1}{2}\beta\}$ is an open set in (X, h) containing A. Since A is compact, there exists $Q \in h$ such that $Q[A] \subset V^{-1}(\,]\frac{1}{2}\beta, \infty[\,)$ (see Murdheshwar and Naimpally 1966, p. 55). Since

$$\inf V(X) = 0, \text{ we have } \inf\{V(x) \mid x \in X - V^{-1}(\,]\tfrac{1}{2}\beta, \infty[\,)\} = 0$$

whence $\inf\{V(x) \mid x \in X - Q[A]\} = 0$ because $X - Q[A] \supset X - V^{-1}(\,]\frac{1}{2}\beta, \infty[\,)$. This contradicts the positive definiteness of V in (X, h^{-1}). Therefore $\inf\limits_{x \in A} V(x) = 0$. Since A is compact and the restriction of V to A is lower semicontinuous, the infimum will be attained, i.e. there exists $a \in A$ such that $V(a) = 0$.

Corresponding to Theorem 2 of Tan and Wong (1977) we state the following theorem in quasi-uniform spaces:

Theorem 3 — Let (X, h) be a quasi-uniform space and A a closed subset of X. Then A is countably compact if and only if for every lower semicontinuous function V of $X \to R_0$ with $\inf V(A) = 0$, $V(x) = 0$ for some $x \in A$.

The proof may be established following Tan and Wong (1977). This is possible because they have not used any property of uniformity in their proof.

Corollary 1 — Let (X, h) be a quasi-uniform space and let A be a closed subset of X. Suppose that for every lower semi-continuous function V of X into R_0 such that V is positive definite mod A and $\inf V(A) = 0$, $V(x) = 0$ for some $x \in A$. Then A is countably compact.

Since every topological space is quasi-uniformizable we mention here that our Theorem 3 is the fullest generalization of Theorem 2 of Tan and Wong (1977). Further we may mention that the above corollary is a partial converse to Theorem 2.

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