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ESSENTIALLY (R), ESSENTIALLY (H_1) AND ESSENTIALLY SPECTRALOID OPERATORS ON HILBERT SPACE

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Sufficient conditions for an operator to be essentially (R), essentially (H_1) and essentially spectraloid are obtained. It is shown that R is a proper subset of $e(R)$, $e(G_1)$ is a proper subset of $e(H_1)$, but classes $e(R)$ and $e(G_1)$, $e(H_1)$ and H_1, $e(S)$ and S are non-comparable.

§1. Let $B(H)$ denote the set of all bounded linear transformations from Hilbert space H into H. Let $\sigma(T), \pi_{00}(T)$, $\tilde{\sigma}(T)$, con $\sigma(T)$, $\bar{W}(T)$, $r(T)$ and $| W(T) |$ respectively denote the spectrum, the set of all isolated points in $\sigma(T)$ that are eigen values of finite multiplicity, the hen-spectrum [complement of the unbounded component of the complement of $\sigma(T)$ (Fujii 1971, 1973)] the convex hull of the spectrum, the closure of the numerical range, the spectral radius and the numerical radius of an operator T. An operator

$$T \in R \text{ if } \| (T - zI)^{-1} \| = 1/d(z, W(T)), z \notin \bar{W}(T) \qquad \text{(Luecke 1972b)}$$

and

$$T \in H_1 \text{ if } \| (T - zI)^{-1} \| = 1/d(z, \tilde{\sigma}(T)), z \notin \tilde{\sigma}(T) \qquad \text{(Fujii 1971, 1973)}$$

Let π be the quotient map from $B(H)$ onto the Calkin algebra $B(H)/K$, where K denotes the set of all compact operators in $B(H)$. An operator $T \in B(H)$ is essentially R, H_1 or a spectraloid according as $\pi(T)$ is an element of R, H_1 or spectraloid. We denote each of these sets by $e(R)$, $e(H_1)$ and $e(S)$ respectively. Let $\sigma_e(T)$, $\sigma_{le}(T)$, $W_e(T)$, $\tilde{\sigma}_e(T)$, $r_e(T)$ and $| W_e(T) |$ denote the essential spectrum, the left essential spectrum, the essentially numerical range (Fillmore *et al.* 1972), the essential hen-spectrum, the essential spectral radius and the essential numerical radius of an operator T.

Luecke (1975) proved the following basic results:

Theorem A — If $T = A \oplus B$ on $H \oplus H$, then

(i) $\sigma_e(T) = \sigma_e(A) \cup \sigma_e(B)$

(ii) $W_e(T) = \text{con } (W_e(A) \cup W_e(B))$

(iii) $\| \pi(T) \| = \max \{ \| \pi(A) \|, \| \pi(B) \| \}$.

Using these properties sufficient conditions for an operator to be essentially G_1 and essentially convexoid are obtained. It is further shown by Lueke (1975) that $e(G_1)$ and G_1; $e(C)$ and C, are non-comparable, where $e(G_1)$ and $e(C)$ denote the classes of essentially G_1 and essentially convexoid operators.

In this note, we obtain sufficient conditions for an operator to be essentially R, essentially H_1 and essentially spectraloid. It is remarkable to note that class R is a proper subset of $e(R)$, while classes $e(R)$ and $e(G_1)$, $e(H_1)$ and H_1, $e(S)$ and S are non-comparable.

§2. It is known (Acharya 1980) that if $T = A \oplus B$ be defined on $H \oplus H$, then

(i) $B \in R$ with $\bar{W}(A) \subseteq \bar{W}(B)$ implies that $T \in R$

(ii) $B \in H_1$ with $\bar{W}(A) \subseteq \tilde{\sigma}(B)$ implies that $T \in H_1$.

We have the sufficient conditions for an operator to be in $e(R)$ and $e(H_1)$ as follows:

Theorem 1 — If $T = A \oplus B$ on $H \oplus H$, where B is essentially R with

$$W_e(A) \subseteq W_e(B)$$

then T is essentially R.

Theorem 2 — If $T = A \oplus B$ on $H \oplus H$, where B is essentially H_1 with

$$W_e(A) \subseteq \tilde{\sigma}_e(B)$$

then T is essentially H_1.

Proofs for both the Theorems can be constructed on the same lines as in Luecke (1975, Theorem 3). For completeness we give the proof for Theorem 2 as follows:

PROOF: Here $\tilde{\sigma}_e(T) = \tilde{\sigma}_e(A) \cup \tilde{\sigma}_e(B) = \tilde{\sigma}_e(B)$.

For $$z \notin \tilde{\sigma}(B),\ \| (\pi(A) - zI)^{-1} \| \leqslant 1/d(z, W_e(A)) \leqslant 1/d(z, \tilde{\sigma}_e(B)).$$

Now
$$\begin{aligned} &\| (\pi(T) - zI)^{-1} \| \\ &= \max \{ \| (\pi(A) - zI)^{-1} \|, \| (\pi(B) - zI)^{-1} \| \} \\ &= \max \{ \| (\pi(A) - zI)^{-1} \|, 1/d(z, \tilde{\sigma}_e(B)) \} \\ &= 1/d(z, \tilde{\sigma}_e(B)) \\ &= 1/d(z, \tilde{\sigma}_e(T)). \end{aligned}$$

Therefore, T is essentially H_1.

If $T = A \oplus B$ on $H \oplus H$ and B is a spectraloid with $|W(A)| \leqslant r(B)$, then T is spectraloid (Acharya 1980). We have the following:

Theorem 3 — If $T = A \oplus B$ on $H \oplus H$ and B is essentially spectraloid with $|W_e(A)| \leqslant r_e(B)$, then T is essentially spectraloid.

PROOF : Since B is essentially spectraloid,

$$r_e(B) = |W_e(B)|.$$

Now, $$r_e(T) = \max\{r_e(A), r_e(B)\} = r_e(B)$$

and $$|W_e(T)| = \max\{|W_e(A)|, |W_e(B)|\} = |W_e(B)| = r_e(B).$$

Thus, $r_e(T) = |W_e(T)|$. Hence T is essentially spectraloid.

§3. According to Putnam (1968)

$$\partial\sigma(T) \subseteq \sigma_{le}(T) \cup \pi_{00}(T)$$

where ∂M denotes the boundary of a set M. It is known that $T \in R$ if and only if $\partial W(T) \subseteq \sigma(T)$ (Luecke 1972b). It is not difficult to observe that $T \in e(R)$ if and only if $\partial W_e(T) \subseteq \sigma_e(T)$. Now we use these results to show the following:

Theorem 4 — Class R is a proper subset of $e(R)$.

PROOF : Let $T \in R$. Hence $\partial W(T) \subseteq \sigma(T)$.

Now $$\partial W(T) \subseteq \partial\sigma(T) \subseteq \sigma_{le}(T) \cup \pi_{00}(T) \qquad \text{(Putnam 1968)}.$$

Further $\partial W(T) \cap \pi_{00}(T) = \phi$. Hence

$$\partial W(T) \subseteq \sigma_{le}(T) \subseteq \sigma_e(T) \subseteq W_e(T) \text{ and } W_e(T)$$

is a convex set which is again a subset $\bar{W}(T)$. Therefore, $\partial W(T) = \partial W_e(T)$ or $\partial W_e(T) \subseteq \sigma_e(T)$, i.e. $T \in e(R)$.

To show that this inclusion is proper, consider $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus 0$ on $H = M \oplus M^{\perp}$, where dimension of M is two. Here $\pi(T) = 0$ so that T is essentially R, but $T \notin R$.

Using the technique of Luecke (1975, Theorem 10) we give a non-trivial example to show the following:

Theorem 5 — $e(G_1)$ is a proper subset of $e(H_1)$.

PROOF : Consider $T = A \oplus N$ on $(M_1 \oplus M_2) \oplus M_3$, (each M_i has infinite dimensions) with $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and N, a normal operator with $\sigma(N) = C$, where C is unit circle in the complex plane.

Now
$$\begin{aligned}
&\| \pi(T - zI)^{-1} \| \\
&= \max \{ \| \pi(A - zI)^{-1} \|, \| \pi(N - zI)^{-1} \| \} \\
&\geqslant \| \pi(A - zI)^{-1} \| \\
&= \| (\pi(A) - z)^{-1} \| \\
&= \left\| \pi \begin{pmatrix} -\frac{1}{z} & \frac{1}{z^2} \\ 0 & -\frac{1}{z} \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} -\frac{1}{z} & \frac{1}{z^2} \\ 0 & -\frac{1}{z} \end{pmatrix} \right\| \\
&> \frac{1}{|z|^2}
\end{aligned}$$

and $\sigma_e(T) = \{0\} \cup C$. Choose $z = 1/10 \notin \sigma_e(T)$.

Then $1/d(z, \sigma_e(T)) = 10$. But $\| \pi(T - zI)^{-1} \| > 100$. Hence $T \notin e(G_1)$. However $\partial W_e(T) \subseteq \sigma_e(T)$ implies that $T \in e(R) \subseteq e(H_1)$.

Corollary 1 — Classes $e(G_1)$ and $e(R)$ are non-comparable.

With the help of technique given in examples in Theorems 6 and 7 of Luecke (1975) it is easy to see that:

Corollary 2 — Classes H_1 and $e(H_1)$ are non-comparable.

Corollary 3 — Classes S and $e(S)$ are non-comparable.

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