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ON THE GENERALIZED TYPE AND GENERALIZED LOWER TYPE OF AN ENTIRE FUNCTION WITH INDEX PAIR (p, q)

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Generalized (p, q)-type and generalized lower (p, q)-type of an entire function with respect to the proximate order with index pair (p, q) are defined and their coefficient characterizations are obtained.

§1. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad ...(1.1)$$

be an entire function. Set

$$M(r) \equiv M(r, f) = \max_{|z|=r} |f(z)|$$

and

$$\mu(r) \equiv \mu(r, f) = \max_{n \geqslant 0} \{ |a_n| r^n \}$$

$M(r)$ and $\mu(r)$ are called the maximum modulus and the maximum term respectively.

The concept of (p, q)-order, lower (p, q)-order, (p, q)-type and lower (p, q)-type of $f(z)$ having an index pair (p, q), $(p \geqslant q \geqslant 1)$, was introduced by Juneja *et al.* (1976, 1977). Thus $f(z)$ is said to be of (p, q)-order ρ and lower (p, q)-order λ if

$$\lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p]} M(r)}{\log^{[q]} r} = \begin{matrix}\rho \\ \lambda\end{matrix} \qquad ...(1.2)$$

and the function $f(z)$ having (p, q)-order ρ $(b < \rho < \infty)$ is said to be of (p, q)-type T and lower (p, q)-type t if

$$\lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p-1]} M(r)}{(\log^{[q-1]} r)^{\rho}} = \begin{matrix}T(p, q) \equiv T \\ t(p, q) \equiv t\end{matrix} \qquad ...(1.3)$$

where $b = 1$ if $p = q$ and $b = 0$ if $p > q$. $\log^{[0]} x = x$ and $\log^{[n]} x = \log(\log^{[n-1]} x)$ for $0 < \log^{[n-1]} x < \infty$. For the definition of index-pair etc. (see Juneja *et al.* 1976).

The growth of a function $f(z)$ can be studied in terms of its order ρ and type T, but these concepts are inadequate to compare the growth of those functions which are of the same order and of infinite type. Hence, for a refinement of the above growth scale, one may utilize proximate order the concept of which is (Nandan *et al.* 1980) as follows:

A function $\rho(r)$ defined on $(0, \infty)$ is said to be a proximate order of an entire function with index pair (p, q) if it satisfies the properties: $\lim_{r\to\infty} \rho(r) = \rho$ and $\lim_{r\to\infty} \Lambda_{[q]}(r)\, \rho'(r) = 0$ where $\Lambda_{[q]}(r) = \log^{[q]} r \ldots \log r \,.\, r$.

We now define the generalized (p, q)-type T^* and generalized lower (p, q)-type t^* of $f(z)$ with respect to a given proximate order $\rho(r)$ as

$$\lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} M(r)}{(\log^{[q-1]} r)^{\rho(r)}} = \begin{matrix} T^* \\ t^* \end{matrix} \; (0 \leqslant t^* \leqslant T^* \leqslant \infty). \qquad \ldots(1.4)$$

A proximate order $\rho(r)$ is called a proximate order of an entire function $f(z)$ with index pair (p, q) if T^* is non-zero and finite and the function $f(z)$ is said to be of perfectly regular (p, q) growth with respect to its proximate order $\rho(r)$ if $T^* = t^*$.

In the present paper we obtain coefficient characterizations of generalized (p, q)-type T^* and generalized lower (p, q)-type t^* of the entire function $f(z)$.

§2. Since, (Nandan *et al.* 1980) $(\log^{[q-1]} r)^{\rho(r)}$ is a monotonically increasing function of r for $0 < r_0 < r < \infty$, so we define a single valued real function $\chi(t)$ of t for $t > t_0$ such that

$$t = (\log^{[q-1]} r)^{\rho(r)-A} \Leftrightarrow \log^{[q-1]} r = \chi(t). \qquad \ldots(2.1)$$

Then we have the following:

Theorem 1 — Let $\rho(r)$ be a proximate order with index pair (p, q) and let $\chi(t)$ be defined as in (2.1). Then

$$\lim_{t\to\infty} \frac{d \log \chi(t)}{d \log t} = \frac{1}{\rho - A} \qquad \ldots(2.2)$$

and for every η such that $0 < \eta < \infty$

$$\lim_{t\to\infty} \frac{\chi(\eta t)}{\chi(t)} = \eta^{1/(\rho-A)} \qquad \ldots(2.3)$$

where $\quad A = 1 \quad$ when $\quad (p, q) = (2, 2)$

$\qquad\quad = 0 \quad$ otherwise.

PROOF: $$\frac{d \log \chi(t)}{d \log t} = \frac{d\,(\log^{[q]} r)}{d\,\{(\rho(r) - A) \log^{[q]} r\}}$$

$$= 1/[\rho(r) - A + \Lambda_{[q]}(r)\, \rho'(r)].$$

Passing to the limits we obtain (2.2).

Again,

$$\frac{\chi(\eta t)}{\chi(t)} = \eta^{1/(\rho(r)-A)},$$

taking limits we get (2.3).

Remark : This theorem generalizes a result of Levin (1968) which was obtained for $(p, q) = (2, 1)$.

Theorem 2 — Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ be an entire function having proximate order $\rho(r)$ with index pair (p, q). Let T^* and t^* be the generalized (p, q)-type and generalized lower (p, q)-type of $f(z)$ with respect to a proximate order $\rho(r)$. Then

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log^{[p-1]} \mu(r)}{(\log^{[q-1]} r)^{\rho(r)}} = \frac{T^*}{t^*}. \qquad ...(2.4)$$

Using Theorem 11 due to Valiron (1949, pp. 32) and Lemma 1 of Juneja *et al.* (1976), the theorem can be easily proved, so we omit the proof.

Theorem 3 — If $f(z) = \sum_{n=1}^{\infty} a_n z^n$ is an entire function with proximate order $\rho(r)$ and (p, q)-order ρ with index pair (p, q), then the generalized (p, q)-type T^* of $f(z)$ with respect to the proximate order $\rho(r)$ is given by

$$T^*/M = \limsup_{n\to\infty} \left[\frac{\chi\,(\log^{[p-2]} n)}{\log^{[q-2]} \{-(1/n) \log |a_n|\}}\right]^{\rho-A} \qquad ...(2.5)$$

where

$$M = \begin{cases} (\rho - 1)^{\rho-1}/\rho^{\rho} & \text{if } (p, q) = (2, 2) \\ 1/e\rho & \text{if } (p, q) = (2, 1) \\ 1 & \text{for all other index pairs } (p, q). \end{cases}$$

PROOF : From (2.4) for every $\epsilon > 0$ and for all

$r > r_0\ (0 < r_0 = r_0(\epsilon) < r < \infty)$

$\log M(r) < \exp^{[p-2]} \{(T^* + \epsilon)(\log^{[q-1]} r)^{\rho(r)}\}$.

Using Cauchy's estimate, this gives for all r such that $0 < r_0 < r < \infty$,

$$\log |a_n| \leqslant \exp^{[p-2]} \{(T^* + \epsilon)(\log^{q-1} r)^{\rho(r)}\} - n \log r. \qquad ...(2.6)$$

Now choose r such that

$$(\log^{[q-1]} r)^{\rho(r)-A} = \frac{1}{T^* + \epsilon} \log^{[p-2]} (n/\rho). \qquad ...(2.7)$$

For $(p, q) \neq (2, 2)$, (2.7) is reduced to

$$(\log^{[q-1]} r)^{\rho(r)} = \frac{1}{(T^* + \epsilon)} \log^{[p-2]} (n/\rho)$$

which gives that

$$t = \frac{1}{T^* + \epsilon} \log^{[p-2]} (n/\rho) \text{ and } \log^{[q-1]} r = \chi\left(\frac{1}{T^* + \epsilon} \log^{[p-2]} (n/\rho)\right).$$

Using the results (2.6) yields

$$\frac{\chi(\log^{[p-2]} n)}{\log^{[q-2]}\left\{-\frac{1}{n}\log|a_n|\right\}} < \frac{\chi(\log^{[p-2]} n)}{\chi\left(\frac{1}{T^* + \epsilon}\log^{[p-2]}(n/\rho)\right) + o(1)}.$$

Passing to limits, we have (using (2.3))

$$\limsup_{n\to\infty}\left[\frac{\chi(\log^{[p-2]} n)}{\log^{[q-1]}\{-(1/n)\log|a_n|\}}\right]^{\rho} \leqslant T^*(p \geqslant 3). \qquad \text{...(2.8)}$$

For $(p, q) = (2, 2)$, the eqn. (2.7) becomes

$$(\log r)^{\rho(r)-1} = n/[\rho(T^* + \epsilon)]$$

which implies that

$$t = n/\rho(T^* + \epsilon) \quad \text{and} \quad \log r = \chi(n/\rho(T^* + \epsilon)).$$

Hence, (2.6) is written as

$$\frac{\chi(n)}{-\frac{1}{n}\log|a_n|} < \frac{\chi(n)}{\chi\left(\frac{n}{\rho(T^* + \epsilon)}\right)\left[1 - \frac{\{n/(T^* + \epsilon)\}^{P(n)}}{\rho^{1+P(n)}\chi(n/\rho(T^* + \epsilon))}\right]}$$

where

$$P(n) = 1/(\rho(r) - 1) \quad \text{and} \quad 1 + P(n) = \rho(r)/(\rho(r) - 1).$$

Since

$$\lim_{n\to\infty}\frac{\chi(n)}{\chi(n/\rho(T^* + \epsilon))} = (\rho T^*)^{1/(\rho-1)} \qquad (\text{since } \epsilon \text{ is very small})$$

and

$$\lim_{n\to\infty}\frac{(n/(T^* + \epsilon))^{P(n)}}{\rho^{1+P(n)}\chi(n/\rho(T^* + \epsilon))} = \frac{1}{\rho}$$

so

$$\limsup_{n\to\infty}\left[\frac{n\chi(n)}{-\log|a_n|}\right]^{\rho-1} \leqslant \frac{\rho^{\rho}}{(\rho - 1)^{\rho-1}} T^*. \qquad \text{...(2.9)}$$

Again, for $(p, q) = (2, 1)$, (2.7) is reduced to

$$n/\rho(T^* + \epsilon) = r^{\rho(r)}$$

which gives

$$t = r^{\rho(r)} \Leftrightarrow r = \chi(t).$$

Equation (2.6) is converted into

$$\frac{\chi(n)}{|a_n|^{-1/n}} < \frac{\chi(n)}{e^{-1/\rho}\chi(n/\rho(T^* + \epsilon))}.$$

Passing to limits we have

$$\limsup_{n\to\infty} \left(\frac{\chi(n)}{|a_n|^{-1/n}}\right) \leqslant T^* e\rho. \quad \text{...(2.10)}$$

Equations (2.8), (2.9) and (2.10) combine into

$$\limsup_{n\to\infty} \left[\frac{\chi(\log^{[p-2]} n)}{\log^{[q-2]}\{-(1/n)\log|a_n|\}}\right]^{\rho-A} \leqslant T^*/M. \quad \text{...(2.11)}$$

To prove the reverse inequality, let

$$\limsup_{n\to\infty} \left[\frac{\chi(\log^{[p-2]} n)}{\log^{[q-2]}\{-(1/n)\log|a_n|\}}\right]^{\rho-A} = \beta/M.$$

For any $\epsilon > 0$, we have for all $n > n_0 = n_0(\epsilon)$

$$|a_n| r^n < \exp\left[-n \exp^{[q-2]}\left(\chi\left(\frac{M}{\alpha}\log^{[p-2]} r\right)\right) + n \log r\right]$$

where $\alpha = \beta + \epsilon.$

So,

$$\log \mu(r) < \max_{n\geqslant 0}\left[-n \exp^{[q-2]}\left(\chi\left(\frac{M}{\alpha}\log^{[p-2]} n\right)\right) + n\log r\right]. \quad \text{...(2.12)}$$

For $(p, q) \neq (2, 1)$ and $(2, 2)$, using (2.2) it can be easily seen that the maximum value on the right-hand side is attained for

$$n = \left[\exp^{[p-2]}\left(\alpha\left\{\log^{[q-2]}\left(\frac{\rho}{1+\rho}\log r\right)\right\}^{\rho(r)}\right)\right].$$

Thus, for r sufficiently large we get from (2.12)

$$\frac{\log^{[p-1]}\mu(r)}{(\log^{[q-1]} r)^{\rho(r)}} < \alpha\frac{[\log^{[q-2]}(\rho(1+\rho)^{-1}\log r)]^{\rho(r)}}{(\log^{[q-1]} r)^{\rho(r)}} + o(1).$$

Proceeding to limits

$$T^* \leqslant \alpha. \quad \text{...(2.13)}$$

Consider when $(p, q) = (2, 1)$. Let $n = \alpha(re^{-1/\rho})/M$, eqn. (2.12) is then reduced to

$$\frac{\log \mu(r)}{r^{\rho(r)}} < \frac{\alpha}{\rho M} e^{-\rho(r)/\rho}$$

and passing to limits we get

$$T^* \leqslant \alpha. \qquad \text{...(2.14)}$$

If $(p, q) = (2, 2)$, in order to get the maximum value of the right-hand side of the inequality (2.12) n is given by

$$n = \frac{\alpha}{M}\left(\frac{\rho - 1}{\rho}\right)^{\rho(r)-1} (\log r)^{\rho(r)-1}$$

which reduces (2.12) to

$$\frac{\log \mu(r)}{(\log r)^{\rho(r)}} < \frac{\alpha}{M} \frac{(\rho - 1)^{\rho(r)-1}}{\rho^{\rho(r)}}.$$

On taking limits we get

$$T^* \leqslant \alpha. \qquad \text{...(2.15)}$$

(2.13), (2.14) and (2.15) give

$$T^* \leqslant \alpha = (\beta + \epsilon).$$

Since this inequality holds for every $\epsilon > 0$, so $T^* \leqslant \beta$. This and (2.11) together prove the theorem.

Taking $\rho(r) = \rho$ and $\chi(t) = t^{1/(\rho-A)}$, we have the following corollary which gives a formula for the (p, q)-type T of the entire function $f(z)$.

Corollary — Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function having the (p, q)-order ρ and (p, q) type $T(0 < T < \infty)$. Then T is given by

$$\limsup_{n\to\infty} \frac{\log^{[p-2]} n}{[\log^{[q-2]}\{-(1/n)\log |a_n|\}]^{\rho-A}} = T/M.$$

Theorem 4 — Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function having the proximate order $\rho(r)$ and (p, q)-order ρ such that

$$\phi(n) = |a_n/a_{n+1}|$$

forms a non-decreasing function of n for $n > n_0$. Then the generalized lower (p, q)-type t^* of $f(z)$ is given by

$$\frac{t^*}{M} = \liminf_{n\to\infty}\left[\frac{\chi(\log^{[p-2]} n)}{\log^{[q-2]}\{-(1/n)\log |a_n|\}}\right]^{\rho-A} \qquad \text{...(2.16)}$$

where M and A are the same as given in Theorem 3.

PROOF: Since by hypothesis, $\phi(n)$ is a non-decreasing function of n for $n > n_0$, we have $\phi(n) > \phi(n-1)$ for infinitely many values of n; otherwise $f(z)$ ceases to be an entire function. So $\phi(n) \to \infty$ as $n \to \infty$.

When $\phi(n) > \phi(n-1)$, the term $a_n z^n$ becomes maximum and then

$$\mu(r) = |a_n| r^n, \nu(r) = n \quad \text{for} \quad \phi(n-1) \leqslant r < \phi(n).$$

First, let $0 < t^* < \infty$, in view of theorem 2, for any ϵ satisfying $t^* > \epsilon > 0$ and for all $r > r_0 = r_0(\epsilon)$ we get

$$\log \mu(r) > \exp^{[p-2]} [(t^* - \epsilon) (\log^{[q-1]} r)^{\rho(r)}]. \quad \text{...(2.17)}$$

Let $\quad a_{n_1} z^{n_1}$ and $a_{n_2} z^{n_2}$ $(n_1 > n_0, \phi(n_1 - 1) > r_0)$.

be two consecutive maximum terms of $f(z)$. Then since $\phi(n)$ is a non-decreasing function of n for $n > n_0$, we have for $n_1 \leqslant n \leqslant n_2 - 1$,

$$\phi(n_1) = \phi(n_1 + 1) = \ldots = \phi(n) = \ldots = \phi(n_2 - 1) \quad \text{...(2.18)}$$

and

$$|a_n| r^n = |a_{n_2}| r^{n_2} \quad \text{for} \quad r = \phi(n). \quad \text{...(2.19)}$$

Hence, (2.17), (2.18) and (2.19) give

$$\log |a_n| + n \log \phi(n) > \exp^{[p-2]} [(t^* - \epsilon) (\log^{[q-1]} \phi(n))^{\rho(\phi(n))}]$$

or,

$$X \equiv \frac{\{\chi (\log^{[p-2]} n)\}^{\rho - A}}{\exp [(\rho - A) \log^{[q-1]} \{-(1/n) \log |a_n|\}]}$$

$$> \frac{\{\chi (\log^{[p-2]} n)\}^{\rho - A}}{\exp [(\rho - A) \log^{[q-1]} \{\log \phi(n) - (1/n) \exp^{[p-2]} \{(t^* - \epsilon) (\log^{[q-1]} \phi(n))^{\rho(\phi(n))}\}\}]}. \quad \text{...(2.20)}$$

We note that the minimum value of the function

$$S(r) = \frac{(\chi (\log^{[p-2]} n))^{\rho - A}}{\exp [(\rho - A) \log^{[q-1]} \{\log r - (1/n) \exp^{[p-2]} ((t^* - \epsilon) (\log^{[q-1]} r)^{\rho(r)})\}]}$$

is attained at a point $r = r_n$ satisfying

$$\frac{E_{[p-2]} \{(t^* - \epsilon) (\log^{[q-1]} r)^{\rho(r)}\}}{\Lambda_{[q-1]}(r)} = n/r\rho. \quad \text{...(2.21)}$$

For $(p, q) = (2.1)$, (2.21) gives $r^{\rho(r)} = n/(t^* - \epsilon)\rho \Leftrightarrow r = \chi(n/(t^* - \epsilon)\rho)$. Hence

$$X > \min_{0<r<\infty} S(r) = \min \frac{(\chi(n))^\rho}{\exp [\rho \{\log r - (t^* - \epsilon) r^{\rho(r)}/n\}]}$$

$$= e [\chi(n)/\chi(n/(t^* - \epsilon)\rho)]^\rho$$

$$\sim e\rho(t^* - \epsilon). \quad \text{...(2.22)}$$

For $(p, q) = (2, 2)$, (2.21) becomes

$$(\log r)^{\rho(r)-1} = \frac{n}{\rho(t^* - \epsilon)} \Leftrightarrow \log r = \chi(n/(t^* - \epsilon)\,\rho).$$

Hence,

$$\min_{0<r<\infty} S(r) = (\chi(n)/\chi(n/(t^* - \epsilon)\,\rho))^{\rho-1}\,\{\rho/(\rho-1)\}^{\rho-1}.$$

For $(p, q) \neq (2, 2)$ and $(2, 1)$, (2.21) is reduced to

$$(\log^{[q-1]} r)^{\rho(r)} = \frac{1}{t^* - \epsilon} \log^{[p-2]}(n/\rho)$$

$$\Leftrightarrow \log^{[q-1]} r = \chi\left(\frac{1}{t^* - \epsilon} \log^{[p-2]}(n/\rho)\right). \quad \text{So}$$

$$\min_{0<r<\infty} S(r) = (\chi(\log^{[p-2]} n))^{\rho}/\exp\{\rho \log^{[q]}(re^{-1/\rho})\}$$

$$\simeq \{\chi(\log^{[p-2]} n)/(\log^{[q-1]} r)\}^{\rho}$$

$$= \left\{\chi(\log^{[p-2]} n)/\chi\left(\frac{1}{t^* - \epsilon}\log^{[p-2]}(n/\rho)\right)\right\}^{\rho}$$

$$\sim t^* - \epsilon. \qquad \text{...(2.24)}$$

(2.20), (2.22), (2.23) and (2.24) combine into

$$\liminf_{n\to\infty} \chi \geqslant t^*/M. \qquad \text{...(2.25)}$$

The inequality (2.25) is obvious if $t^* = 0$. When $t^* = \infty$, above arguments with an arbitrarily large number in place of $(t^* - \epsilon)$ leads to

$$\liminf_{n\to\infty} X = \infty.$$

We now prove that strict inequality cannot hold in (2.25). For if it holds, then there exists a number $\delta (\delta > t^*)$ such that

$$\frac{\delta}{M} = \liminf_{n\to\infty}\left[\frac{\chi(\log^{[p-2]} n)}{\log^{[q-2]}\{-n \log|a_n|\}}\right]^{\rho-A}.$$

Let δ_1 be such that $\delta > \delta_1 > t^*$, then for all $n > n_0$

$$\log|a_n| > -n \exp^{[q-2]}\left[\frac{\chi(\log^{[p-2]} n)}{(\delta_1/M)^{1/(\rho-A)}}\right].$$

Therefore, for sufficiently large r and n we have

$$\log M(r) > -n \exp^{[q-2]}\left[\frac{\chi(\log^{[p-2]} n)}{(\delta_1/M)^{1/(\rho-A)}}\right] + n \log r. \qquad \text{...(2.26)}$$

For $(p, q) = (2, 1)$, choose $n = [\rho\delta_1 r^{\rho(r)}]$, then in view of Theorem 1,

$$\log M(r) > -n \log\left[\frac{\chi(n)}{(e\rho\delta_1)^{1/\rho}}\right] + n \log \chi(n/\rho\delta_1)$$

or,

$$\frac{\log M(r)}{r^{\rho(r)}} > \delta_1.$$

Passing to limits

$$t^* \geqslant \delta_1. \qquad \ldots(2.27)$$

In case $(p, q) = (2, 2)$, choose

$$(\log r)^{\rho(r)-1} = \frac{Mn}{\delta_1 \{(\rho - 1)/\rho\}^{\rho-1}} = t,$$

then (2.26) is reduced to

$$\log M(r) > n [\log r - \chi(n) (M/\delta_1)^{1/(\rho-1)}]$$

$$\sim \frac{n}{\rho} \log r$$

or

$$\frac{\log M(r)}{(\log r)^{\rho(r)}} > \delta_1$$

which gives on passing to limits

$$t^* \geqslant \delta_1. \qquad \ldots(2.28)$$

Further, consider $(p, q) \neq (2, 1)$ and $(2, 2)$ if n is given by

$$\log^{[p-2]} (n/\rho) = \delta_1 (\log^{[q-1]} r/e^{\epsilon})^{\rho(r/e^{\epsilon})} \Leftrightarrow \log^{[q-1]} r/e^{\epsilon} = \chi\left(\frac{\log^{[p-2]} (n/\rho)}{\delta_1}\right);$$

then

$$\log M(r) > n \left\{\log r - \exp^{[q-2]}\left[\frac{\chi (\log^{[p-2]} n)}{\delta_1^{1/\rho}}\right]\right\}$$

$$= n \left[\epsilon + \exp^{[q-2]} \left\{\frac{\chi (\log^{[p-2]} (n/\rho))}{\delta_1^{1/\rho}}\right\} - \exp^{[q-2]} \left\{\frac{\chi (\log^{[p-2]} n)}{\delta_1^{1/\rho}}\right\}\right]$$

or,

$$\frac{\log^{[p-1]} M(r)}{(\log^{[q-1]} r)^{\rho(r)}} > \delta_1 + o(1).$$

Proceeding to limits we have

$$t^* \geqslant \delta_1. \qquad \ldots(2.29)$$

So (2.27), (2.28) and (2.29) are formed into

$$t^* \geqslant \delta_1$$

which is a contradiction. Hence the proof of the theorem is complete.

Corollary — Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function having the (p, q)-order ρ and lower (p, q)-type $t (0 \leqslant t < \infty)$ such that $\phi(n)$ is non-decreasing function of n for $n > n_0$, then

$$t/M = \liminf_{n \to \infty} \frac{\log^{[p-2]} n}{\{\log^{[q-2]} (-(1/n) \log | a_n |)\}^{\rho - A}}.$$

Remarks

(i) Notations:

$$\log^{[m]} X = \exp^{[-m]} X = \log (\log^{[m-1]} X) = \exp (\exp^{[-m-1]} X), m = 0, \pm 1, \pm 2, \ldots .$$

(ii) The results obtained by Levin (1968) and Srivastava (1970) who obtained them separately for $(p, q) = (2, 1)$ and $(p, q) = (2, 2)$, are the special cases of our result in theorems 1 and 3.

(iii) With $\rho(r) = \rho$, $\chi(t) = t^{1/(\rho - A)}$, Theorem 3 and Theorem 4, for $(p, q) = (2, 1)$, yield the results obtained separately by Boas (1954) and Shah (1951) and for $(p, q) = (2, 2)$, these theorems are due to Rahman (1957).

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