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A NOTE ON RIESZ MEAN

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In the present paper, three theorems are proved which generalize the results of Das (1969) and Varshney and Prasad (1969).

§1. Let $\Sigma^* a_n$ be a given series with sequence of $\{s_n\}$ for its nth partial sum. We suppose that sequence $\{\lambda_n\}$ to be a sequence of non-decreasing, nonnegative numbers tending to infinity with n. Further let R_n denote the $(R, \lambda_n, 1)$ mean of the sequence $\{s_n\}$ defined

$$R_n = \frac{1}{\lambda_n} \sum_{\nu=0}^{n} (\lambda_\nu - \lambda_{\nu-1})\, s_\nu, \ (\lambda_{-1} = 0). \qquad \text{...(1.1)}$$

If $R_n \to s$ as $n \to \infty$, the sequence $\{s_n\}$ is said to be summable $(R, \lambda_n, 1)$ and if in addition $\{R_n\}$ is of bounded variation, then it is said to be absolutely summable or summable $| R, \lambda_n, 1 |$.

Let $\{p_n\}$ be a sequence of constants such that $P_n = \sum_{k=0}^{n} p_k \neq 0 \ (n \geqq 0)$ and $P_{-1} = p_{-1} = 0$. Then the (N, p) mean of $\{s_n\}$ is defined by

$$t_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu} s_\nu. \qquad \text{...(1.2)}$$

when $\Sigma\, a_n$ is absolutely (N, p) summable, we write, for brevity $\Sigma\, a_n \in | N, p |$. Let T_n be the (N, p) mean of the sequence $\{na_n\}$.

We define the sequence of constants $\{c_n\}$ formally by

$$(\Sigma\, p_n x^n)^{-1} = \Sigma\, c_n x^n, \ (c_{-1} = 0). \qquad \text{...(1.3)}$$

We write for the sequence $\{f_n\}$

$$f_n^{(1)} = f_0 + f_1 + f_2 + \dots + f_n$$

$$f^{(2)} = f_0^{(1)} + f_1^{(1)} + \dots + f_n^{(1)}.$$

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*Summation without limit is from 0 to ∞.

§2. The main object of this paper is to give three general theorems which generalize the following results.

Theorem A (Das 1969, Theorem 9) — Let $\sum a_n \in \left| N, \frac{1}{n+1} \right|$ and

$\{n^{1-\alpha} a_n\} \in | R, e^{n^{\alpha}}, 1 | \ (0 < \alpha < 1)$

then

$\Sigma\, a_n \in | c, 0 |$, or absolutely covergent.

Theorem B [Varshney and Prasad 1969] — If the series $\Sigma\, a_n$ is summable $| N, k |$ $(k \geqq 1)$ and the sequence $\{n^{\delta} a_n\}$ is of bounded variation for some $\delta (0 < \delta < 1)$, then the series $\Sigma\, a_n$ is absolutely convergent.

§3. We establish the following:

Theorem 1 — Let $X(\rho)$ be some function of the integer ρ such that

$$P_{\rho} C^{(1)}_{X(\rho)} = O(1) \qquad \text{...(3.1)}$$

$$\sum_{n=\nu+1}^{\infty} | C_n | = O(| C^{(1)}_{\nu} |) \qquad \text{...(3.2)}$$

$$P_{\rho} \sum_{\nu=\rho}^{\rho+X(\rho)} | C^{(1)}_{\nu-\rho} | = O(\rho). \qquad \text{...(3.3)}$$

Then

$$\sum \frac{| T_n |}{n} < \infty \Rightarrow \sum a_n \in | R, \lambda_n, 1 |$$

if and only if

$$\rho P_{\rho} \sum_{\nu=\rho}^{\rho+X(\rho)} | C^{(1)}_{\nu-\rho} | \left| \frac{\Delta(\lambda_{\nu-1})}{\nu \lambda_{\nu}} \right| = O(1). \qquad \text{...(3.4)}$$

Theorem 2 — Let $\frac{p_n}{p_{n+1}} \leqq \frac{p_{n+1}}{p_{n+2}} \leqq 1,\ p_n > 0,$

$$\frac{P_{\rho}}{P_{X(\rho)}} = O(1) \qquad \text{...(3.5)}$$

$$X(\rho) = O(\rho). \qquad \text{...(3.6)}$$

Then

$$\Sigma\, a_n \in | N, p | \Rightarrow \Sigma\, a_n \in | R, \lambda_n, 1 |$$

if and only if

$$\rho P_\rho \sum_{\nu=\rho}^{\rho+X(\rho)} | C_{\nu-\rho}^{(1)} | \left| \frac{\Delta \lambda_{\nu-1}}{\nu \lambda_\nu} \right| = O(1). \qquad ...(3.7)$$

Theorem 2*A* — Let the conditions (3.5) and (3.6) of Theorem 2 hold and let further the following conditions also hold:

$$\frac{\Delta (\lambda_{\nu-1})}{\lambda_\nu} \quad \text{is monotonic decreasing,} \qquad ...(3.8)$$

$$\frac{X(\rho)\, \Delta (\lambda_{\nu-1})}{\lambda_\rho} = O(1). \qquad ...(3.9)$$

Then

$$\Sigma\, a_n \in |\, N, p \,| \Rightarrow \Sigma\, a_n \in |\, R, \lambda_n, 1 \,| .$$

Theorem 3 — Let $\Sigma\, a_n \in |\, N, \delta \,|$ for $\delta > 0$ (however large) and

$$\{n^{1-\alpha} a_n\} \in |\, R, e^{n^\alpha}, 1 \,| \quad \text{for} \quad 0 < \alpha < 1. \qquad ...(3.10)$$

Then $\Sigma\, a_n$ is absolutely convergent.

§4. We need the following lemmas for the proof of Theorem 1.

Lemma 1 —

$$\sum_{n=1}^{\infty} \frac{| T_n |}{n} < \infty \Rightarrow \sum a_n \in |\, R, \lambda_n, 1 \,| \qquad ...(4.1)$$

if and only if

$$J_\rho = \rho P_\rho \sum_{n=\rho}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \cdot \lambda_{n-1}} \left| \sum_{\nu=\rho}^{n} \frac{\lambda_{\nu-1}}{\nu} C_{\nu-\rho} \right| = O(1)$$

uniformly in $\rho \geqq 1$.

PROOF: Now using the inversion formula

$$\nu a_\nu = \sum_{\rho=1}^{\nu} c_{\nu-\rho} T_\rho P_\rho$$

we obtain

$$R_n - R_{n-1} = \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \,.\, \lambda_{n-1}} \sum_{\nu=1}^{n} a_\nu \lambda_{\nu-1}$$

(*equation continued on p.* 1437)

$$= \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \,.\, \lambda_{n-1}} \sum_{\nu=1}^{n} \frac{\lambda_{\nu-1}}{\nu} \sum_{\rho=1}^{\nu} C_{\nu-\rho} P_\rho T_\rho$$

$$= \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \,.\, \lambda_{n-1}} \sum_{\rho=1}^{n} \rho P_\rho \frac{T_\rho}{\rho} \sum_{\nu=\rho}^{n} \frac{\lambda_{\nu-1}}{\nu} C_{\nu-\rho}.$$

Thus (4.1) is true if and only if uniformly in $\rho \geqq 1$,

$$J_\rho = \rho P_\rho \sum_{n=\rho}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \,.\, \lambda_{n-1}} \left| \sum_{\nu=\rho}^{n} \frac{\lambda_{\nu-1}}{\nu} C_{\nu-\rho} \right| = O(1).$$

This completes the proof.

Lemma 2 — We write $m = \min \{n, \rho + X(\rho)\}$.

Let (3.2), (3.3), (3.4) hold. Then

$$\rho P_\rho \sum_{n=\rho}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \,.\, \lambda_{n-1}} \left| \sum_{\nu=\rho}^{m} \triangle \left(\frac{\lambda_{\nu-1}}{\nu}\right) \sum_{r=0}^{\nu} C_{r-\rho} \right| = O(1)$$

uniformly in $\rho \geqq 1$.

PROOF : Now

$$\sum_{n=\rho}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \,.\, \lambda_{n-1}} \left| \sum_{\nu=\rho}^{m} \triangle \left(\frac{\lambda_{\nu-1}}{\nu}\right) \sum_{r=0}^{\nu} C_{r-\rho} \right|$$

$$\leqq \sum_{n=\rho}^{\rho+X(\rho)} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \,.\, \lambda_{n-1}} \left| \sum_{\nu=\rho}^{n} \triangle \left(\frac{\lambda_{\nu-1}}{\nu}\right) C^{(1)}_{\nu-\rho} \right|$$

$$+ \sum_{n=\rho+X(\rho)+1}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \,.\, \lambda_{n-1}} \left| \sum_{\nu=\rho}^{m} \triangle \left(\frac{\lambda_{\nu-1}}{\nu}\right) C^{(1)}_{\nu-\rho} \right|$$

$$= \sum_{n=\rho}^{\rho+X(\rho)} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \,.\, \lambda_{n-1}} \left| \sum_{\nu=\rho}^{n} \triangle \left(\frac{\lambda_{\nu-1}}{\nu}\right) C^{(1)}_{\nu-\rho} \right|$$

$$+ \sum_{n=\rho+X(\rho)+1}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \,.\, \lambda_{n-1}} \left| \sum_{\nu=\rho}^{\rho+X(\rho)} \triangle \left(\frac{\lambda_{\nu-1}}{\nu}\right) C^{(1)}_{\nu-\rho} \right|$$

$$\leqq \sum_{\nu=\rho}^{\rho+X(\rho)} \Delta\left(\frac{\lambda_{\nu-1}}{\nu}\right) C^{(1)}_{\nu-\rho} \left| \sum_{n=\nu}^{\rho+X(\rho)} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \cdot \lambda_{n-1}} \right|$$

$$+ \sum_{n=\rho+X(\rho)+1}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \cdot \lambda_{n-1}} \sum_{\nu=\rho}^{\rho+X(\rho)} \Delta\left(\frac{\lambda_{\nu-1}}{\nu}\right) \left| C^{(1)}_{\nu-\rho} \right|$$

$$= \sum_{\nu=\rho}^{\rho+X(\rho)} \Delta\left(\frac{\lambda_{\nu-1}}{\nu}\right) | C^{(1)}_{\nu-\rho} | \sum_{n=\nu}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \cdot \lambda_{n-1}}$$

$$= \sum_{\nu=\rho}^{\rho+X(\rho)} \left\{ \Delta\left(\frac{\lambda_{\nu-1}}{\nu}\right) + \frac{\lambda_{\nu-1}}{\nu^2} \right\} O\left(\frac{1}{\lambda_{\nu-1}}\right) | C^{(1)}_{\nu-\rho} |$$

$$= \sum_{\nu=\rho}^{\rho+X(\rho)} \frac{\Delta(\lambda_{\nu-1})}{\nu \lambda_{\nu-1}} | C^{(1)}_{\nu-\rho} | + \sum_{\nu=\rho}^{\rho+X(\rho)} \frac{1}{\nu^2} | C^{(1)}_{\nu-\rho} |$$

$$= O\left(\frac{1}{\rho P_\rho}\right) + O\left(\frac{\rho}{\rho^2 P_\rho}\right) = O\left(\frac{1}{\rho P_\rho}\right)$$

by the relations (3.2), (3.3) and (3.4).

This completes the proof.

Lemma 3 — Let (3.1) and (3.4) hold. Then

$$\sum_{n=\rho}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \cdot \lambda_{n-1}} | C^{(1)}_{m-\rho} | \frac{\lambda_m}{m+1} = O\left(\frac{1}{\rho P_\rho}\right)$$

when $m = \min\{n, \rho + X(\rho)\}$.

PROOF: Now

$$\sum_{n=\rho}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \cdot \lambda_{n-1}} | C^{(1)}_{m-\rho} | \frac{\lambda_m}{m+1}$$

$$\leqq \sum_{n=\rho}^{\rho+X(\rho)} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \cdot \lambda_{n-1}} \frac{\lambda_m}{m+1} | C^{(1)}_{m-\rho} |$$

$$+ \sum_{n=\rho+X(\rho)+1}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \cdot \lambda_{n-1}} \frac{\lambda_m}{m+1} | C^{(1)}_{m-\rho} |$$

(*equation continued on p.* 1439)

$$= \sum_{n=\rho}^{\rho+X(\rho)} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \cdot \lambda_{n-1}} \frac{\lambda_n}{n+1} \mid C_{n-\rho}^{(1)} \mid$$

$$+ \sum_{n=\rho+X(\rho)+1}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \cdot \lambda_{n-1}} \frac{\lambda_{\rho+X(\rho)}}{\rho + X(\rho)} \mid C_{X(\rho)}^{(1)} \mid$$

$$= \sum_{n=\rho}^{\rho+X(\rho)} \frac{\lambda_n - \lambda_{n-1}}{(n+1)\lambda_{n-1}} \mid C_{n-\rho}^{(1)} \mid$$

$$+ \frac{\lambda_{\rho+X(\rho)}}{\rho + X(\rho)} \mid C_{X(\rho)}^{(1)} \mid \sum_{n=\rho+X(\rho)}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \cdot \lambda_{n-1}}$$

$$= \sum_{n=\rho}^{\rho+X(\rho)} \frac{\Delta \lambda_n}{(n+1)\lambda_{n-1}} \mid C_{n-\rho}^{(1)} \mid$$

$$+ \frac{\lambda_{\rho+X(\rho)}}{\rho + X(\rho)} \mid C_{X(\rho)}^{(1)} \mid \sum_{n=\rho+X(\rho)}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \cdot \lambda_{n-1}}$$

$$= \sum_{n=\rho}^{\rho+X(\rho)} \frac{\Delta \lambda_n}{(n+1)\lambda_{n-1}} \mid C_{n-\rho}^{(1)} \mid + \frac{\lambda_{\rho+X(\rho)}}{\rho + X(\rho)} \mid C_{X(\rho)}^{(1)} \mid \cdot \frac{1}{\lambda_{\rho+X(\rho)}}$$

$$= O\left(\frac{1}{\rho P_\rho}\right) + O\left(\frac{1}{\rho P_\rho}\right)$$

$$= O\left(\frac{1}{\rho P_\rho}\right)$$

by the conditions (3.1) and (3.4).

§5. *Proof of Theorem* 1 — It follows from (Das 1969, Theorem 6) on general infinite series that

$$\sum_{n=1}^{\infty} \frac{\mid T_n \mid}{n} = O(1) \qquad \ldots(5.1)$$

if and only if $\sum_{n=1}^{\infty} \mid t_n - t_{n-1} \mid = O(1)$.

Therefore for the proof of the theorem, it is enough to show that

$$\sum_{n=1}^{\infty} \frac{|T_n|}{n} = O(1) \Rightarrow \{R_n\} \in \text{bounded variation.} \qquad \ldots(5.2)$$

From Lemma 1, the necessary and sufficient condition for the result (5.2) is

$$J_\rho = \sum_{n=\rho}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \cdot \lambda_{n-1}} \left| \sum_{\nu=\rho}^{n} \frac{\lambda_{\nu-1}}{\nu} C_{\nu-\rho} \right| = O\left(\frac{1}{\rho P_\rho}\right).$$

Now

$$J_\rho = \sum_{n=\rho}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \cdot \lambda_{n-1}} \left| \left\{ \sum_{\nu=\rho}^{m} + \sum_{\nu=m+1}^{n} \right\} \frac{\lambda_{\nu-1}}{\nu} C_{\nu-\rho} \right|$$

$$= J_{(1)} + J_{(2)} \text{ (say)}$$

where

$$J_{(1)} = \sum_{n=\rho}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \cdot \lambda_{n-1}} \left| \sum_{\nu=\rho}^{m} \frac{\lambda_{\nu-1}}{\nu} C_{\nu-\rho} \right|$$

$$J_{(2)} = \sum_{n=\rho}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \cdot \lambda_{n-1}} \left| \sum_{\nu=m+1}^{n} \frac{\lambda_{\nu-1}}{\nu} C_{\nu-\rho} \right|.$$

Now applying Abel's transformation to inner sigma of $J_{(1)}$ and then using the relation (1.3) we get

$$J_{(1)} = \sum_{n=\rho}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \cdot \lambda_{n-1}} \left| \sum_{\nu=\rho}^{m} \triangle\left(\frac{\lambda_{\nu-1}}{\nu}\right) \sum_{\mu=0}^{\nu} C_{\mu-\rho} + \frac{\lambda_m}{m+1} \sum_{\mu=0}^{m} C_{\mu-\rho} \right|$$

$$\leqq J_{(1)}^{(1)} + J_{(1)}^{(2)} \text{ (say)}$$

where

$$J_{(1)}^{(1)} = \sum_{n=\rho}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \cdot \lambda_{n-1}} \left| \sum_{\nu=\rho}^{m} \triangle\left(\frac{\lambda_{\nu-1}}{\nu}\right) \sum_{\mu=0}^{\nu} C_{\mu-\rho} \right|$$

and $$J_{(1)}^{(2)} = \sum_{n=\rho}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \cdot \lambda_{n-\rho}} \left| C_{m-\rho}^{(1)} \right| \frac{\lambda_{m-1}}{m+1}.$$

By Lemmas 2 and 3 we get

$$J^{(1)}_{(1)} = O\left(\frac{1}{\rho P_\rho}\right),$$

and
$$J^{(2)}_{(1)} = O\left(\frac{1}{\rho P_\rho}\right)$$

respectively. Therefore for the proof of the Theorem, it is enough to show that

$$J_{(2)} = O\left(\frac{1}{\rho P_\rho}\right).$$

Now using change of summation, the relations (3.1), (3.2) and the fact that

$$m = \min\{n, X(\rho) + \rho\},$$

we get

$$J_{(2)} \sum_{\nu=\rho+X(\rho)}^{\infty} \frac{\lambda_{\nu-1}}{\nu} \mid C_{\nu-\rho} \mid \sum_{n=\nu}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \cdot \lambda_{n-1}}$$

$$= \sum_{\nu=\rho+X(\rho)}^{\infty} \frac{\lambda_{\nu-1}}{\nu} \mid C_{\nu-\rho} \mid \frac{1}{\lambda_{\nu-1}}$$

$$= \sum_{\nu=\rho+X(\rho)}^{\infty} \frac{\mid C_{\nu-\rho} \mid}{\nu}$$

$$= \frac{1}{\rho + X(\rho)} O(\mid C^{(1)}_{X(\rho)} \mid)$$

$$= O\left(\frac{1}{\rho P_\rho}\right).$$

This completes the proof.

Proof of Theorem 2 — It is known from Das (1969) that

$$\sum a_n \in \mid N, p \mid \Leftrightarrow \sum_{n=1}^{\infty} \frac{\mid T_n \mid}{n} = O(1).$$

Since $C^{(1)}_n \geqq 0$, the relation (3.3) of Theorem 1 reduces to the condition

$$P_\rho C^{(2)}_{X(\rho)} = O(\rho). \qquad \ldots(5.3)$$

But from (3.5) and (3.6) we have

$$C^{(2)}_{X(\rho)} = O\left(\frac{X(\rho)}{P_{X(\rho)}}\right).$$

Therefore (5.3) reduces to

$$\frac{X(\rho)\, P_\rho}{P_{X(\rho)}} = O(1).$$

Since all the conditions of Theorem 1 are satisfied, the proof of Theorem 2 follows from Theorem 1.

Proof of Theorem 2A — It can be easily seen that the relations (3.5), (3.8) and (3.9) imply the necessary and sufficient condition of Theorem 1 (i.e.) the relation (3.4). The proof follows from Theorem 2.

Corollary — $\Sigma\, a_n \in \mid N, \delta \mid \Rightarrow \Sigma\, a_n \in \mid R, e^{n^\alpha}, 1 \mid$ for $\delta > 0$, $0 < \alpha < 1$.

PROOF : Let us take up $X(\rho) = [\rho^{1-\alpha}] + 2$.

So that

$$X(\rho) = O(\rho)$$

and $$P_\rho = O(P_{X(\rho)})$$

and other conditions of Theorem 2A are satisfied.

The proof follows from Theorem 2A.

Proof of Theorem 3 — We know that (Das 1969, Theorem 9)

$$\Sigma\, a_n \in \mid R, e^{n^\alpha}, 1 \mid \Rightarrow \Sigma \mid a_n \mid < \infty,$$

if and only if the relation (3.10) holds.

Now the proof follows from the Corollary of Theorem 2A.

Remark 1 : Here we show examples to satisfy the conditions from (3.1) to (3.3).

Example 1 — Let $p_n = \dfrac{1}{n+1}$, $X(n) = n$.

Then (Das 1969, Lemma 5) we have

$$C_n^{(1)} = O\left(\frac{1}{\log n}\right),\ C_n^{(2)} = O\left(\frac{n}{\log n}\right)$$

so that the condition (3.1) reduces to $O(1)$. (3.3) reduces to

$$\log(n+1) \sum_{\nu=n}^{2n} C_{\nu-n}^{(1)} = \log(n+1)\, C_n^{(2)}$$

$$= \log(n+1) \,.\, O\left(\frac{n}{\log(n+1)}\right)$$

$$= O(n).$$

Example 2 — Let $p_n = A_n^{\alpha-1}$ $X(n) = n$

so that $C_n = A_n^{-(\alpha+1)}$ and $C_n^{(1)} = A_n^{-\alpha}$.

Then the condition (3.1) reduces to

$$P_n C_{X(n)}^{(1)} = O(1).$$

And (3.3) reduces to

$$A_n^{\alpha} \sum_{\nu=n}^{2n} C_{\nu-n}^{(1)} = A_n^{\alpha} C_n^{(2)}$$

$$= A_n^{\alpha} . A_n^{1-\alpha}$$

$$= O(n).$$

Remark 2 : In Theorems 1, 2 and 2A, we have established a very general theorem on inclusion relation i.e.

$$| N, p | \Rightarrow | R, \lambda_n, 1 | .$$

As a Corollary to Theorem 2A we have obtained the inclusion

$$(**) \; | N, \delta | \Rightarrow | R, e^{n^{\alpha}}, 1 | \; (0 < \alpha < 1), \delta > 0.$$

Since $| N, \delta | \subset | N . \delta' |$ (see Mahapatra 1979) $\delta' > \delta > 0$ we get (**) is trivial in the case $0 < \delta < 1$ in view of Theorem 8 of Das (1969). But in the case $\delta > 1$, our Theorem is sharper than the result of Das (1969). Again if we put $\delta = 1$ in Theorem 3 we get Theorem 9 due to Das (1969). It is also more general than that of Varshney and Prasad (1969) in the sense that they took the condition $\{n^{1-\alpha} a_n\} \in BV$ instead of the lighter condition (3.10). Also they have taken δ as positive integer greater than one, whereas in our case $\delta > 0$.

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