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ON TRANSLATIVITY OF THE PRODUCT OF RIESZ SUMMABILITY METHODS

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In the present paper, sufficient condition under which a summability method $(\bar{N}, r)$ be translative is investigated. The definition of the product of two Riesz summability methods is given, and the set of necessary and sufficient conditions for $(\bar{N}, r)$ $(\bar{N}, q)$ being translative is established. A counter example is also given to show that even if $(\bar{N}, r)$ and $(\bar{N}, q)$ are both regular and translative, the product need not be.

1. INTRODUCTION

Given a series

$$\sum_{n=0}^{\infty} a_n. \qquad \text{...(1)}$$

We will throughout write $\{S_n\}$ for the sequence of its partial sums; that is to say

$$S_n = a_0 + a_1 + \ldots + a_n.$$

We will write r, q to denote the sequence $\{r_n\}$, $\{q_n\}$; we will write throughout

$$R_n = r_0 + r_1 + \ldots + r_n,$$

and will take $R_{-1} = r_{-1} = 0$, $R_n \neq 0$ for all $0 \geqslant n$.

We will write $\frac{r_{n+1}}{r_n} \downarrow$ to mean that $\frac{r_{n+1}}{r_n}$ is non-increasing, and $\frac{r_{n+1}}{r_n} \uparrow$ to mean that $\frac{r_{n+1}}{r_n}$ is non-decreasing. We also write $a_n \asymp b_n$ to mean that a_n lies between two positive multiples of b_n. We shall use throughout for any sequence, $\triangle U_n = U_n - U_{n+1}$, $\triangle^2 U_n = \triangle(\triangle U_n)$, etc.

Each sequence $r_0, r_1, r_2, \ldots$ for which $R_n = r_0 + r_1 + \ldots + r_n \neq 0$ for each n determines a Riesz transformation $(\bar{N}, r)$,

$$t_n = \frac{1}{R_n} \sum_{k=0}^{n} r_k S_k, \quad n = 0, 1, 2, \ldots . \qquad \text{...(2)}$$

We make similar definition with regard to other letters in place of r. It follows from Toeplitz's theorem (Hardy 1949, Theorem 2) that the necessary and sufficient conditions for the regularity of the method $(\bar{N}, r)$ are that

$$|R_n| \to \infty \text{ as } n \to \infty \qquad \dots(3)$$

and

$$\sum_{k=0}^{n} |r_k| = O(|R_n|). \qquad \dots(4)$$

The product of two Riesz methods $(\bar{N}, r)$ $(\bar{N}, q)$ is expressed as the $(\bar{N}, r)$ transform of $(\bar{N}, q)$ transform of $\{S_n\}$ and is given by the sequence-to-sequence transformation

$$t_n = \sum_{k=0}^{n} a_{n,k} S_k \qquad \dots(5)$$

where

$$a_{n,k} = \frac{q_k}{R_n} \sum_{v=k}^{n} \frac{r_v}{Q_v}, \quad 0 \leqslant k \leqslant n \qquad \dots(6)$$

and

$$a_{n,k} = 0 \text{ for } k > n. \qquad \dots(7)$$

A sequence-to-sequence method A is called translative to the left, if the limitability of $S_0, S_1, \dots, S_n, \dots$ implies the limitability of $0, S_0, S_1, \dots, S_{n-1}$ to the same limit, it is translative to the right, if the converse holds. A is translative, if it is translative to the rignt and left.

In this paper we will obtain necessary and sufficient conditions for $(\bar{N}, r)$ $(\bar{N}, q)$ to be translative, and to show that even if $(\bar{N}, r)$, $(\bar{N}, q)$ are both translative, the product need not be. These results will be concluded in sections 5 and 6.

2. Subsidiary Results

Lemma 2.1 — Let the summability method A be given by the sequence-to-sequence transformation

$$B_n = \sum_{k=0}^{\infty} Y_{n,k} S_k. \qquad \dots(8)$$

Suppose that

(a) $$\sum_{k=0}^{\infty} | Y_{n,k} | = O(1); \qquad ...(9)$$

(b) For every fixed k,

$$Y_{n,k} \to 0 \text{ as } n \to \infty; \qquad ...(10)$$

(c) There is some sequence $\{f_k\}$ converging to 1 such that

$$w_n = \sum_{k=0}^{\infty} Y_{n,k} f_k \to 1 \text{ as } n \to \infty. \qquad ...(11)$$

Then

$$\sum_{k=0}^{\infty} Y_{n,k} \to 1 \text{ as } n \to \infty. \qquad ...(12)$$

PROOF : Using Theorem 4 of Hardy (1949) and taking the particular sequence given in the hypothesis, $f_k - 1 \to 0$ as $k \to \infty$, we have

$$\sum_{k=0}^{\infty} Y_{n,k}(f_k - 1) \to 0.$$

This together with (11) yield the result.

Corollary 2.1 — Suppose that the regular summability method A given by the sequence-to-sequence transformation

$$U_n = \sum_{k=0}^{n} C_{n,k} S_k \qquad ...(13)$$

such that

$$C_{n,n} \neq 0 \text{ for all } n \geqslant 0 \qquad ...(14)$$

and

$$\sum_{k=0}^{n} C_{n,k} = 1 \text{ for all } n \geqslant 0. \qquad ...(15)$$

Let $\bar{U}_n$ denote the transform of $\{S_{n-1}\}$, and that U_n, $\bar{U}_n$ are connected by the relations:

$$\bar{U}_{n+1} = \sum_{k=0}^{n+1} a_{n+1,k} U_k \qquad \text{...(16)}$$

$$U_n = \sum_{k=0}^{n} b_{n,k} \bar{U}_k. \qquad \text{...(17)}$$

Then A is translative to the left if, and only if

$$\sum_{k=0}^{\infty} | a_{n+1,k} | = O(1) \qquad \text{...(18)}$$

and

$$a_{n+1,k} \to 0 \text{ as } n \to \infty \text{ for every fixed } k. \qquad \text{...(19)}$$

Also, A is translative to the right if, and only if

$$\sum_{k=0}^{\infty} | b_{n,k} | = O(1) \qquad \text{...(20)}$$

and

$$b_{n,k} \to 0 \text{ as } n \to \infty \text{ for every fixed } k. \qquad \text{...(21)}$$

PROOF : Take the special case in which $S_k = 1$ (all $k \geqslant 0$).

Using this and (15), we have from (13) that

$$U_n = 1, \text{ all } n \geqslant 0. \qquad \text{...(22)}$$

Also by regularity we have

$$\bar{U}_{n+1} \to 1 \text{ as } n \to \infty \qquad \text{...(23)}$$

which by (16) and (22) gives

$$\sum_{k=0}^{n+1} a_{n+1,k} \to 1 \text{ as } n \to \infty \qquad \text{...(24)}$$

which by Toeplitz's theorem gives the result for left translativity. As for right translative, let (20) and (21) be given. Using (22), (23) and Lemma 2.1, we have from (17) that

$$\sum_{k=0}^{n} b_{n,k} \to 1 \text{ as } n \to \infty. \qquad \text{...(25)}$$

This with (20) and (21) imply that (17) is regular, and thus A is translative to the right. Conversely, if A is translative to the right then (17) is regular. For this it is necessary that (20) and (21) hold. This completes the proof.

3. Riesz Methods

In this section we state some well-known results that are necessary for our use, and will restrict ourselves to the case in which $r_n > 0$ (all $n \geqslant 0$), $R_n \to \infty$ as $n \to \infty$. The following Lemma is due to Garabedian and Randels (1938, Theorem 4).

Lemma 3.1 — Necessary and sufficient conditions for the regular $(\bar{N}, r)$ be translative to the right are

$$\sum_{v=1}^{n} \left| \triangle \frac{r_{v-1}}{r_v} \right| R_v = O(R_n) \qquad \ldots(26)$$

and

$$\frac{r_n R_{n+1}}{r_{n+1} R_n} = O(1). \qquad \ldots(27)$$

It is worth noting that if $r_n > 0$, then the condition (27) is equivalent to the following condition

$$\frac{r_n}{r_{n+1}} = O(1). \qquad \ldots(28)$$

Let $\{U_n\}$, $\{\bar{U}_n\}$ be respectively the $(\bar{N}, r)$ transform of $\{S_n\}$, $\{S_{n-1}\}$. Obtain $\bar{U}_{n+1}$ in term of S_n, and use the identity

$$S_n = \frac{R_n U_n - R_{n-1} U_{n-1}}{r_n} \qquad \ldots(29)$$

to obtain $\bar{U}_{n+1}$ in terms of U_n. Using Corollary 2.1, one can easily prove the following result:

Lemma 3.2 — Necessary and sufficient conditions for the regular $(\bar{N}, r)$ be translative to the left are

$$\sum_{v=1}^{n} R_{v-1} \left| \triangle \frac{r_v}{r_{v-1}} \right| = O(R_{n+1}) \qquad \ldots(30)$$

and

$$\frac{r_{n+1} R_n}{r_n R_{n+1}} = O(1). \qquad \ldots(31)$$

We remark that if $r_n > 0$, then a sufficient condition for (31) is

$$\frac{r_{n+1}}{r_n} = O(1). \qquad ...(32)$$

But (32) is not necessary for (31). For this, let $R_n = e^{n^2}$, then $R_n \to \infty$ as $n \to \infty$, and $r_n = e^{n^2} - e^{(n-1)^2} > 0$ (for all $n \geqslant 1$), $r_0 = R_0 = 1$.

Therefore

$$\frac{r_{n+1}R_n}{r_nR_{n+1}} = \frac{e^{2n+1} - 1}{e^{2n+1} - e^2} = O(1).$$

but
$$\frac{r_{n+1}}{r_n} = \frac{e^{2n+1} - 1}{1 - e^{-2n+1}} \neq O(1).$$

4. Some Special Cases

In this section we prove the following two theorems.

Theorem 4.1 — Let $r_n > 0$ (all $n \geqslant 0$), $R_n \to \infty$. If $r_n \asymp r_{n+1}$, then the conditions (28) and (31) are satisfied, and that the conditions (26) and (30) are equivalent.

Proof: The hypothesis implies that (28) holds and

$$\frac{r_{n+1}}{r_n} = O(1). \qquad ...(33)$$

Using (28) and (33), it can be easily shown that

$$R_n \asymp R_{n+1}. \qquad ...(34)$$

Thus, (31) follows from (33) and (34). Next, observe that

$$\left| \Delta \frac{q_{v+1}}{q_v} \right| = \frac{q_{v+2}}{q_v} \left| \Delta \frac{q_v}{q_{v+1}} \right|.$$

This with (28), (33) and (34) implies that (26) and (30) are equivalent.

Theorem 4.2 — Let $r_n > 0$ (all $n \geqslant 0$), $R_n \to \infty$ as $n \to \infty$, then a sufficient condition for $(\bar{N}, r)$ be translative is that $\frac{r_{n+1}}{r_n}$ should be ultimately monotonic (in either sense).

Proof: (i) Assume that for $n > N$, $\frac{r_{n+1}}{r_n} \uparrow$. We will show that the conditions of Lemmas 3.1 and 3.2 are all satisfied. (28) which is equivalent to (27) is easily satisfied. If $\frac{r_{n+1}}{r_n} = O(1)$, then (31) follows from (32). If not, then $\frac{r_{n+1}}{r_n} \to \infty$ as $n \to \infty$. Let M be any given constant such that $M > 1$, then there exist n_0 such that

$$\frac{r_n}{r_{n+1}} \leqslant \frac{1}{M} \quad \text{for} \quad n > n_0. \qquad \text{...(35)}$$

This implies

$$R_n \leqslant r_n\left(1 + \frac{1}{M} + \frac{1}{M^2} + \ldots + \frac{1}{M^{n-n_0+1}}\right) + R_{n_0}. \qquad \text{...(36)}$$

Given any $\epsilon > 0$, choose M so that $\frac{1}{M} + \frac{1}{M^2} + \ldots + \frac{1}{M^{n-n_0+1}} < \epsilon$, using this and (35), we have

$$r_n \to \infty \quad \text{as} \quad n \to \infty \qquad \text{...(37)}$$

which gives

$$R_{n_0} < \epsilon r_n \quad \text{for sufficiently large } n. \qquad \text{...(38)}$$

Using (38), and the fact that $R_n > r_n$ we have from (36) that $R_n \sim r_n$ for sufficiently large n.

This gives (31).

As for (30), we have

$$\triangle \frac{r_v}{r_{v-1}} \leqslant 0 \quad \text{for} \quad v > N; \qquad \text{...(39)}$$

so that

$$\sum_{v=1}^{n} R_{v-1}\left|\triangle \frac{r_v}{r_{v-1}}\right| = \sum_{v=1}^{N-1} R_{v-1}\left|\triangle \frac{r_v}{r_{v-1}}\right| - \sum_{v=N}^{n} R_{v-1} \triangle \frac{r_v}{r_{v-1}}$$

$$= \frac{r_{n+1}}{r_n} R_n - R_{n+1} + C$$

where C is some constant.

This with (31) gives (30).

Finally, using (39), we have

$$\triangle \frac{r_{v-1}}{r_v} \geqslant 0 \quad \text{for} \quad v > N;$$

so that

$$\sum_{v=1}^{n} R_v\left|\triangle \frac{r_{v-1}}{r_v}\right| = \sum_{v=1}^{N-1} R_v\left|\triangle \frac{r_{v-1}}{r_v}\right| - \sum_{v=N}^{n} R_v \triangle \frac{r_{v-1}}{r_v}$$

$$= CR_n + R_{n-1} - \frac{r_n}{r_{n+1}} R_n$$

where C is some constant.

Using this and (28), condition (26) follows.

(ii) Assume that $\frac{r_{n+1}}{r_n}\downarrow$ for $n > N$. Then

$$r_{n+1} = O(r_n)$$

which gives (32). Since $\frac{r_{n+1}}{r_n}\downarrow$, then $\frac{r_n}{r_{n+1}}\uparrow$ for $n > N$.

If (28) does not hold, then we would have $\frac{r_n}{r_{n+1}} \to \infty$ as $n \to \infty$, which implies that

$$\frac{r_{n+1}}{r_n} \to 0 \text{ as } n \to \infty.$$

Hence $\Sigma_{n=0}^{\infty} r_n$ would converge, which would contradict $R_n \to \infty$ as $n \to \infty$. So (28) holds. As for (26) and (30) we can argue as in the first part of this proof, but with appropriate changes in sign.

5. Translativity of $(\bar{N}, r)(\bar{N}, q)$

We now have material for a proof of the following theorem.

Theorem 5.1 — Let $r_n > 0$, $R_n \to \infty$, $q_n > 0$, $Q_n \to \infty$. Then the necessary and sufficient conditions that $(\bar{N}, r)(\bar{N}, q)$ be translative are

$$\sum_{k=0}^{n} |\lambda_{n,k}| = O(1) \qquad \ldots(40)$$

and

$$\sum_{k=0}^{n} |\beta_{n,k}| = O(1) \qquad \ldots(41)$$

where

$$\lambda_{n,n} = \frac{R_n r_{n+1} Q_n q_{n+1}}{R_{n+1} r_n Q_{n+1} q_n} \qquad \ldots(42)$$

$$\lambda_{n,k} = \frac{R_k}{R_{n+1}} \left\{ \Delta \frac{r_{k+1} q_{k+1} Q_k}{r_k q_k Q_{k+1}} + \frac{Q_k r_{k+2}}{Q_{k+2} r_k} \Delta \frac{q_{k+1}}{q_k} + \Delta \left(\frac{Q_k}{r_k} \Delta \frac{q_{k+1}}{q_k} \right) \sum_{v=k+2}^{n} \frac{r_{v+1}}{Q_{v+1}} \right\} \quad (0 \leqslant k \leqslant n-1). \qquad \ldots(43)$$

and

$$\beta_{n,n} = \frac{1}{\lambda_{n,n}} \qquad \ldots(44)$$

$$\beta_{n,k} = \frac{R_{k+1}}{R_n}\left\{\Delta\left[\frac{q_k}{q_{k+1}}\left(\frac{r_k Q_{k+1}}{r_{k+1}Q_k}+1\right)\right]\right.$$

$$\left. + \Delta\left(\frac{Q_{k+1}}{r_{k+1}}\,\Delta\,\frac{q_k}{q_{k+1}}\right)\sum_{v=k+2}^{n}\frac{r_v}{Q_v}\right\} \quad (0 \leqslant k \leqslant n-1). \qquad \ldots(45)$$

Further let $(\bar{N}, r)$, $(\bar{N}, q)$ be regular, and that $\left\{\frac{r_{n+1}}{r_n}\right\}$, $\left\{\frac{q_{n+1}}{q_n}\right\}$ are ultimately monotonic (in either sense), (thus by Theorem 4.2, it follows that $(\bar{N}, r)$; $(\bar{N}, q)$ are translative), then $(\bar{N}, r)$ $(\bar{N}, q)$ is translative if, and only if

$$\sum_{k=0}^{n-1} |\lambda_{n,k}| = O(1) \qquad \ldots(46)$$

and

$$\sum_{k=0}^{n-1} |\beta_{n,k}| = O(1). \qquad \ldots(47)$$

PROOF : Let $\{U_n\}$, $\{\bar{U}_n\}$ be respectively the $(\bar{N}, q)$ transform of $\{S_n\}$, $\{S_{n-1}\}$. Let $\{t_n\}$, $\{\bar{t}_n\}$ be respectively the $(\bar{N}, r)$ $(\bar{N}, q)$ transform of $\{S_n\}$, $\{S_{n-1}\}$. Then

$$U_n = \frac{1}{Q_n}\sum_{v=0}^{n} q_v S_v \qquad \ldots(48)$$

which gives

$$\bar{U}_{n+1} = \frac{1}{Q_{n+1}}\sum_{v=0}^{n} q_{v+1} S_v. \qquad \ldots(49)$$

And

$$t_n = \frac{1}{R_n}\sum_{v=0}^{n} r_v U_v \qquad \ldots(50)$$

which gives

$$\bar{t}_{n+1} = \frac{1}{R_{n+1}}\sum_{v=0}^{n} r_{v+1}\bar{U}_{v+1}. \qquad \ldots(51)$$

Using (48) to obtain S_n in terms of U_n. Using this in (49) to obtain $\bar{U}_{n+1}$ in terms of U_n. From (50) obtain U_n in terms of t_n to get $\bar{U}_{n+1}$ in terms of t_n. Putting this in (51) we have

$$\bar{t}_{n+1} = \sum_{k=0}^{n} \lambda_{n,k} t_k \qquad \ldots(52)$$

where $\lambda_{n,k}$ as given in (42) and (43).

In a similar fashion we can easily find t_n in terms of $\bar{t}_n$. The result is

$$t_n = \sum_{k=0}^{n} \beta_{n,k} \bar{t}_{k+1}, \text{ where } \beta_{n,k} \text{ as given in (44) and (45).} \qquad \ldots(53)$$

The first part of the theorem follows from Corollary 2.1 if we prove that

$$\lambda_{n,k} \to 0 \text{ as } n \to \infty \text{ for fixed } k \qquad \ldots(54)$$

and

$$\beta_{n,k} \to 0 \text{ as } n \to \infty \text{ for fixed } k. \qquad \ldots(55)$$

For (54) to be satisfied we need only to show that for fixed k,

$$\frac{1}{R_{n+1}} \sum_{v=k+2}^{n} \frac{r_{v+1}}{Q_{v+1}} \to 0 \text{ as } n \to \infty.$$

We have $Q_n \to \infty$ as $n \to \infty$. Hence given any $\epsilon > 0$, there is an N such that

$$\frac{1}{|Q_{n+1}|} < \epsilon \text{ for } n > N.$$

Thus

$$\frac{1}{R_{n+1}} \sum_{v=k+2}^{n} \frac{r_{v+1}}{Q_{v+1}} = \frac{1}{R_{n+1}} \sum_{v=k+2}^{N} \frac{r_{v+1}}{Q_{v+1}} + \frac{1}{R_{n+1}} \sum_{v=N+1}^{n} \frac{r_{v+1}}{Q_{v+1}}$$

$$< \frac{1}{R_{n+1}} \sum_{v=k+2}^{N} \frac{r_{v+1}}{Q_{v+1}} + \epsilon \to 0 \text{ as } n \to \infty.$$

In a similar way (55) is satisfied.

For the final part of the theorem, it follows from the translativity of $(\bar{N}, r)$; $(\bar{N}, q)$ that $\lambda_{n,n} = O(1)$ and $\beta_{n,n} = O(1)$. Hence (46) and (47) follow from (40) and (42) respectively. This completes the proof.

6. EXAMPLE

It seems that the set of conditions of Theorem 5.1 are so complicated, but the question is whether these conditions are valid. The main point of this section is to give an example to show that even if $\left\{\frac{r_{n+1}}{r_n}\right\}$ and $\left\{\frac{q_{n+1}}{q_n}\right\}$ are monotonic (thus $(\bar{N}, r)$ and $(\bar{N}, q)$ are translative), (40) and (42) need not be satisfied, so that $(\bar{N}, r)$ $(\bar{N}, q)$ need not be translative.

Let $(\bar{N}, r)$ be $(C, 1)$, and $(\bar{N}, q)$ be regular. Let also

$$q_{2K} = \frac{1}{K+1}, \; q_{2K+1} = \frac{1}{\sqrt{(K+1)(K+2)}}$$

and

$$q_{2K+2} = \frac{1}{K+2}.$$

Then

$$\frac{q_{2K+1}}{q_{2K}} = \frac{\sqrt{K+1}}{\sqrt{K+2}}$$

and

$$\left.\begin{aligned}&\left[\Delta \frac{q_{k+1}}{q_1}\right]_{k=2K} = 0\\ &\left[\Delta \frac{q_{k+1}}{q_k}\right]_{k=2K+1} = \frac{\sqrt{K^2+4K+3}-(K+2)}{\sqrt{(K+1)(K+2)}}.\end{aligned}\right\} \qquad \ldots(56)$$

So $\left\{\frac{q_{k+1}}{q_k}\right\}$ is monotonic.

Also

$$\left[\Delta^2 \frac{q_{k+1}}{q_k}\right]_{k=2K} = \frac{(K+2)-\sqrt{(K+1)(K+3)}}{\sqrt{(K+2)(K+3)}} > 0 \qquad \ldots(57)$$

and

$$\left[\Delta^2 \frac{q_{k+1}}{q_k}\right]_{k=2K+1} = \frac{\sqrt{(K+1)(K+3)}-(K+2)}{\sqrt{(K+2)(K+3)}} < 0. \qquad \ldots(58)$$

Thus we have seen from (57) and (58) that $\Delta^2 \frac{q_{k+1}}{q_k}$ change in sign with respect to k even or odd. We note also that

$$Q_k \asymp \log(k+1) \asymp \log k \qquad (k \text{ large enough}).$$

First we show that (40) need not be satisfied. For $0 \leqslant k \leqslant n-1$, we have from (43) that

$$\lambda_{n,k} = \left(\frac{k+1}{n+2}\right)\left\{\Delta \frac{q_{k+1}Q_k}{q_k Q_{k+1}} + \frac{Q_k}{Q_{k+2}} \Delta \frac{q_{k+1}}{q_k} + \Delta\left[Q_k \Delta \frac{q_{k+1}}{q_k}\right] \sum_{\nu=k+2}^{n} \frac{1}{Q_{\nu+1}}\right\}$$

Using (56) we have

$$\left[\frac{Q_k}{Q_{k+2}} \Delta \frac{q_{k+1}}{q_k}\right]_{k=2K} = 0.$$

Note that

$$\Delta \frac{q_{2K+1}^2}{q_{2K}} = \frac{1}{(K+2)^{3/2}(\sqrt{K+1} + \sqrt{K+2})}$$

we have

$$\left[\Delta \frac{q_{k+1}Q_k}{q_k Q_{k+1}}\right]_{k=2K} = -\frac{q_{2K+1}^2}{q_{2K}} \Delta \frac{1}{Q_{2K+1}} - \frac{1}{Q_{2K+2}} \Delta^2 \frac{q_{2K+1}}{q_{2K}}$$

which is non-positive and is equal to $O\left(\frac{1}{K^2 \log K}\right)$.

Using (56) and (57) we have that

$$\left[\Delta\left(Q_k \Delta \frac{q_{k+1}}{q_k}\right)\right]_{k=2K} > \frac{C \log K}{K^2}$$

where C is some positive constant.

Write

$$\lambda_{n,2K} = \gamma_{n,K} + \delta_{n,K},$$

where $\gamma_{n,K}$ is the contribution of the first two terms of $\lambda_{n,K}$ and $\delta_{n,K}$ is the contribution of the last term. The result that

$$\sum_{K=0}^{n} |\lambda_{n,2K}| \neq O(1)$$

would follow at once, if we show that

$$\sum_{K=0}^{n} |\gamma_{n,K}| = O(1)$$

and

$$\sum_{K=0}^{n} |\delta_{n,K}| \neq O(1).$$

$$\sum_{K=0}^{n} |\gamma_{n,K}| = \frac{1}{n+2} \sum_{K=0}^{n} (2K+1) \,.\, O\left(\frac{1}{K^2 \log K}\right)$$

$$= O\left\{\frac{\log(\log n)}{n+2}\right\} = O(1).$$

And

$$\sum_{K=0}^{n} |\delta_{n,K}| > \frac{1}{n+2} \sum_{K=1}^{n} (2K+1) \,.\, \frac{C \log K}{K^2} \sum_{v=2K+2}^{n} \frac{1}{Q_{v+1}} \asymp \log n \neq O(1).$$

But

$$\sum_{k=0}^{n} |\lambda_{n,k}| \geqslant \sum_{K=0}^{n} |\lambda_{n,2K}|$$

therefore (40) need not be satisfied.

As for (41), we have from (45) that for $0 \leqslant k \leqslant n-1$,

$$\beta_{n,k} = \left(\frac{k+2}{n+1}\right) \left\{\triangle\left[\frac{q_k}{q_{k+1}}\left(\frac{Q_{k+1}}{Q_k}+1\right)\right]\right.$$

$$\left. + \triangle\left[Q_{k+1} \triangle \frac{q_k}{q_{k+1}}\right] \sum_{v=k+2}^{n} \frac{1}{Q_v}\right\}.$$

Now

$$\triangle\left[\frac{q_k}{q_{k+1}}\left(\frac{Q_{k+1}}{Q_k}+1\right)\right]_{k=2K} = \left[\triangle \frac{q_k}{Q_k}\right]_{k=2K}$$

$$= \frac{\dfrac{1}{K+1} - \dfrac{1}{\sqrt{(K+1)(K+2)}}}{Q_{2K+1}}$$

$$+ \frac{1}{(K+1)\sqrt{(K+1)(K+2)}\, Q_{2K} Q_{2K+1}} = A + B, \text{ say}$$

$$A \asymp \frac{1}{K^3 \log 2K} \asymp \frac{1}{k^3 \log k}$$

$$B \asymp \frac{1}{k^2 (\log k)^2},$$

so $A + B < \dfrac{\log k}{k^2}$ (k large enough).

On the other hand,

$$\triangle \left[Q_{k+1} \triangle \frac{q_k}{q_{k+1}} \right]_{k=2K} = \frac{\sqrt{(K+1)(K+3)} - (K+2)}{\sqrt{(K+1)(K+2)}} Q_{k+2}$$

which is non-positive and is equal to $O\left(\dfrac{\log k}{k^2}\right)$. So as before we can show that $\Sigma_{k=0}^{n} \mid \beta_{n,k} \mid \neq O(1)$, therefore (41) need not be satisfied, and the proof is complete.

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