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INTERPOLATION BY GENERALIZED POLYNOMIALS

YEN TZU FU

Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506

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In this paper, the author studies the cubic spline interpolation and approximation by generalized polynomials with four terms. A theorem on existence and uniqueness is proved.

1. INTRODUCTION

Cubic spline interpolation is the approximation of a given function by smooth piecewise cubic functions with interpolation property (Ahlberg *et al.* 1967, Cheney 1966, Schoenberg 1973).

Let $a = x_0 < x_1 < \ldots < x_n = b$ be a partition on $[a, b]$. Let f be a function defined on $[a, b]$. A proof of the following theorem may be found in Rivlin (1969, pp. 106–107).

Theorem 1.1 — For given U_0, U_n, there exists a unique function $s(x) \in C^2[a, b]$ such that in each interval $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$, $s(x)$ agrees with a polynomial of degree at most 3 and $s(x)$ satisfies

$$s(x_i) = f(x_i), \; i = 0, 1, \ldots, n$$

and

$$s'(x_i) = u_i, \; i = 0, n.$$

Throughout this paper $n, \alpha_0, \alpha_1, \ldots, \alpha_n$ will denote integers such that $n > 1$ and $0 \leqslant \alpha_0 < \alpha_1 < \ldots < \alpha_n$.

Definition 1.2 — Let $\{g_\alpha\}_{\alpha=0}^{\infty}$ be a sequence of functions, real-valued, non-negative and continuous on $[0, R]$ and analytic on $(0, R]$ where R is a positive constant. Further suppose that g_α is not a constant function if $\alpha \geqslant 1$, g_0 is not identically zero and $g_\alpha(0) = 0$ unless g_α is a constant. Then $\{g_\alpha\}_{\alpha=0}^{\infty}$ is said to have property $\mathscr{D}$ if and only if the following hold:

(i) For every set of non-zero real numbers $\{C_0, C_1, \ldots, C_n\}$ and for every choice of integers $\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ the number of zeros, counted with due regard to

Present address : Department of Mathematics, Indiana State University, Evansville, 8600 University Boulevard, Evansville, Indiana 47712, USA

multiplicity in $(0, R]$, of the sum $\Sigma_{k=0}^{n} C_k g_{\alpha_k}$ is at most equal to the number of variations of sign in the sequence $\{C_0, C_1, ..., C_n\}$.

(ii) For every set of non-zero numbers $\{C_0, C_1, ..., C_n\}$ and for every choice of integers $\{\alpha_0, \alpha_1, ..., \alpha_n\}$ the number of zeros, counted with due regard to multiplicity in $(0, R]$ of the sum $\Sigma_{k=0}^{n} C_k g'_{\alpha_k}$ is at most equal to the number of variations of sign in the sequence $\{C_0, C_1, ..., C_n\}$. (cf. Bell and Shah 1975).

Definition 1.3 — Let $\{g_\alpha\}_{\alpha=0}^{\infty}$ be a sequence of functions with property $\mathcal{D}$. For any set of non-zero finite real numbers $\{C_0, C_1, ..., C_n\}$, $\Sigma_{k=0}^{n} C_k g_{\alpha_k}$ is said to be a generalized polynomial of $(n + 1)$ terms.

The purpose of this paper is to replace the piecewise cubic polynomials in Theorem 1.1 by generalized polynomials of four terms, and to prove a theorem on existence and uniqueness.

2. Interpolation Theorems by Generalized Polynomials with Four Terms

Let $\{g_\alpha\}_{\alpha=0}^{\infty}$ be a sequence of functions with property $\mathcal{D}$ as defined in Definition 1.2. Let

$$V(x) = [g_{\alpha_0}(x)\, g_{\alpha_1}(x)\, g_{\alpha_2}(x)\, g_{\alpha_3}(x)]$$

be a row vector with components $g_{\alpha_k}(x)$, $g_{\alpha_k} \in \{g_\alpha\}_{\alpha=0}^{\infty}$, $k = 0, 1, 2, 3$.

Similarly, define row vectors V_i, V'_i as follows,

$$V_i = [g_{\alpha_0}(x_i)\, g_{\alpha_1}(x_i)\, g_{\alpha_2}(x_i)\, g_{\alpha_3}(x_i)],\ i = 0, 1, ..., n,$$

$$V'_i = [g'_{\alpha_0}(x_i)\, g'_{\alpha_1}(x_i)\, g'_{\alpha_2}(x_i)\, g'_{\alpha_3}(x_i)],\ i = 0, 1, ..., n,$$

where $0 < x_0 < ... < x_n \leqslant R$.

Let

$$\begin{vmatrix} V(x) \\ V_i \\ V'_{i-1} \\ V'_i \end{vmatrix}$$

be the determinant with row vectors $V(x)$, V_i, V'_{i-1}, V'_i defined as above.

Lemma 2.1 — If x_{i-1} and x_i are not both double zeros for the same function

$$g_{\alpha_k},\ k = 0, 1, 2, 3; \qquad \ldots(1)$$

then

$$\begin{vmatrix} V_{i-1} \\ V_i \\ V'_{i-1} \\ V'_i \end{vmatrix} \neq 0 \quad \text{for } i = 1, 2, \ldots, n.$$

PROOF : By the property $\mathcal{D}$ in Definition 1.2 there exist no common zeros in $(0, R]$ for any two functions in the sequence $\{g_\alpha\}_{\alpha=0}^{\infty}$. Otherwise the following generalized polynomial with no variation of sign

$$g_{\alpha_l}(x) + g_{\alpha_k}(x)$$

would have a zero in $(0, R]$ if $g_{\alpha_l}(x)$ and $g_{\alpha_k}(x)$ have a common zero in $(0, R]$. Thus we may assume that

$$g_{\alpha_0}(x_{i-1}) \neq 0.$$

The property $\mathcal{D}$ also implies that the determinant

$$\begin{vmatrix} g_{\alpha_0}(x_{i-1}) & g_{\alpha_k}(x_{i-1}) \\ g_{\alpha_0}(x_i) & g_{\alpha_k}(x_i) \end{vmatrix} = 0$$

if and only if

$$g_{\alpha_k}(x_{i-1}) = 0 \quad \text{and} \quad g_{\alpha_k}(x_i) = 0,\ k = 1, 2, 3. \qquad \ldots(2)$$

Moreover, there exists at most one function g_{α_k} $0 < k \leqslant 3$ such that (2) holds, since there exist no common zeros in $(0, R]$ for any two members in the sequence $\{g_\alpha\}_{\alpha=0}^{\infty}$. Thus we may assume that

$$\begin{vmatrix} g_{\alpha_0}(x_{i-1}) & g_{\alpha_l}(x_{i-1}) \\ g_{\alpha_0}(x_i) & g_{\alpha_l}(x_i) \end{vmatrix} \neq 0 \quad \text{for} \quad l = 1, 2.$$

Similar consideration as above leads to the conclusion that

$$\begin{vmatrix} g_{\alpha_0}(x_{i-1}) & g_{\alpha_1}(x_{i-1}) & g_{\alpha_k}(x_{i-1}) \\ g_{\alpha_0}(x_i) & g_{\alpha_1}(x_i) & g_{\alpha_k}(x_i) \\ g'_{\alpha_0}(x_{i-1}) & g'_{\alpha_1}(x_{i-1}) & g'_{\alpha_k}(x_{i-1}) \end{vmatrix} = 0, \ \ 1 < k \leqslant 3$$

if and only if

$$g_{\alpha_k}(x_{i-1}) = 0, \ g_{\alpha_k}(x_i) = 0 \ \text{ and } \ g'_{\alpha_k}(x_{i-1}) = 0, \ 1 < k \leqslant 3. \qquad \ldots(3)$$

Also, there exists at most one g_{α_k} $0 < k \leqslant 3$ satisfying (3).

If there exists none of g_{α_k} $k = 0, 1, 2, 3$ satisfying (3), then define the function

$$F(x) = \begin{vmatrix} V_{i-1} \\ V_i \\ V_{i-1} \\ V(x) \end{vmatrix} \quad \text{for } \ x \in [x_0, x_n].$$

Then $F(x)$ is a generalized polynomial with at least two terms. Further

$$\begin{vmatrix} V_{i-1} \\ V_i \\ V'_{i-1} \\ V'_i \end{vmatrix} = 0$$

implies that $F(x)$ has two double zeros x_{i-1} and x_i, which contradicts the property $\mathscr{D}$ of the sequence $\{g_\alpha\}_{\alpha=0}^{\infty}$.

In case there exists a function, say g_{α_3}, satisfying (3), then by (1)

$$g'_{\alpha_3}(x_i) \neq 0.$$

Define

$$H(x) = \begin{vmatrix} V_{i-1} \\ V_i \\ V(x) \\ V'_i \end{vmatrix} \quad \text{for } \ x \in [x_0, x_n].$$

Again, that

$$\begin{vmatrix} V_{i-1} \\ V_i \\ V'_{i-1} \\ V'_i \end{vmatrix} = 0$$

leads to the same contradiction as above. Thus the lemma is proved.

Corollary 2.2 — If condition (1) of Lemma 2.1 holds and neither x_{i-1} nor x_i is a triple zero for

$$g_{\alpha_k},\ k = 0, 1, 2, 3. \qquad \ldots(4)$$

Then

$$\begin{vmatrix} V_{i-1} \\ V_i \\ V'_i \\ V''_i \end{vmatrix} \neq 0 \text{ and } \begin{vmatrix} V_{i-1} \\ V_i \\ V'_{i-1} \\ V''_{i-1} \end{vmatrix} \neq 0.$$

PROOF : If there exists none of g_{α_k}, $k = 0, 1, 2, 3$; such that

$$g_{\alpha_k}(x_{i-1}) = 0,\ g_{\alpha_k}(x_i) = 0 \text{ and } g'_{\alpha_k}(x_i) = 0. \qquad \ldots(5)$$

Then

$$\begin{vmatrix} V_{i-1} \\ V_i \\ V'_i \\ V''_i \end{vmatrix} = 0$$

implies that the following generalized polynomial

$$F(x) = \begin{vmatrix} V_{i-1} \\ V_i \\ V'_i \\ V(x) \end{vmatrix}$$

has a zero x_{i-1} and a triple zero x_i which contradicts the property $\mathcal{D}$ of the sequence $\{g_\alpha\}_{\alpha=0}^{\infty}$.

Suppose there exists a function, say g_{α_3}, such that (5) holds. By (4)

$$g'_{\alpha_3}(x_i) \neq 0.$$

Then

$$\begin{vmatrix} V_{i-1} \\ V_i \\ V'_i \\ V''_i \end{vmatrix} = 0$$

implies that the following generalized polynomial

$$H(x) = \begin{vmatrix} V_{i-1} \\ V_i \\ V(x) \\ V''_i \end{vmatrix}$$

has a zero x_{i-1} and a triple zero x_i which contradicts the property $\mathcal{D}$ of the sequence $\{g_\alpha\}_{\alpha=0}^{\infty}$.

That

$$\begin{vmatrix} V_{i-1} \\ V_i \\ V'_{i-1} \\ V''_{i-1} \end{vmatrix} \neq 0$$

may be proved in a similar way.

Theorem 2.3 — For given $0 < x_{i-1} < x_i \leqslant R$, α_0, α_1, α_2, α_3 and constants f_{i-1}, f_i, f'_{i-1} and f'_i, if x_{i-1}, x_i and g_{α_k}, $k = 0, 1, 2, 3$; satisfy (1) and (4), then there exists a unique generalized polynomial

$$F(x) = \Sigma_{k=0}^{3} C_k g_{\alpha_k}(x)$$

such that

$$F(x_{i-1}) = f_{i-1},\ F(x_i) = f_i,\ F'(x_{i-1}) = f'_{i-1} \text{ and } F'(x_i) = f'_i. \qquad \ldots(6)$$

PROOF : Let

$$\triangle_{i-1} = \begin{vmatrix} V_{i-1} \\ V_i \\ V'_{i-1} \\ V'_i \end{vmatrix}.$$

Define

$$F(x) = \begin{vmatrix} V(x) \\ V_i \\ V'_{i-1} \\ V'_i \end{vmatrix} \frac{f_{i-1}}{\triangle_{i-1}} + \begin{vmatrix} V_{i-1} \\ V(x) \\ V'_{i-1} \\ V'_i \end{vmatrix} \frac{f_i}{\triangle_{i-1}} + \begin{vmatrix} V_{i-1} \\ V_i \\ V(x) \\ V'_i \end{vmatrix} \frac{f'_{i-1}}{\triangle_{i-1}} + \begin{vmatrix} V_{i-1} \\ V_i \\ V'_{i-1} \\ V(x) \end{vmatrix} \frac{f'_i}{\triangle_{i-1}}.$$

By Lemma 2.1

$$\triangle_{i-1} \neq 0.$$

Clearly, $F(x)$ is a generalized polynomial of the form $\Sigma_{k=0}^{3} C_k g_{\alpha_k}(x)$ which satisfies (6). The uniqueness follows from the fact that

$$\triangle_{i-1} \neq 0.$$

Theorem 2.4 — Let $0 < x_0 < \ldots < x_n \leqslant R$, let $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ be given non-negative integers and $f \in C'_{[x_0, x_n]}$. Suppose further that $x_{i-1}, x_i, i = 1, 2, \ldots, n$; and $g_{\alpha_k}, k = 0, 1, 2, 3$; satisfy (1) and (4) and any generalized polynomials g, h of the form $\Sigma_{k=0}^{3} C_k g_{\alpha_k}(x)$ satisfy the following conditions:

(iii) If $g(a) = 0,\ g'(a) = 0$ and $g(b) = 0$, then there exists an odd number of inflection points between a and b.

(iv) If $g(a) = 0, g'(a) = 0, g(b) = 0$ and

$$h(a) = 0,\ h(b) = 0,\ h'(b) = 0$$

then

$$\left|\frac{g''(b)}{g'(b)}\right| > \left|\frac{h''(b)}{h'(a)}\right|.$$

Then there exists a unique function $F \in C^2_{[x_0, x_n]}$ such that

$$F(x_i) = f(x_i)\ i = 0, 1, \ldots, n,\ F'(x_0) = f'(x_0),\ F'(x_n) = f'(x_n);$$

and in each interval $[x_{i-1}, x_i]$ $i = 1, 2, \ldots, n$ F agrees with a generalized polynomial containing at most four terms g_{α_0}, g_{α_1}, g_{α_2} and g_{α_3}.

Remarks : (1) By the property $\mathcal{D}$ of the sequence $\{g_\alpha\}_{\alpha=0}^{\infty}$, the generalized polynomial of four terms can have at most three zeros. Condition (iii) implies that for any double zero a and single zero b of such a generalized polynomial g the following holds:

$$g''(a) \cdot g''(b) \leqslant 0 \quad \text{and} \quad g''(a) \neq 0.$$

(2) The graphs of the generalized polynomials

$$\frac{g(x)}{g'(b)} \text{ and } \frac{h(x)}{h'(a)}$$

in (iv) are as indicated in Fig. 1 and Fig. 2, respectively.

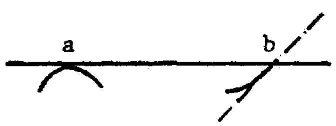

FIG. 1.

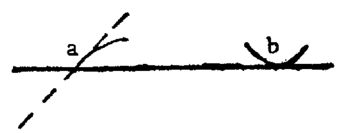

FIG. 2.

Condition (iv) implies that the curvature of $\dfrac{g(x)}{g'(b)}$ at b is greater than that of $\dfrac{h(x)}{h'(a)}$ at b.

Proof of Theorem 2.4 — Define

$$F(x) = \begin{vmatrix} V(x) \\ V_i \\ V'_{i-1} \\ V'_i \end{vmatrix} \frac{f_{i-1}}{\Delta_{i-1}} + \begin{vmatrix} V_{i-1} \\ V(x) \\ V'_{i-1} \\ V'_i \end{vmatrix} \frac{f_i}{\Delta_{i-1}} + \begin{vmatrix} V_{i-1} \\ V_i \\ V(x) \\ V'_i \end{vmatrix} \frac{U_{i-1}}{\Delta_{i-1}} + \begin{vmatrix} V_{i-1} \\ V_i \\ V'_{i-1} \\ V(x) \end{vmatrix} \frac{U_i}{\Delta_{i-1}}$$

for $x \in [x_{i-1}, x_i]$ ...(7)

where

$$\Delta_{i-1} = \begin{vmatrix} V_{i-1} \\ V_i \\ V'_{i-1} \\ V'_i \end{vmatrix} \qquad [i = 1, 2, \ldots, n)$$

and $f_i = f(x_i)$, $i = 0, 1, \ldots, n$; $U_0 = f'(x_0)$, $U_n = f'(x_n)$ with $(n-1)$ quantities U_i $[i = 1, 2, \ldots, (n-1)]$ to be determined.

Upon equating $F''(x_i)$ as calculated on $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$, we have the following $(n-1)$ linear equations:

$$\begin{vmatrix} V_i'' \\ V_i \\ V'_{i-1} \\ V'_i \end{vmatrix} \frac{f_{i-1}}{\Delta_{i-1}} + \begin{vmatrix} V_{i-1} \\ V_i'' \\ V'_{i-1} \\ V'_i \end{vmatrix} \frac{f_i}{\Delta_{i-1}} + \begin{vmatrix} V_{i-1} \\ V_i \\ V_i'' \\ V'_i \end{vmatrix} \frac{U_{i-1}}{\Delta_{i-1}} + \begin{vmatrix} V_{i-1} \\ V_i \\ V'_{i-1} \\ V_i'' \end{vmatrix} \frac{U_i}{\Delta_{i-1}}$$

$$= \begin{vmatrix} V_i'' \\ V_{i+1} \\ V'_i \\ V'_{i+1} \end{vmatrix} \frac{f_i}{\Delta_i} + \begin{vmatrix} V_i \\ V_i'' \\ V'_i \\ V'_{i-1} \end{vmatrix} \frac{f_{i+1}}{\Delta_i} + \begin{vmatrix} V_i \\ V_{i+1} \\ V_i'' \\ V'_{i+1} \end{vmatrix} \frac{U_i}{\Delta_i} + \begin{vmatrix} V_i \\ V_{i+1} \\ V'_i \\ V_i'' \end{vmatrix} \frac{U_{i+1}}{\Delta_i}$$

$$i = 1, 2, \ldots, (n-1). \qquad \ldots(8)$$

The matrix of the system (8) is tridiagonal with elements

$$\left.\begin{aligned} a_{j,j-1} &= \frac{\begin{vmatrix} V_{j-1} \\ V_j \\ V_j'' \\ V'_j \end{vmatrix}}{\Delta_{j-1}}, && j = 2, 3, \ldots, (n-1) \\ a_{j,j} &= \frac{\begin{vmatrix} V_{j-1} \\ V_j \\ V'_{j-1} \\ V_j'' \end{vmatrix}}{\Delta_{j-1}} - \frac{\begin{vmatrix} V_j \\ V'_{j+1} \\ V_j'' \\ V'_{j+1} \end{vmatrix}}{\Delta_j}, && j = 1, 2, \ldots, (n-1) \text{ and} \\ a_{j,j+1} &= -\frac{\begin{vmatrix} V_j \\ V_{j+1} \\ V'_j \\ V_j'' \end{vmatrix}}{\Delta_j} && j = 1, 2, \ldots, (n-2). \end{aligned}\right\} \ldots(9)$$

Each of the following four generalized polynomials

$$g_1(x) = \frac{\begin{vmatrix} V_{j-1} \\ V_j \\ V'_{j-1} \\ V(x) \end{vmatrix}}{\Delta_{j-1}}, \; j = 1, 2, \ldots, (n-1); \quad g_2(x) = -\frac{\begin{vmatrix} V_j \\ V_{j+1} \\ V(x) \\ V'_{j+1} \end{vmatrix}}{\Delta_j}, \; j = 1, 2, \ldots, (n-1);$$

$$h_1(x) = \frac{\begin{vmatrix} V_{j-1} \\ V_j \\ V(x) \\ V'_j \end{vmatrix}}{\Delta_{j-1}}, \; j = 2, 3, \ldots, (n-1); \quad h_2(x) = -\frac{\begin{vmatrix} V_j \\ V_{j+1} \\ V'_j \\ V(x) \end{vmatrix}}{\Delta_j}, \; j = 1, 2, \ldots, (n-2);$$

...(10)

has a double and a single zero with the derivative at the single zero either 1 or -1. The graphs of g_1, g_2, h_1 and h_2 at their zeros are shown in (a), (b), (c) and (d), Fig. 3, respectively.

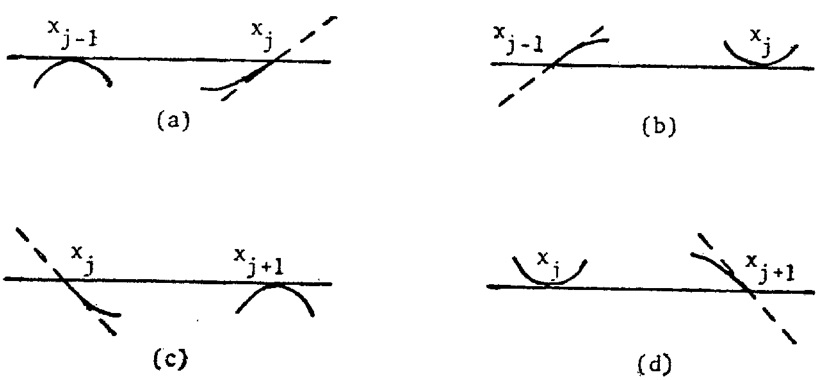

FIG. 3.

By applying (iii) to g_1, g_2, h_1 and h_2, we have

$$g_1''(x_j) \geqslant 0 \quad j = 1, 2, \ldots, (n-1); \quad g_2''(x_j) \geqslant 0 \quad j = 1, 2, \ldots, (n-1)$$

$$h_1''(x_j) > 0 \quad j = 2, 3, \ldots, (n-1); \quad h_2''(x_j) > 0 \quad j = 1, 2, \ldots, (n-2)$$

...(11)

By applying (iv) to g_k and $h_k (k = 1, 2,)$, we have

$$\left.\begin{aligned} g_1^*(x_j) &= \left|\frac{g_1''(x_j)}{g'(x_j)}\right| > \left|\frac{h_1''(x_j)}{h_1'(x_{j-1})}\right| = h_1^*(x_j) \quad j = 2, 3, \ldots, (n-1); \\ g_2^*(x_j) &= \left|\frac{g_2''(x_j)}{g_2'(x_j)}\right| > \left|\frac{h_2''(x_j)}{h_2'(x_{j+1})}\right| = h_2^*(x_j) \quad j = 1, 2, \ldots, (n-2) \end{aligned}\right\} \ldots(12)$$

That every entry in the matrix of the linear system (8) is non-negative follows from (9), (10) and (11). (12) implies that the matrix is diagonally dominant. Since a tridiagonal, diagonally dominant with non-negative entries is regular (see Rivlin 1969, pp. 107), system (8) has unique solution $\{U_1, U_2, \ldots, U_{n-1}\}$. With $U_0 = f'(x_0)$, $U_n = f'(x_n)$ and $f_i = f(x_i)$ $i = 0, 1, 2, \ldots, n$ the function (7) is uniquely constructed which satisfies every requirement for Theorem 2.4. Thus the theorem is proved.

(3) *Example* — Let $f(x) = \sin x$; $x_0 = \frac{\pi}{10}$, $x_1 = \frac{\pi}{2}$, $x_2 = \pi$, $g_{\alpha_0}(x) = 1$,

$g_{\alpha_1}(x) = x^{1/4}$, $g_{\alpha_2}(x) = x^{2/3}$ $g_{\alpha_3}(x) = x^{4/5}$.

The generalized polynomial constructed as in Theorem 2.4 is the following (coefficients are rounded off at fifth digit):

$$F(x) = \begin{cases} 8.11367 - 23.19826x^{1/4} + 48.84553x^{2/3} - 32.85175x^{4/5} & x \in \left[\frac{\pi}{10}, \frac{\pi}{2}\right] \\ -.28215 - 5.74643x^{1/4} + 26.47984x^{2/3} - 19.55697x^{4/5} & x \in \left[\frac{\pi}{2}, \pi\right] \end{cases}$$

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