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ABSOLUTE RIESZ SUMMABILITY FACTORS FOR A FOURIER SERIES AND ITS CONJUGATE SERIES

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In this paper the author discusses two general theorems on the absolute Riesz summability factors for a Fourier series and its conjugate series and obtains appropriate conditions under which the above summability can be ensured. The theorems exhibit the competency of a Riesz method as an instrument to study absolute Cesàro summability. Results due to Mohanty (1950) follow as corollaries to his theorems.

1. DEFINITIONS AND NOTATIONS

Let $\lambda = \lambda(\omega)$ be a differentiable, monotonic-increasing function of ω tending to infinity with ω. A given infinite series $\Sigma\, a_n$ is said to summable $|\, R, \lambda, r\,|$, $r > 0$ and we write $\Sigma\, a_n \in |\, R, \lambda, r\,|$, if

$$\int_A^{\infty} \frac{\lambda'(\omega)}{\{\lambda(\omega)\}^{r+1}} \left| \sum_{n<\omega} \{\lambda(\omega) - \lambda(n)\}^{r-1} \lambda_n a_n \right| d\omega < \infty$$

where A is a finite positive number (Obrechkoff 1928, 1929; Mohanty 1951). It is known (Hyslop 1936) that $|\, R, \omega, k\,| \sim |\, C, k\,|$.

Let $f \in L(-\pi, \pi)$ be a 2π-periodic function. Let the Fourier series of f at a point x, be

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} A_n(x).$$

Then the conjugate series of the above Fourier series is

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) = \sum_{n=1}^{\infty} B_n(x).$$

Throughout this paper we shall use the following notations:

$$\phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t)\}$$

$$\psi(t) = \tfrac{1}{2}\{f(x+t) - f(x-t)\}$$

$$\phi^*(t) = \phi(t) - \phi(+0)$$

$$e(\omega) = \exp(\log \omega)^{1+\delta}, \quad \delta \geqslant 0$$

$$g(\omega, t) = \sum_{n<\omega} \{e(\omega) - e(n)\}^{\beta-1} e(n)\, n^{\alpha-1}(\log n)^{-\alpha\delta} \cos nt$$

$$\bar{g}(\omega, t) = \sum_{n<\omega} \{e(\omega) - e(n)\}^{\beta-1} e(n)\, n^{\alpha-1}(\log n)^{-\alpha\delta} \sin nt$$

$$Q(\omega) = \{e(\omega) - e(m)\}^{\beta-1} e(m)\, m^{\alpha-1}(\log m)^{-\alpha\delta}$$

where m is an integer such that $0 < \omega - m \leqslant 1$.

k denotes a suitable constant chosen for convenience in analysis. K, K_1, K_2 and K_3 stand for absolute constants, possibly different each time.

2. Introduction

Mohanty (1950) established the following two results pertaining to absolute Cesàro summability factors for a Fourier series and its conjugate series.

Theorem A — Let $0 < \alpha < 1$. Then $\int_0^\pi t^{-\alpha} \mid d\phi(t) \mid < \infty$ implies that

$$\Sigma\, n^\alpha A_n(x) \in \mid C, \beta \mid, \ \beta > \alpha.$$

Theorem B — Let $0 < \alpha < 1$. Then $\psi(+0) = 0$ and $\int_0^\pi t^{-\alpha} \mid d\psi(t) \mid < \infty$ imply that $\Sigma\, n^\alpha B_n(x) \in \mid C, \beta \mid, \ \beta > \alpha$.

The purpose of this paper is to establish the following two general results on the absolute Riesz summability factors for a Fourier series and its conjugate series.

In what follows we prove the following:

Theorem 1 — Let $0 < \alpha < 1$ and $\delta \geqslant 0$. Then $\int_0^\pi t^{-\alpha} \mid d\phi(t) \mid < \infty$ implies that $\sum_{n=2}^{\infty} n^\alpha A_n(x) (\log n)^{-\alpha\delta} \in \mid R, \exp(\log \omega)^{1+\delta}, \beta \mid \ \beta > \alpha$.

Theorem 2 — Let $0 < \alpha < 1$ and $\delta \geqslant 0$. Then $\psi(+0) = 0$ and $\int_0^\pi t^{-\alpha} \mid d\psi(t) \mid < \infty$ imply that $\sum_{n=2}^{\infty} n^\alpha B_n(x) (\log n)^{-\alpha\delta} \in \mid R, \exp(\log \omega)^{1+\delta}, \beta \mid, \ \beta > \alpha$.

3. Estimates

We require the following order estimates to prove our theorems.

For $0 < \alpha < \beta \leqslant 1$,

$$\begin{matrix} g(\omega, t) \\ \bar{g}(\omega, t) \end{matrix} = \begin{cases} O(\omega^{\alpha} e^{\beta}(\omega) (\log \omega)^{-\delta(\alpha+1)}) + Q(\omega), \\ O(t^{-\beta} \omega^{\alpha-\beta} e^{\beta}(\omega) (\log \omega)^{\delta(\beta-\alpha-1)}) + Q(\omega). \end{cases}$$

We give a proof for the estimates for g only, the proof for $\bar{g}$ is similar.

PROOF : $g(\omega, t) \leqslant \sum\limits_{n<\omega} \{e(\omega) - e(n)\}^{\beta-1} e(n)\, n^{\alpha-1}(\log n)^{-\alpha\delta}$

$$= \sum_{n \leqslant \sqrt{\omega}} + \sum_{\sqrt{\omega}<n<\omega} = R_1 + R_2, \text{ say.}$$

$$R_1 \leqslant \{e(\omega) - e(\sqrt{\omega})\}^{\beta-1} e(\sqrt{\omega})\, (\sqrt{\omega})^{\alpha}\, (\log \sqrt{\omega})^{-\alpha\delta}$$

$$= \{(\omega - \sqrt{\omega})\, e'(\omega_1)\}^{\beta-1} e(\sqrt{\omega})\, (\sqrt{\omega})^{\alpha}\, (\log \sqrt{\omega})^{-\alpha\delta}, \; (\sqrt{\omega} < \omega_1 < \omega),$$

$$\leqslant K \omega^{\alpha/2} e^{\beta}(\omega) (\log \omega)^{\delta(\beta-\alpha-1)};$$

$$R_2 \leqslant \int_{\sqrt{\omega}}^{\omega} \{e(\omega) - e(u)\}^{\beta-1} e(u)\, u^{\alpha-1} (\log u)^{-\alpha\delta}\, du + Q(\omega)$$

$$\leqslant K \omega^{\alpha}(\log \sqrt{\omega})^{-\delta(\alpha+1)} \{e(\omega) - e(\sqrt{\omega})\}^{\beta} + Q(\omega)$$

$$\leqslant K \omega^{\alpha}(\log \omega)^{-\delta(\alpha+1)} e^{\beta}(\omega) + Q(\omega).$$

To prove the second estimate we put $\omega_1 = \left[\omega - \frac{k}{t}\right]$. Now

$$g(\omega, t) = \{ \sum_{n \leqslant \omega_1} + \sum_{\omega_1+1}^{m} \} (e(\omega) - e(n))^{\beta-1} e(n)\, n^{\alpha-1}(\log n)^{-\alpha\delta} \cos nt$$

$$= R_3 + R_4, \text{ say.}$$

As $\left\{\left((e(\omega) - e(n))\right)^{\beta-1} e(n)\, n^{\alpha-1}(\log n)^{-\alpha\delta}\right\}$ is ultimately monotonic-increasing in $n < \omega$,

$$R_3 = O[\{e(\omega) - e(\omega_1)\}^{\beta-1} e(\omega_1)\, \omega_1^{\alpha-1} (\log \omega_1)^{-\alpha\delta} \max_{2 \leqslant a < b \leqslant \omega_1} |\sum_{a}^{b} \cos nt\, |\,]$$

$$= O\{t^{-\beta} \omega^{\alpha-\beta} e^{\beta}(\omega) (\log \omega)^{\delta(\beta-\alpha-1)}\},$$

and similarly we can have

$$R_4 = O\{t^{-\beta} \omega^{\alpha-\beta} e^{\beta}(\omega) (\log \omega)^{\delta(\beta-\alpha-1)}\}.$$

4. PROOF OF THEOREM 1

Without any loss of generality we assume $0 < \alpha < \beta < 1$. Integrating by parts, we have

$$A_n(x) = \frac{2}{\pi} \int_0^{\pi} \phi(t) \cos nt\, dt = -\frac{2}{\pi} \int_0^{\pi} \frac{\sin nt}{n}\, d\phi(t).$$

The series $\sum_{n=2}^{\infty} n^{\alpha} A_n(x) (\log n)^{-\alpha\delta} \in | R\, e(\omega), \beta |$, iff

$$I = \int_2^{\infty} \frac{(\log \omega)^{\delta}}{\omega e^{\beta}(\omega)} \left| \int_0^{\pi} \sum_{n<\omega} \{e(\omega) - e(n)\}^{\beta-1} e(n)\, n^{\alpha} (\log n)^{-\alpha\delta} \phi(t) \cos nt\, dt \right| d\omega$$

$$= \int_2^{\infty} \frac{(\log \omega)^{\delta}}{\omega e^{\beta}(\omega)} \left| \int_0^{\pi} d\phi(t)\, \bar{g}(\omega, t)\, dt \right| d\omega$$

is convergent.

We observe that in order to prove our theorem it is enough to show that

$$\int_0^{\pi} | d\phi(t) |\, p(t) < \infty, \qquad \ldots(4.1)$$

where $$p(t) = \int_2^{\infty} \frac{(\log \omega)^{\delta}}{\omega e^{\beta}(\omega)} | \bar{g}(\omega, t) |\, d\omega.$$

On writing $T = \frac{k}{t} (\log k/t)^{\delta}$ and subdividing the range of integration in $p(t)$, we have

$$p(t) \leqslant K_1 \int_2^{T} \omega^{\alpha-1} (\log \omega)^{-\alpha\delta}\, d\omega + K_2 t^{-\beta} \int_T^{\infty} \omega^{\alpha-\beta-1} (\log \omega)^{\delta(\beta-\alpha)}\, d\omega$$

$$+ K_3 \int_2^{\infty} \frac{(\log \omega)^{\delta}}{\omega\, e^{\beta}(\omega)} Q(\omega)\, d\omega$$

$$\leqslant K_1 T^{\alpha} (\log T)^{-\alpha\delta} + K_2 t^{-\beta} T^{\alpha-\beta} (\log T)^{\delta(\beta-\alpha)}$$

$$+ K_3 \sum_{m=2}^{\infty} \int_m^{m+1} \{e(\omega) - e(m)\}^{\beta-1} \frac{e(m) (\log \omega)^{\delta} m^{\alpha-1}\, d\omega}{\omega e^{\beta}(\omega) (\log m)^{\alpha\delta}}$$

$$\leqslant K_1 t^{-\alpha} \left(\log \frac{k}{t}\right)^{\alpha\delta} \left\{\log \frac{k}{t} + \delta \log\log \frac{k}{t}\right\}^{-\alpha\delta}$$

$$+ K_2 t^{-\alpha} \left(\log \frac{k}{t}\right)^{\delta(\alpha-\beta)} \{\log k/t + \delta \log\log k/t\}^{\delta(\beta-\alpha)}$$

$$+ K_3 \sum_{m=2}^{\infty} m^{\alpha-1} e^{-\beta}(m) (\log m)^{-\alpha\delta} \{e(m+1) - e(m)\}^{\beta}$$

$$\leqslant K_1 t^{-\alpha}\left\{1+\frac{\delta \log\log k/t}{\log k/t}\right\}^{-\alpha\delta} + K_2 t^{-\alpha}\left\{1+\frac{\delta \log\log k/t}{\log k/t}\right\}^{\delta(\beta-\alpha)}$$

$$+ K_3 \sum_{m=2}^{\infty} m^{\alpha-1} e^{-\beta}(m) (\log m)^{-\alpha\delta}\left[\frac{e(m_1)(1+\delta)(\log m_1^{\delta})}{m_1}\right]^{\beta}$$

$$\text{for } m < m_1 < m+1,$$

$$\leqslant K_1 t^{-\alpha} + K_2 t^{-\alpha} + K_3 \sum_{2}^{\infty} \frac{(\log m)^{\delta(\beta-\alpha)}}{m^{1+\beta-\alpha}}$$

$$\leqslant K t^{-\alpha} + K.$$

Hence uniformly in $0 < t < \pi$,

$$p(t) = O(t^{-\alpha}).$$

Thus from (4.1) and the hypothesis

$$\int_0^{\pi} t^{-\alpha} \mid d\phi(t) \mid < \infty.$$

This completes the proof of Theorem 1.

5. Proof of Theorem 2

Integrating by parts and remembering that $\psi(+0) = 0$, we have

$$B_n(x) = -\frac{2}{\pi}\psi(\pi)\frac{\cos n\pi}{n} + \frac{2}{\pi}\int_0^{\pi}\frac{\cos nt}{n}\, d\psi(t).$$

The proof of this theorem is almost similar to that of Theorem 1 except that in place of the estimates for $g(\omega, t)$ we have to use the estimates for $\bar{g}(\omega, t)$ and the hypotheses of this theorem.

6. Corollaries

Mohanty (1951) established that if $\alpha > 0$, $\eta > 0$, then necessary and sufficient conditions that $\int_0^{\eta} t^{-\alpha} \mid d\psi(t) \mid < \infty$, and $\psi(+0) = 0$, are that

(i) $t^{-\alpha}\psi(t) \in BV(0, \eta)$,

and (ii) $\dfrac{\mid \psi(t) \mid}{t^{\alpha+1}} \in L(0, \eta)$.

Replacing $\phi(t)$ in the hypothesis of theorem 1 by $\phi^*(t) = \phi(t) - \phi(+0)$, so that $\phi^*(+0) = 0$, and using the above result along with $\phi^*(t)$ in place of $\psi(t)$; and $\delta = 0$, we see that Theorems 1 and 2 are respectively equivalent to Corollaries 1 and 2.

Corollary 1 — Let $0 < \alpha < 1$. Then

(i) $t^{-\alpha}\,\phi^*(t) \in BV(0, \pi)$ and

(ii) $\dfrac{|\,\phi^*(t)\,|}{t^{\alpha+1}} \in L(0, \pi)$ imply that $\sum_{n=2}^{\infty} n^{\alpha} A_n(x) \in |\,C, \beta\,|,\ \beta > \alpha.$

Corollary 2 — Let $0 < \alpha < 1$. Then

(i) $t^{-\alpha}\,\psi(t) \in BV(0, \pi)$ and

(ii) $\dfrac{|\,\psi(t)\,|}{t^{\alpha+1}} \in L(0, \pi)$ imply that $\sum_{n=2}^{\infty} n^{\alpha} B_n(x) \in |\,C, \beta\,|,\ \beta > \alpha.$

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