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ON NÖRLUND SUMMABILITY OF JACOBI SERIES

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In this paper the authors prove a new theorem on Nörlund summability of Jacobi series. This theorem is an improvement over an earlier result of Pandey and Beohar (1978) and also on that of Prasad and Saxena (1979).

1. INTRODUCTION

Let $f(x)$ be defined in the closed interval $[-1, 1]$ such that the function

$$(1-x)^{\alpha}(1+x)^{\beta} f(x) \in L[-1, 1];\ \alpha > -1, \beta > -1.$$

The Jacobi series corresponding to this function is

$$f(x) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x) \qquad \ldots(1.1)$$

where

$$a_n = \frac{(2n+\alpha+\beta+1)\,\Gamma(n+1)\,\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}$$

$$\times \int_{-1}^{+1} (1-x)^{\alpha}(1+x)^{\beta} f(x)\, P_n^{(\alpha,\beta)}(x)\, dx \qquad \ldots(1.2)$$

and $P_n^{(\alpha,\beta)}$ are the Jacobi polynomials.

We write

$$F(\phi) \equiv \{f(\cos\phi) - A\}\left(\sin\frac{\phi}{2}\right)^{2\alpha+1}\left(\cos\frac{\phi}{2}\right)^{2\beta+1}$$

A being a fixed constant.

The object of the present paper is to prove a theorem on Nörlund summability of Jacobi series, which improves the result of Gupta (1970), Choudhary (1970), Pandey and Beohar (1978), Prasad and Saxena (1979).

We prove the following:

Theorem — Let $\{p_n\}$ be a nonnegative, non-increasing sequence such that $\sum_{0}^{n} p_k = P_n$ and $n^{(2\alpha+1)/2}/P_n \to 0$ as $n \to \infty$ and

$$\sum_{a}^{n} \frac{P_k}{k^{(2\alpha+3)/2}} = O\left(\frac{P_n}{n^{(2\alpha+1)/2}}\right) \quad ...(1.3)$$

a being a fixed positive integer. If

$$\psi(t) \equiv \int_0^t | F(\phi) | \, d\phi = o(t^{2\alpha+2}) \text{ as } t \to 0 \quad ...(1.4)$$

then the series (1.1) is summable (N, p_n) at $x = 1$ to the sum A provided $-\frac{1}{2} \leqslant \alpha < \frac{1}{2}$; $\beta > -\frac{1}{2}$ and the antipole condition

$$\int_{-1}^{b} (1 + x)^{(2\beta-3)/4} \, | f(x) | \, dx < \infty \quad ...(1.5)$$

b fixed, is satisfied

2. Lemmas

The following lemmas are required for the proof of our theorem.

Lemma 1 — For α, β arbitrary and real and C a fixed positive constant

$$P_n^{(\alpha,\beta)}(\cos\theta) = \begin{cases} \theta^{-(2\alpha+1)/2} \, O(n^{-1/2}) & \text{for } \frac{c}{n} \leqslant \theta \leqslant \pi/2 \\ O(n^{\alpha}) & \text{for } 0 \leqslant \theta \leqslant \frac{c}{n} \end{cases} \quad ...(2.1)$$

[For the proof see Szegö (1959, p. 167).]

Lemma 2 — If $\alpha > -1$, $\beta > -1$; $\frac{c}{n} \leqslant \theta \leqslant \pi - \frac{c}{n}$

$$P_n^{(\alpha,\beta)}(\cos\theta) = n^{-1/2} u(\theta) \left[\cos\left\{\left(n + \frac{\alpha + \beta + 1}{2}\right)\theta + \nu\right\} + \frac{O(1)}{n \sin\theta}\right] \quad ...(2.2)$$

where

$$u(\theta) = \frac{1}{\sqrt{\pi}} (\sin \tfrac{1}{2}\theta)^{-(2\alpha+1)/2} (\cos \tfrac{1}{2}\theta)^{-(2\beta+1)/2}, \; \nu = -(\alpha + \tfrac{1}{2}) \frac{\pi}{2}$$

[see Szegö (1959, p. 196)].

Lemma 3 — Let

$$N_n(\phi) = \frac{2^{\alpha+\beta+1}}{P_n} \sum_{k=0}^{n} p_k \lambda_{n-k} P_{n-k}^{(\alpha+1,\beta)}(\cos\theta)$$

where

$$\lambda_n = \frac{2^{-(\alpha+\beta+1)}\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\,\Gamma(n+\beta+1)} \cong \frac{2^{-(\alpha+\beta+1)}}{\Gamma(\alpha+1)}\, n^{\alpha-1}.$$

For $0 \leqslant \theta \leqslant \frac{1}{n}$

$$N_n(\phi) = O(n^{2\alpha+2}). \qquad \ldots(2.3)$$

For $\frac{1}{n} \leqslant \theta \leqslant \pi - \frac{1}{n}$

$$N_n(\phi) = \frac{1}{P_n}\, O\left[n^{(2\alpha+1)/2}P(1/\phi)/(\sin \tfrac{1}{2}\phi)^{(2\alpha+3)/2}\,(\cos \tfrac{1}{2}\phi)^{(2\beta+1)/2}\right]$$

$$+\, O\left[n^{(2\alpha-1)/2}/(\sin \tfrac{1}{2}\phi)^{(2\alpha+5)/2}\,(\cos \tfrac{1}{2}\phi)^{(2\beta+3)/2}\right]. \qquad \ldots(2.4)$$

[For the proof of Lemma reference may be made to Pandey and Beohar (1978).]

Lemma 4 — The antipole condition

$$\int_{-1}^{b} (1+x)^{(2\beta-3)/4}\,|\,f(x)\,|\,dx < \infty$$

implies that

$$(\cos \tfrac{1}{2}\phi)^{(2\beta-1)/2}\,|\,f(\cos\phi) - A\,|\,d\phi \qquad \ldots(2.5)$$

which further implies that

$$\int_{0}^{\frac{1}{2}} t^{(2\beta-1)/2}\,|\,f(-\cos t) - A\,|\,dt = o(1) \qquad \ldots(2.6)$$

[see Gupta (1970) for its proof].

3. Proof of the Theorem

The nth partial sum of the series (1.1) at this point $x = 1$ is given by Obrechkoff (1936) as

$$s_n(1) = 2^{\alpha+\beta+1}\lambda_n \int_{0}^{\pi} (\sin \tfrac{1}{2}\phi)^{2\alpha+1}\,(\cos \tfrac{1}{2}\phi)^{2\beta+1}\, f(\cos\phi)\, P_n^{(\alpha+1,\beta)}(\cos\phi)\, d\phi.$$

Consequently

$$s_n(1) - A = 2^{\alpha+\beta+1}\lambda_n \int_{0}^{\pi} (\sin \tfrac{1}{2}\phi)^{2\alpha+1}\,(\cos \tfrac{1}{2}\phi)^{2\beta+1}$$

$$\times\, F(\phi)\, P_n^{(\alpha+1,\beta)}(\cos\phi)\, d\phi. \qquad \ldots(3.1)$$

The Nörlund mean of the series (1.1) at $x = 1$ is given by

$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_k S_{n-k}(1).$$

Therefore

$$t_n - A = \frac{1}{P_n} \sum p_k 2^{\alpha+\beta+1} \lambda_{n-k} \int_0^{\pi} F(\phi)\, P_{n-k}^{(\alpha+1,\beta)} (\cos \phi)\, d\phi$$

$$= \int_0^{\pi} F(\phi)\, N_n(\phi)\, d\phi$$

$$= \int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi-(1/n)} + \int_{\pi-(1/n)}^{\pi}$$

$$= I_1 + I_2 + I_3 + I_4, \text{ say} \qquad \ldots(3.2)$$

δ being a suitable constant.

$$I_1 = \int_0^{1/n} F(\phi)\, N_n(\phi)\, d\phi$$

$$|\, I_1 \,| = O(n^{2\alpha+2})\, o(n^{-(2\alpha+2)}), \text{ by } (1.5)$$

$$= o(1).$$

In order to estimate I_2 we employ the asymtotic relation given in (2.4). Thus

$$I_2 = O \int_{1/n}^{\delta} |\, F(\phi) \,| \frac{(n^{(2\alpha+1)/2}}{P_n} P(1/\phi)\, (\sin \tfrac{1}{2}\phi)^{-(2\alpha+3)/2}\, d\phi$$

$$+ O \int_{1/n}^{\delta} |\, F(\phi) \,| \,.\, (n^{(2\alpha-1)/2}\, (\sin \tfrac{1}{2}\phi)^{-(2\alpha+5)/2}\, d\phi$$

$$= I_{2\cdot 1} + I_{2\cdot 2}, \text{ say.} \qquad \ldots(3.3)$$

Now

$$I_{2\cdot 1} = O\left(\frac{n^{(2\alpha+1)/2}}{P_n}\right) \int_0^{1/n} \frac{|\, F(\phi) \,|\, P(1/\phi)}{\phi^{(2\alpha+3)/2}}\, d\phi$$

$$= O\left(\frac{n^{(2\alpha+1)/2}}{P_n}\right) \left[o\left(\frac{t^{2\alpha+2} P(1/t)}{t^{(2\alpha+3)/2}}\right)_{1/n}^{\delta} + o \int_{1/n}^{\delta} \phi^{2\alpha+2} \frac{d}{d\phi} \left(\frac{P(1/\phi)}{\phi^{(2\alpha+3)/2}}\right) d\phi \right]$$

$$= o(1) + O\left(\frac{n^{(2\alpha+1)/2}}{P_n}\right)\left[o \int_{1/\delta}^{n} \frac{1}{x^{2\alpha+2}} \frac{d}{dx} \mid (P(x) . x^{(2\alpha+3)/2} \mid dx\right.$$

$$\left.= o(1) + O\left(\frac{n^{(2\alpha+1)/2}}{P_n}\right) \sum_{a}^{n} \frac{1}{k^{(2\alpha+3)/2}} \mid \triangle \{P(k) . k^{(2\alpha+3)/2}\} \mid \right] + O(1)$$

where $a = \left[\frac{1}{\delta}\right] + 1,\ n \geqslant [1/t]$

$$= o(1) + O\left(\frac{n^{(2\alpha+1)/2}}{P_n}\right) o \sum \frac{P_k}{k^{(2\alpha+3)/2}}$$

$$= o(1) \text{ by (1.3)} \qquad \ldots(3.4)$$

$$I_{2\cdot 2} = \int_{1/n}^{\delta} \mid F(\phi)\ (n^{(2\alpha-1)/2})\ (\sin \tfrac{1}{2}\phi)^{-(2\alpha+5)/2}\ d\phi$$

$$= O(n^{(2\alpha-1)/2}) \int_{1/n}^{\delta} \frac{F(\phi)\ d\phi}{\phi^{(2\alpha+5)/2}}$$

$$= O(n^{(2\alpha-1)/2}) \left[\left\{\frac{1}{\phi^{(2\alpha+5)/2}}\ o(\phi^{2\alpha+2})\right\}_{1/n}^{\delta} + o \int_{1/n}^{\delta} (\phi^{(2\alpha-3)/2}\ d\phi + O(1)\right.$$

$$= O(n^{(2\alpha-1)/2})\ o(\phi^{(2\alpha-1)/2})$$

$$= o(1). \qquad \ldots(3.5)$$

Considering I_3, we have

$$I_3 = O \int_{\delta}^{\pi-(1/n)} \frac{\mid F(\phi) \mid P(1/\phi)}{(\sin \frac{1}{2}\phi)^{(2\alpha+3)/2}\ (\cos \frac{1}{2}\phi)^{(2\beta+1)/2}} \left(\frac{n^{(2\alpha+1)/2}}{P_n}\right) d\phi$$

$$+ O(n^{(2\alpha-1)/2} \int_{\delta}^{\pi-(1/n)} \mid F(\phi) \mid \frac{d\phi}{(\sin \frac{1}{2}\phi)^{(2\alpha+5)/2}\ (\cos \frac{1}{2}\phi)^{(2\beta+1)/2}}$$

$$= O\left(\frac{n^{(2\alpha-1)/2}}{P_n}\right) \int_{\delta}^{\pi-(1/n)} \mid F(\cos\phi) - A \mid (\cos \tfrac{1}{2}\phi)^{(2\beta-1)/2} \cos \tfrac{1}{2}\phi\ d\phi$$

$$+ O(n^{(2\alpha-1)/2} \int_{\delta}^{\pi-(1/n)} \mid F(\cos\phi) - A \mid (\cos \tfrac{1}{2}\phi)^{(2\beta-1)/2}\ d\phi$$

$$= O\left(\frac{n^{(2\alpha+1)/2}}{P_n}\right) + O(n^{(2\alpha-1)/2})$$

$$= o(1) \text{ as } n \to \infty. \qquad \ldots(3.6)$$

Finally coming to I_4

$$I_4 = \int_{\pi-(1/n)}^{\pi} F(\phi) \frac{2^{\alpha+\beta+1}}{P_n} \sum_{k=0}^{n} p_k \lambda_{n-k} P_{n-k}^{(\alpha+1,\beta)} (\cos \phi)\, d\phi$$

$$= \frac{2^{\alpha+\beta+1}}{P_n} \sum_{k=0}^{n} \int_0^{1/n} | F(\pi - \theta) | \lambda_{n-k} p_k P_{n-k}^{(\alpha+1,\beta)} (\cos \phi)\, d\phi$$

$$= O\left(\frac{1}{P_n}\right) \sum_{k=0}^{n} p_k (n - k)^{\alpha+1} \int_0^{1/n} O(n - k)^{\beta} | F(- \cos \phi) - A |$$

$$\times (\sin \tfrac{1}{2}\phi)^{2\beta+1} (\cos \tfrac{1}{2}\phi)^{2\alpha+1}\, d\phi$$

$$= O(n^{\alpha+\beta+1}) \int_0^{1/n} | F(- \cos \phi) - A | \phi^{\beta-1}\, d\phi$$

$$= o(1) \text{ by (2.6).}$$

Thus the theorem is completely established.

Example — If we take

$$f(\phi) = \phi^p (\sin \tfrac{1}{2}\phi)^{2\alpha} (\cos \tfrac{1}{2}\phi)^{2\beta} \sin \phi,\ p > 0$$

which is the case when $x = \cos \phi$, then we get

$$\int_0^t | F(\phi) |\, d\phi = O(t^{2\alpha+2+\gamma}),\ \gamma > 0$$

$$= o(t^{2\alpha+2}) \text{ as } O(t^{\gamma}) \to 0, \text{ as } t \to 0.$$

Thus $F(\phi)$ satisfies the condition (1.4) of our theorem.

Also as $n^{(2\alpha+1)/2}/P_n$ is decreasing we take $P_n = n^p$ such that $p > \alpha + \frac{1}{2} > 0$ or let

$$P_n = n^{(2\alpha+1+2\delta)/2} \text{ as } \alpha > -\tfrac{1}{2}, \delta > 0.$$

Again

$$\sum_a^n \frac{P_k}{k^{(2\alpha+3)/2}} = \sum \frac{k^p}{k^{(2\alpha+3)/2}}$$

$$= \sum_a^n \frac{k^{(2\alpha+1+2\delta)/2}}{k^{(2\alpha+3)/2}}$$

$$= \sum_{a}^{n} \frac{1}{k^{1-\delta}}$$

$$= O(n^{\delta})$$

$$= O\left(\frac{n^{(2\alpha+1+2\delta)/2}}{n^{(2\alpha+1)/2}}\right) = O\left(\frac{P_n}{n^{(2\alpha+1)/2}}\right).$$

Thus condition (1.3) of our theorem is satisfied. From the above various examples can be constructed, by choosing suitable values.

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