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ON THE ABSOLUTE MATRIX SUMMABILITY OF THE SERIES CONJUGATE TO A FOURIER SERIES

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In the present paper the authors investigate sufficient conditions under which a summability method of a certain general type absolutely sums the conjugate series of a Fourier series of a function $f(t)$ at $t = x$, which satisfies the conditions

(i) $\psi(t) \in BV(0, \pi)$ and (ii) $\frac{\psi(t)}{t} \in L(0, \pi)$,

where $\psi(t) = \frac{1}{2}\{f(x + t) - f(x - t)\}$. The main theorem includes a theorem of Pati, who considered the problem for the special case of Nörlund summability.

§1. Let

$$\Sigma\, a_k \qquad ...(1.1)$$

be a given infinite series. Let $A = (\alpha_{n,k})$ be an infinite matrix with real or complex elements. The series-to-series transformation is given by

$$t_n = \sum_{k=0}^{\infty} \alpha_{n,k} a_k \qquad (n = 0, 1, 2, 3, \ldots .)$$

The series $\Sigma\, a_k$ is said to be summable (A), if

$$\sum_{n=0}^{\infty} t_n \qquad ...(1.2)$$

converges to s. The series is said to be absolutely summable A or in short summable $| A |$ if (1.2) converges absolutely. The method (A) is said to be conservative or absolutely conservative according as

$$\Sigma\, a_n \in (C, 0) \Rightarrow \Sigma\, t_n \in (C, 0)$$

or

$$\Sigma\, | a_n | \in (C, 0) \Rightarrow \Sigma\, t_n \in | C, 0 | .$$

In the present paper, we will be concerned with the case in which A is absolutely conservative. It is known that (Knopp and Lorentz 1949, Mears 1942) the method is absolutely conservative if and only if, for $k \geqq 0$

$$\sum_{n=0}^{\infty} |\alpha_{n,k}| = O(1). \qquad ...(1.3)$$

It may be remarked that A is absolutely regular if and only if (1.3) holds and further, for all $k > 0$

$$\sum_{n=0}^{\infty} \alpha_{n,k} = 1. \qquad ...(1.4)$$

Let $f(t)$ be a poriodic function with period 2π and integrable in the sense of Lebesgue over $(-\pi, \pi)$. Let

$$f(t) \sim \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt). \qquad ...(1.5)$$

The series conjugate to (1.5), at $t = x$, is, given by

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx). \qquad ...(1.6)$$

We set

$$\phi(t) = \tfrac{1}{2} \{f(x + t) + f(x - t)\}$$
$$\psi(t) = \tfrac{1}{2} \{f(x + t) - f(x - t)\}.$$

§2. Kuttner and Sahney (1972, Theorem) have obtained sufficient conditions under which the series (1.5) at $t = x$ is summable $|A|$ whenever $\phi(t) \in BV(0, \pi)$. Their theorem reads as follows:

Theorem A — Let $A = (\alpha_{n,k})$ be an absolutely conservative series-to-series transformation, with $\alpha_{n,k} \geqq 0$ for all n, k. Suppose that either (a) for each fixed n, there is a positive integer $r(n)$ such that $\alpha_{n,k}$ is non-decreasing for $1 \leqq k \leqq r(n)$ and non-increasing for $k \geqq r(n)$, or (b) for each fixed n, there is a positive integer $s(n)$, such that $\frac{\alpha_{n,k}}{k}$ is non-decreasing for $1 \leqq k \leqq s(n)$, and non-increasing for $k \geqq s(n)$. Suppose also in case (a) that, for $K \geqq 1$

$$\sum_{r(n) \geqq 2K} \frac{1}{r(n)} \sum_{k=r(n)-K}^{r(n)+K} \alpha_{n,k} = O(1) \qquad ...(2.1)$$

and in case (b) that, for $K \geqq 1$

$$\sum_{s(n) \geqq 2K} \frac{1}{s(n)} \sum_{k=s(n)-K}^{s(n)+K} \alpha_{n,k} = O(1). \qquad ...(2.2)$$

Then the series (1.5) at $t = x$ is summable $|A|$ whenever $\phi(t) \in BV(0, \pi)$.

Our main theorem reads as follows:

Theorem 1 — Suppose that all the conditions on the matrix element $\alpha_{n,k}$ hold good as in the case of Theorem A. Then the series (1.6) is summable $|A|$ whenever

$$\psi(t) \in BV(0, \pi) \qquad \ldots(2.3)$$

and

$$\frac{\psi(t)}{t} \in L(0, \pi). \qquad \ldots(2.4)$$

At this stage we present some remarks made by Kuttner and Sahney (1972) with regard to conditions (2.1), (2.2) and some others noted below for the sake of completeness of this paper.

For $K \geqq 1$,

$$\sum_{r(n) \geqq 2K} \sum_{k=r(n)-K}^{r(n)+K} \frac{\alpha_{n,k}}{k} = O(1). \qquad \ldots(2.5)$$

$$\sum_{r(n) \geqq 2K} \frac{\alpha_{n,r(n)}}{r(n)} = O\left(\frac{1}{K}\right) \qquad \ldots(2.6)$$

$$\sum_{s(n) \geqq 2K} \sum_{k=s(n)-K}^{s(n)+K} \frac{\alpha_{n,k}}{k} = O(1) \qquad \ldots(2.7)$$

$$\sum_{s(n) \geqq 2K} \frac{\alpha_{n,s(n)}}{s(n)} = O\left(\frac{1}{K}\right). \qquad \ldots(2.8)$$

Remark

(2.1) $\sim$ (2.5) (2.2) $\sim$ (2.7)

(2.1) $\Rightarrow$ (2.6) and (2.2) $\Rightarrow$ (2.8).

Throughout the present paper it is assumed that

$$K = \left[\frac{\pi}{t}\right] \text{ and } 0 < t \leqq \pi.$$

§3. We shall require the following lemmas for the proof of our theorems.

Lemma 1 [Mohanty and Ray (1969, Lemma 2)] — If $\eta > 0$, the necessary and sufficient condition that $\psi(t) \in BV(0, \eta)$ and $\frac{\psi(t)}{t} \in L(0, \eta)$ are that

$$\int_0^{\eta} t^{-\beta} \mid dH(t) \mid < \infty \quad \text{and} \quad H(+0) = 0,$$

where

$$H(t) = t^{\beta}\psi(t),\ \beta > 0.$$

Lemma 2 (Bosanquet and Kestelman 1939) — Suppose that $f_n(x)$ is measurable in (a, b), where $b - a \leqq \infty$, for $n = 1, 2, 3, \ldots$. Then a necessary and sufficient condition that for every function $\lambda(x)$ integrable (L) over (a, b) the functions $f_n(x)\,\lambda(x)$ should be integrable (L) over (a, b) and

$$\sum_{n=1}^{\infty} \left| \int_a^b \lambda(x) f_n(x)\, dx \right| < \infty$$

is that

$$\sum_{n=1}^{\infty} \mid f_n(x) \mid$$

should be essentially bounded for x in (a, b).

Lemma 3 — Let $A = (\alpha_{n,k})$ be series-to-series transformation such that

$$\sum_{n=0}^{\infty} \mid \alpha_{n,0} \mid < \infty \qquad \ldots(3.1)$$

and such that for every fixed n

$$L_n(t) = t \sum_{k=1}^{\infty} \alpha_{n,k} \int_t^{\pi} \frac{\sin ku}{u}\, du \qquad \ldots(3.2)$$

converges boundedly in t. Then in order that the series (1.6) should be summable $\mid A \mid$ whenever $\psi(t) \in BV(0, \pi)$ and $\frac{\psi(t)}{t} \in L(0, \pi)$, it is sufficient that

$$\sum_{n=0}^{\infty} \mid L_n(t) \mid = O(1) \qquad \ldots(3.3)$$

and necessary that the sum (3.3) should be essentially bounded.

Proof of Lemma 3 : By virtue of Lemma 1, the conditions imposed on $\psi(t)$ in the hypothesis can be replaced by (taking $\beta = 1$, without loss of generality) the following equivalent condition

$$\int_0^{\pi} t^{-1} \mid dH(t) \mid < \infty \quad \text{and} \quad H(+0) = 0. \qquad \ldots(3.4)$$

Integrating by parts, we get

$$B_n(x) = \frac{2}{\pi} \int_0^\pi \int_t^\pi \frac{\sin nu}{u}\, du\, dH(t) \qquad \ldots(3.5)$$

the integrated part vanishes, since $H(+0) = 0$ and

$$\int_t^\pi \frac{\sin nu}{u}\, du \qquad \text{is bounded.}$$

Writing $f_n(t) = L_n(t)$ and $\lambda(t) = t^{-1}dH(t)$, we get the result at once by an appeal to Lemma 2. In the sufficiency part of Lemma 3, the boundedness of (3.3) can be replaced by just the essential boundedness of (3.3), if we assume that, for every fixed n,

$$\sum_{k=1}^{\infty} | \alpha_{n,k} - \alpha_{n,k+1} | < \infty, \qquad \ldots(3.6)$$

in other words, that t_n is defined whenever (1.1) converges. It is easy to prove that the essential boundedness of (3.3) is equivalent to its boundedness whenever (3.6) holds.

Lemma 4 — Let $A = (\alpha_{n,k})$ be an absolutely conservative series-to-series transformation. If, for every fixed n, $\frac{\alpha_{n,k}}{k}$ is ultimately nonnegative and non-increasing (and thus in particular, if the hypotheses of the theorem are satisfied) then the hypotheses of Lemma 3 are satisfied.

PROOF OF LEMMA 4 : Relation (3.1) is a special case of (1.3). Now, it will be enough to prove that (3.2) converges boundedly when n is fixed. Suppose for $k \geqq M$, $\frac{\alpha_{n,k}}{k}$ is nonnegative and non-increasing. Then we have uniformly in k_1, k_2 for $K, M \leqq k_1 \leqq k_2$,

$$\left| t \sum_{k_1}^{k_2} \alpha_{n,k} \int_t^\pi \frac{\sin ku}{u}\, du \right| \qquad \ldots(3.7)$$

$$= \left| \sum_{k_1}^{k_2} \alpha_{n,k} \frac{\cos kt - \cos k\tau}{k} \right| \quad (t < \tau < \pi)$$

$$\leqq \frac{2\alpha(n, k_1)^*}{2k_1 \sin \frac{1}{2} t}.$$

*Henceforward we shall write $\alpha(n, k)$ for $\alpha_{n,k}$ for the sake of convenience.

But (1.3) implies that $\alpha(n, k)$ is bounded; hence the expression on the right side of (3.7) is bounded uniformly in the range considered, fo fixed t, $0 < t < \pi$, and tends to zero uniformly in k_2 a $k_1 \to \infty$.

Since M is a constant and $\int_t^\pi \frac{\sin ku}{u} du$ is finite for every k

$$t \sum_{k=1}^{M-1} \left| \alpha_{n,k} \int_t^\pi \frac{\sin ku}{u} du \right|$$

is bounded : also; if $K \geqq M$

$$t \sum_{k=M}^{K} \left| \alpha_{n,k} \int_t^\pi \frac{\sin ku}{u} du \right|$$

$$< Ct \sum_{k=M}^{K} | \alpha_{n,k} | = O(1),$$

by the boundedness of $\alpha_{n,k}$ and definition of K.

Lemma 5 [Kuttner and Sahney (1972, Lemma 3)] — Suppose that $\theta_k \geqq 0$. Suppose that θ_k is non-decreasing for $1 \leqq k \leqq s$, and non-increasing for $k \geqq s$. Then for any fixed positive integers a, b, any t with $0 < t \leqq \pi$,

$$\left| \sum_a^b \theta_k e^{ikt} \right| \leqq A \sum_{k=\max(1, s-K)}^{s+K} \theta_k \qquad \ldots(3.8)$$

where A is an absolute constant.

§4. *Proof of Theorem* 1 — It follows from Lemmas 3 and 4 that it is enough to show that the hypothesis of the theorem imply (3.3).

I. Consider first those values of n (if any) for which $r(n) < 2K$ in case (a), and for which $s(n) < 2K$ in case (b): $\frac{\alpha_{n,k}}{k}$ is given to be non-increasing for $k \geqq 2K$ in case (b), in case (a) we are given that $\alpha_{n,k}$ is non-increasing for $k \geqq 2K$; hence *a fortiori*, so is $\frac{\alpha_{n,k}}{k}$. Thus in either case, since the partial sums of $\Sigma \cos kt$ are $O(t^{-1})$, we have

$$t \sum_{k=2K}^{\infty} \alpha_{n,k} \int_t^{\pi} \frac{\sin ku}{u}\, du$$

$$= \sum_{k=2K}^{\infty} \frac{\alpha_{n,k}}{k} (\cos kt - \cos k\tau), \quad (t < \tau < \pi)$$

$$= O\left[\frac{\alpha(n, 2K)}{K} t^{-1}\right]$$

$$= O\,[\alpha(n, 2K)], \quad \text{since} \quad K = \left[\frac{\pi}{t}\right].$$

We shall next consider those terms in the sum (3.2) for which $k \leqq 2K$.

$$\left| t \sum_{k=1}^{2K-1} \alpha_{n,k} \int_t^{\pi} \frac{\sin ku}{u}\, du \right|$$

$$= t \left| \sum_{k=1}^{2K-1} \alpha_{n,k} \int_{tk}^{\pi k} \frac{\sin \theta}{\theta}\, d\theta \right|.$$

Thus we have

$$L_n(t) = O(\alpha(n, 2K)) + O(t \sum_{k=1}^{2K-1} \alpha_{n,k}).$$

Hence the contribution to the sum (3.3) of those values of n now under consideration is

$$O(\sum_{n=0}^{\infty} \alpha_{n,2K}) + O(t \sum_{k=1}^{2K-1} \sum_{n=0}^{\infty} \alpha_{n,k}) = O(1) \qquad ...(4.1)$$

by (1.3) and the definition of K.

II. We now investigate the remaining values of n.

II(i). Consider first case (b). For any fixed n, we apply Lemma 5 with $\theta_k = \frac{\alpha_{n,k}}{k}$, and take the real part of (3.8). It follows immediately that

$$L_n(t) = t \sum_{k=0}^{\infty} \alpha_{n,k} \int_t^{\pi} \frac{\sin ku}{u}\, du$$

$$= \sum_{k=1}^{\infty} \frac{\alpha_{n,k}}{k} (\cos kt - \cos k\tau)$$

(*equation continued on p.* 1489)

$$= O\left\{ \sum_{k=s(n)-K}^{s(n)+K} \frac{\alpha_{n,k}}{k} \right\}.$$

Thus

$$\sum_{s(n)\geqq 2K} | L_n(t) | = O\left\{ \sum_{s(n)\geqq 2K} \sum_{s(n)-K}^{s(n)+K} \frac{\alpha_{n,k}}{k} \right\} = O(1)$$

in view of (2.7).

II(ii). Now consider the case (a). Since $\alpha_{n,k}$ is non-increasing for $k \geqq r(n)$ so is $\frac{\alpha_{n,k}}{k}$; thus the part of the sum (3.2) for which $k > r(n) - K$ may be dealt with as in case (b). The part for which $k < K$ may be dealt with by using the inequality

$$\left| \int_t^{\pi} \frac{\sin ku}{u}\, du \right| \leqq A$$

as in the proof of (4.1). Thus it remains to show that

$$\sum_{r(n)\geqq 2K} | R_n(t) | = O(1) \qquad \ldots(4.2)$$

where

$$R_n(t) = t \sum_{k=K}^{r(n)-K} \alpha_{n,k} \int_t^{\pi} \frac{\sin ku}{u}\, du.$$

Writing

$$Q_n(t) = \sum_{k=K}^{r(n)-K} \frac{\alpha_{n,k}}{k} \cos kt$$

we have, using mean value theorem,

$$R_n(t) = Q_n(t) - Q_n(\tau) \quad \text{for} \quad t < \tau < \pi.$$

Hence for the validity of (4.2), it remains to show that

$$\sum_{r(n)\geqq 2K} | Q_n(t) | = O(1). \qquad \ldots(4.3)$$

Now

$$Q_n(t) = -\frac{1}{2\sin\frac{1}{2}t}\sum_{k=K}^{r(n)-K}\frac{\alpha_{n,k}}{k}\left(\sin\left(k-\tfrac{1}{2}\right)t-\sin\left(k+\tfrac{1}{2}\right)t\right)$$

$$= -\frac{1}{2\sin\frac{1}{2}t}\left\{\sum_{k=K}^{r(n)-K}\sin\left(k+\tfrac{1}{2}\right)t\,\Delta_k\left(\frac{\alpha_{n,k}}{k}\right)\right.$$

$$+\frac{\alpha_{n,K}}{K}\sin\left(K-\tfrac{1}{2}\right)t$$

$$\left.-\frac{\alpha(n,r(n)-K+1)}{r(n)-K+1}\sin\left(r(n)-K+\tfrac{1}{2}\right)t\right\}.$$

Since

$$\Delta_k\left(\frac{\alpha_{n,k}}{k}\right)=\frac{\alpha(n,k)}{k(k+1)}+\frac{\Delta_k(\alpha_{n,k})}{k+1}$$

it follows that

$$Q_n(t) = O\left[\frac{1}{t}\left\{\sum_{k=K}^{r(n)-K}\frac{\alpha_{n,k}}{k(k+1)}+\sum_{k=K}^{r(n)-K}\frac{|\Delta_k(\alpha_{n,k})|}{k+1}\right.\right.$$

$$\left.\left.+\frac{\alpha_{n,K}}{K}+\frac{\alpha(n,r(n)-K+1)}{r(n)-K+1}\right\}\right]$$

$$= O\left\{R_n^1(t)+R_n^2(t)+R_n^3(t)+R_n^4(t)\right\}.$$

Now*, since $\alpha_{n,k}$ is non-decreasing in the relevant range

$$R_n^2(t) = -\frac{1}{t}\sum_{k=K}^{r(n)-K}\frac{\Delta_k(\alpha_{n,k})}{k+1}$$

$$= \frac{1}{t}\sum_{k=K}^{r(n)-K}\frac{\alpha_{n,k}}{k(k+1)}-\frac{1}{t}\,\frac{\alpha_{n,K}}{K}+\frac{\alpha(n,r(n)-K+1)}{r(n)-K+1}$$

$$= R_n^1(t)-R_n^3(t)+R_n^4(t)$$

*There is no point in presenting the remaining part of the proof as it is similar to the proof given by Kuttner and Sahney (1972, p. 414), however for the convenience of the reader the proof is given here.

so that

$$Q_n(t) = O\{R_n^1(t) + R_n^4(t)\}.$$

So it will be enough to establish the following:

$$\sum_{r(n)\geqq 2K} R_n^{\mu}(t) = O(1) \quad \text{for} \quad \mu = 1, 4. \qquad \ldots(4.4)$$

For $\mu = 1$,

$$\sum_{r(n)\geqq 2K} R_n^{\mu}(t) \leqq \frac{1}{t} \sum_{k=K}^{\infty} \frac{1}{k(k+1)} \sum_{n=0}^{\infty} \alpha_{n,k} = O(1)$$

by (1.3) and definition of K.

For $\mu = 4$, if $r(n) \geqq 2K$,

$$R_n^{\mu}(t) = O\left[\frac{\alpha(n, r(n) - K + 1)}{tr(n)}\right]$$

$$= O\left[\frac{1}{tKr(n)} \sum_{k=r(n)-K+1}^{r(n)} \alpha_{n,k}\right]$$

$$= O\left[\frac{1}{r(n)} \sum_{k=r(n)-K+1}^{r(n)} \alpha_{n,k}\right].$$

Hence, for $\mu = 4$,

$$\sum_{r(n)\geqq 2K} R_n^{\mu}(t) = O(1)$$

by (2.1).

This completes the proof of the theorem.

§5. In this section, we shall deduce some known results on triangular matrix summability and Nörlund summability as corollaries from the main theorem. We recall that given a matrix $\| T \| = (a_{n,k})$ the sequence-to-sequence transformation is given by

$$\sigma_n = \sum_{k=0}^{\infty} a_{n,k} s_k \qquad \ldots(5.1)$$

where s_k is the kth partial sum of Σu_n. We assume that σ_n exists for every $n=0, 1, 2, \ldots$. The series Σu_n is said to be summable (T) if $\lim_{n\to\infty} \sigma_n = s$. The series Σu_n is said to be absolutely summable by T-method, in short summable $| T |$, if

$$\sum_{n=1}^{\infty} | \sigma_n - \sigma_{n-1} | < \infty.$$

The matrix $\| T \|$ is called a triangular matrix, if

$$a_{n,k} = 0 \quad \text{for} \quad k > n.$$

Presently we shall be concerned with such matrices. Now, we proceed to show that the sequence-to-sequence transformation given in (5.1) can be expressed as a series-to-series transformation.

We write

$$\sigma_n - \sigma_{n-1} = \begin{cases} b_n & \text{for} \quad n \geqq 1 \\ b_0 & \text{for} \quad n = 0. \end{cases}$$

By formal computation it can be seen that

$$b_0 = u_0$$

$$b_n = \sum_{k=0}^{n} (A_{n,k} - A_{n-1,k}) \, u_k, \; n \geqq 1$$

where

$$A_{n,k} = \sum_{\nu=k}^{n} a_{n,\nu}$$

and $\quad A_{n,0} = 1 \quad \text{for every} \quad n \geqq 0.$

Thus we have

$$b_n = \begin{cases} u_0, \; n = 0 \\ \sum_{k=0}^{n} \alpha_{n,k} \, u_k, \; n = 1, 2, 3, \ldots \end{cases}$$

where

$$\alpha_{n,k} = A_{n,k} - A_{n-1,k}, \; n \geqq 1.$$

Thus the sequence-to-sequence transformation of $\Sigma \, u_n$ with the triangular matrix $(a_{n,k})$ is essentially equivalent to the series-to-series transformation of $\Sigma \, u_n$ with the triangular matrix $(\alpha_{n,k})$, where

$$\alpha_{n,k} = \sum_{\nu=k}^{n} a_{n,\nu} - \sum_{\nu=k}^{n-1} a_{n-1,\nu}.$$

We need the following Lemma.

Lemma 6 — Suppose that $(a_{n,k})$ is a triangular matrix, and let us write

$$\sum_{\nu=k}^{n} a_{n,\nu} = A_{n,k}$$

and assume that $A_{n,0} = 1$ for every $n \geqq 0$. If

$$\{a_{n,k}\}_{k=0}^{n} \text{ and } \{a_{n-1,k} - a_{n,k+1}\}_{k=0}^{n-1} \qquad ...(5.2)$$

are both nonnegative and non-decreasing sequences with respect to k,

$$\{a_{n,k+1} - a_{n,k}\}_{k=0}^{n-1} \qquad ...(5.3)$$

is non-decreasing with respect to k, and that

$$\sum_{n=k+1}^{\infty} \frac{A_{n,n-k}}{n} < \infty, \quad \text{for every } k \qquad ...(5.4)$$

then

(i) $\{\alpha_{n,k}(= A_{n,k} - A_{n-1,k})\}$ ...(5.5)

is nonnegative and non-decreasing with respect to k,

(ii) $(\alpha_{n,k})$ is absolutely conservative, that is

$$\sum_{n=0}^{\infty} | \alpha_{n,k} | < \infty \qquad ...(5.6)$$

and for $K \geqq 1$

(iii) $$\sum_{n \geqq 2K} \frac{1}{n} \sum_{n-K}^{n+K} \alpha_{n,k} = O(1). \qquad ...(5.7)$$

PROOF : We have

$$\begin{aligned}
\alpha_{n,k} &= \sum_{\nu=k}^{n} a_{n,\nu} - \sum_{\nu=k}^{n-1} a_{n-1,\nu} \\
&= \left(\sum_{\nu=0}^{n} a_{n,\nu} - \sum_{\nu=0}^{k-1} a_{n,\nu}\right) - \left(\sum_{\nu=0}^{n-1} a_{n-1,\nu} - \sum_{\nu=0}^{k-1} a_{n-1,\nu}\right) \\
&= A_{n,0} - A_{n-1,0} + \sum_{\nu=0}^{k-1} (a_{n-1,\nu} - a_{n,\nu}) \\
&= \sum_{\nu=0}^{k-1} (a_{n-1,\nu} - a_{n,\nu+1}) + \sum_{\nu=0}^{k-1} (a_{n,\nu+1} - a_{n,\nu})
\end{aligned}$$

since $A_{n,0} = 1$ for all n.

Now the proof of (5.5) follows immediately by (5.2) and (5.3). Using (5.5) we have

$$\sum_{n=1}^{N} | \alpha_{n,k} | = \sum_{n=1}^{N} \alpha_{n,k}$$

$$= \sum_{n=1}^{N} (A_{n,k} - A_{n-1,k})$$

$$= A_{N,k} - A_{0,k} < 1 \text{ for } N \geqq 1.$$

Thus $(\alpha_{n,k})$ is absolutely conservative.

The inner sum of (5.7) is essentially equal to $\sum_{k=n-K}^{n} \alpha_{n,k}$ and does not exceed

$$\sum_{k=n-K}^{n} a_{n,k}$$

since $\alpha_{n,k} = a_{n,k} - (A_{n-1,k} - A_{n,k+1})$, and $A_{n-1,k} - A_{n,k+1}$ is nonnegative by (5.2). Thus a sufficient condition for (5.7) to hold is that

$$\sum_{n \geqq 2K} \frac{A_{n,n-K}}{n} = O(1). \qquad ...(5.8)$$

Since (5.4) implies (5.8) the lemma follows at once. By virtue of Lemma 6, our theorem includes the following result.

Theorem B (Hota 1973–74) — Let $\| T \| = (a_{n,k})$ be an infinite triangular matrix. Suppose that the conditions (5.2), (5.3) and (5.4) on matrix element $a_{n,k}$ hold good as in the case of Lemma 6. Then the series (1.6) is summable $| T |$ whenever

$$\psi(t) \in BV(0, \pi) \quad \text{and} \quad \frac{\psi(t)}{t} \in L(0, \pi).$$

Application to Nörlund summability — The A method reduces to the familiar Nörlund method when

$$\alpha_{n,k} = \begin{cases} 0 \text{ for } k > n \\ \dfrac{P_{n-k}}{P_n} - \dfrac{P_{n-k-1}}{P_{n-1}}, \ 1 \leqq k \leqq n. \end{cases}$$

It is known [Kuttner and Sahney (1972, p. 415)] that if p_n is nonnegative, non-increasing and

$$\sum_{n=k}^{\infty} \frac{1}{nP_n} = O\left(\frac{1}{P_k}\right), \qquad ...(5.9)$$

then $(\alpha_{n,k})$ as defined in (5.8) satisfies all the conditions imposed on $\alpha_{n,k}$ in Theorem 1.

Hence our theorem includes the following result.

Theorem C — Suppose that $\{p_n\}$ is nonnegative, non-increasing and (5.9) holds, then the series conjugate to the Fourier series of $f(t)$ at $t = x$ is summable $| N, p_n |$, whenever

$$\psi(t) \in BV(0, \pi) \quad \text{and} \quad \frac{\psi(t)}{t} \in L(0, \pi).$$

The assumption that $\{p_n\}$ is nonnegative, non-increasing is, without some further condition, not sufficient for the conclusion, for it has been shown by Mohanty and Ray (1967), that, when $p_n = \frac{1}{n+1}$, it is not true that (1.6) for any function satisfying the condition $\psi(t) \in BV(0, \pi)$ and $\frac{\psi(t)}{t} \in L(0, \pi)$ is absolutely summable (N, p_n). This example also shows that, in our theorem, the assumption that the method is absolutely conservative and that (a) holds would not be sufficient for the conclusion.

We shall state the following results on absolute Nörlund summability for allied series.

Theorem D — Suppose that $p_n > 0$ and that $\frac{p_{n+1}}{p_n}$ is non-decreasing and less than equal to 1 for all n. Suppose that (5.9) holds. Then the conjugate series of the Fourier series of $f(t)$ at $t = x$ is summable $| N, p_n |$ whenever

$$\psi(t) \in BV(0, \pi) \quad \text{and} \quad \frac{\psi(t)}{t} \in L(0, \pi).$$

Theorem E — Suppose that for all n, $p_n \geqq p_{n+1} > 0$, and that $(p_n - p_{n+1})$ is non-increasing. Suppose also that

$$\sum_{n=0}^{K} \frac{P_n}{n+1} = O(P_K) \qquad \ldots(5.10)$$

then the series conjugate to the Fourier series of $f(t)$ at $t = x$ is summable $| N, p_n |$ whenever $\psi(t) \in BV(0, \pi)$ and $\frac{\psi(t)}{t} \in L(0, \pi)$.

It may be noted that, the analogues of Theorem C and Theorem D for Fourier series are due to Dikshit (1970) and Singh (1964) respectively. It may also be noted that Theorem C includes a result due to Pati (1963). It is immediately evident that Theorem C includes Theorem D. The result that Theorem C includes Theorem E follows from the following Lemma due to Kuttner and Sahney (1972, Lemma 4), which shows that, in Theorem E we may replace (5.10) by (5.9).

Lemma 7 — Suppose that $p_0 > 0$, $p_n \geqq 0$. Then (5.9), (5.10) are equivalent. In fact either is equivalent to the assertion:

There is a constant integer $r > 1$, and a constant $\lambda > 1$ such that, for all sufficiently large n

$$P_{rn} \geqq \lambda P_n. \qquad \ldots(5.11)$$

Cesàro summability — The case of Cesàro summability (C, δ) is a Nörlund method with

$$p_n = \binom{n + \delta - 1}{\delta - 1}.$$

If $0 < \delta \leqq 1$, then the conditions of Theorem B are satisfied. Thus our Theorem includes the following result due to Bosanquet and Hyslop.

Theorem (Bosanquet and Hyslop 1937) — If $\psi(t) \in BV(0, \pi)$ and $\frac{\psi(t)}{t} \in L(0, \pi)$, then $(1.6) \in |C, \delta|$, $\delta > 0$.

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