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A FINITE INTEGRAL INVOLVING A JACOBI POLYNOMIAL AND A GENERALIZED H-FUNCTION OF TWO VARIABLES

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In this paper a finite integral involving the product of Jacobi polynomial and a generalized H-function of two variables has been evaluated. As the arguments of the H-function are of generalized character, so a number of integrals involving these functions have been deduced. The integrals would also yield relations for Appell functions, Kampé de Fériet function, Meijer's G-function, etc. on specializing the parameters.

Expansion for the generalized H-function of two variables in a series involving the product of a Jacobi polynomial and an H-function has been obtained by using these integrals

1. INTRODUCTION

Generalization to two variables of Fox's H-function (Fox 1961, p. 408) occurring in this paper has been defined by Mittal and Gupta (1972) and discussed by the author (Singh 1977). It will be represented as follows:

$$H(y, z) = H^{0,n_1:m_2,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}\left[\begin{array}{c|l} y & ((a_{p_1};\alpha_{p_1}, A_{p_1})) : ((c_{p_2}, \gamma_{p_2})); ((e_{p_3}, E_{p_3})) \\ z & ((b_{q_1};\beta_{q_1}, B_{q_1})) : ((d_{q_2}, \delta_{q_2})); ((f_{q_3}, F_{q_3})) \end{array}\right]$$

$$= -\frac{1}{4\pi^2}\int_{L_1}\int_{L_2}\phi(s, t)\,\theta_1(s)\,\theta_2(t)\,y^s z^t\,ds\,dt \qquad ...(1.1)$$

where

$$\phi(s, t) = \frac{\prod_{j=1}^{n_1}\Gamma(1 - a_j + \alpha_j s + A_j t)}{\prod_{j=n_1+1}^{p_1}\Gamma(a_j - \alpha_j - A_j t)\prod_{j=1}^{q_1}\Gamma(1 - b_j + \beta_j s + B_j t)} \qquad ...(1.2)$$

$$\theta_1(s) = \frac{\prod_{j=1}^{m_2}\Gamma(d_j - \delta_j s)\prod_{j=1}^{n_2}\Gamma(1 - c_j + \gamma_j s)}{\prod_{j=m_2+1}^{q_2}\Gamma(1 - d_j + \delta_j s)\prod_{j=n_2+1}^{p_2}\Gamma(c_j - \gamma_j s)} \qquad ...(1.3)$$

$$\theta_2(t) = \frac{\prod_{j=1}^{m_3} \Gamma(f_j - F_j t) \prod_{j=1}^{n_3} \Gamma(1 - e_j + E_j t)}{\prod_{j=m_3+1}^{q_3} \Gamma(1 - f_j + F_j t) \prod_{j=n_3+1}^{p_3} \Gamma(e_j - E_j t)}. \qquad ...(1.4)$$

The conditions under which (1.1) converges are

$$|\arg y| < \tfrac{1}{2}\mu_1\pi \text{ and } |\arg z| < \tfrac{1}{2}\mu_2\pi,$$

where

$$\mu_1 = \sum_1^{n_1} \alpha_j - \sum_{n_1+1}^{p_1} \alpha_j - \sum_1^{q_1} \beta_j + \sum_1^{m_2} \delta_j - \sum_{m_2+1}^{q_2} \delta_j + \sum_1^{n_2} \gamma_j - \sum_{n_2+1}^{p_2} \gamma_j > 0 \qquad ...(1.5)$$

$$\mu_2 = \sum_1^{n_1} A_j - \sum_{n_1+1}^{p_1} A_j - \sum_1^{q_1} \beta_j + \sum_1^{m_3} F_j - \sum_{m_3+1}^{q_3} F_j + \sum_1^{n_3} E_j - \sum_{n_3+1}^{p_3} E_j > 0. \qquad ...(1.6)$$

Because of the large number of parameters, $((a_{p_1}; \alpha_{p_1}, A_{p_1}))$ stands for

$$(a_1; \alpha_1, A_1), \ldots, (a_{p_1}; \alpha_{p_1}, A_{p_1}))$$

and $[a, b; (\alpha, \beta)]$ denotes $(a; \alpha, \beta), (b; \alpha, \beta)$. Also $(h, k, l, m) \geqslant 0$ stands for $h \geqslant 0$, $k \geqslant 0$, $l \geqslant 0$, $m \geqslant 0$.

$$H^{0,n_1}_{p_1,q_1}\left[\begin{array}{c|l} y & ((a_{p_1}; \alpha_{p_1}, A_{p_1})) \\ z & ((b_{q_1}, \beta_{q_1}, B_{q_1})) \end{array}\right]$$

stands for $H(y, z)$ defined by (1.1), when there is a change only in n_1, p_1, q_1; a_i, α_i, A_i; b_j, β_j, B_j $(i = 1, \ldots, p_1; j = 1, \ldots, q_1)$.

2. Main Integral

The integral to be evaluated is

$$\int_{-1}^{1} (1-x)^\rho (1+x)^\sigma P_n^{(\alpha,\beta)}(x)\, H\left[y(1-x)^h(1+x)^\mu, z(1-x)^k(1+x)^\nu\right] dx$$

$$= \frac{2^{\rho+\sigma+1}\Gamma(n+\alpha+1)}{\Gamma(n+1)\,\Gamma(\alpha+1)} \sum_{m=0}^{n} \frac{(-n)_m (\alpha+\beta+n+1)_m}{m!\,(\alpha+1)_m}$$

$$\times H^{0,n_1+2}_{p_1+2,q_1+1}\left[\begin{array}{c|c|l} 2^{h+\mu} & y & (-\rho-m; h, k), (-\rho; \mu, \nu), ((a_{p_1}; \alpha_{p_1}, A_{p_1})) \\ 2^{k+\nu} & z & ((b_{q_1}; \beta_{q_1}, B_{q_1})), (-\rho-\sigma-m-1; h+\mu, k+\nu) \end{array}\right] \qquad ...(2.1)$$

provided that $(h, k, \mu, \nu) \geqslant 0$, (Re β, Re $(\rho + hd_i/\delta_i)$, Re $(\sigma + \mu d_i/\delta_i)$, Re $(\rho + kf_j/F_j)$, Re $(\sigma + \nu f_j/F_j)) - 1$. $(i = 1, \ldots, m_2; j = 1, \ldots, m_3)$, $|\arg y| < \frac{1}{2}\mu_1\pi$, $|\arg z| < \frac{1}{2}\mu_2\pi$.

PROOF: To establish (2.1), expressing the H-function of two variables on the left-hand side as Mellin-Barnes type of double contour integral (1.1), interchanging the order of integration, which is justifiable due to absolute convergence of the integrals involved in the process, we get:

$$\frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \phi(s, t)\, \theta_1(s)\, \theta_2(t)\, y^s z^t$$

$$\times \left\{ \int_{-1}^{1} (1 - x)^{\rho + hs + kt} (1 + x)^{\sigma + \mu s + \nu t} P_n^{(\alpha, \beta)}(x)\, dx \right\} ds\, dt. \quad \ldots(2.2)$$

Evaluating the inner integral with the help of [Erdélyi *et al.* 1954, 284(3)] i.e.

$$\int_{-1}^{1} (1 - x)^{\rho} (1 + x)^{\sigma} P_n^{(\alpha, \beta)}(x)\, dx$$

$$= \frac{2^{\rho+\sigma+1}\Gamma(\rho + 1)\, \Gamma(\sigma + 1)\, \Gamma(n + \alpha + 1)}{\Gamma(\rho + \sigma + 2)\, \Gamma(n + 1)\, \Gamma(\alpha + 1)}$$

$$\times\, {}_3F_2 \left[\begin{matrix} -n, \alpha + \beta + n + 1, \rho + 1; 1 \\ \alpha + 1, \rho + \sigma + 2 \end{matrix} \right] \quad \ldots(2.3)$$

where (Re ρ, Re σ) > -1, we have

$$\frac{\Gamma(n + \alpha + 1)}{(2\pi i)^2}$$

$$\times \int_{L_1} \int_{L_2} \frac{2^{\rho+\sigma+1+(h+\mu)s+(k+\nu)t}\Gamma(\rho + 1 + hs + kt)\, \Gamma(\sigma + 1 + \mu s + \nu t)}{\Gamma(n + 1)\, \Gamma(\alpha + 1)\, \Gamma(\rho + \sigma + 2 + (h + \mu)\, s + (k + \nu)\, t)}$$

$$\times\, {}_3F_2 \left[\begin{matrix} -n, \alpha + \beta + n + 1, \rho + 1 + hs + kt; 1 \\ \alpha + 1, \rho + \sigma + 2 + (h + \mu)\, s + (k + \nu)\, t \end{matrix} \right] \phi(s, t)\, \theta_1(s)\, \theta_2(t)\, y^s z^t ds\, dt. \quad \ldots(2.4)$$

Now expressing the hypergeometric function as a series, changing the order of integration and summation in view of [Carslaw 1950, p. 176, (75)], which is permissible under the conditions given in (2.1) and (2.3) and applying (1, 1), the value of the integral is obtained.

3. Particular Cases

In this section we discuss some interesting particular cases of the result (2.1).

(i) In (2.1) taking $h = k = 0$ and setting $\rho = \alpha$, expressing the H-function on the right-hand side as Mellin-Barnes type double contour integral, interchanging the order of summation and integration, evaluating the series inside the integral with the help of Gauss' theorem (Mittal and Gupta 1972, p. 144) and again using (1.1), we get the result:

$$\int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\sigma} P_n^{(\alpha,\beta)}(x)\, H[y(1+x)^{\mu}, z(1+x)^{\nu}]\, dx$$

$$= \frac{2^{\alpha+\sigma+1}\Gamma(n+\alpha+1)}{\Gamma(n+1)}$$

$$\times H^{0,n_1+2}_{p_1+2,q_1+2}\left[\begin{matrix} 2^{\mu}y \\ 2^{\nu}z \end{matrix}\middle| \begin{matrix} [\beta-\sigma, -\sigma; (\mu,\nu)], ((a_{p_1}; \alpha_{p_1}, A_{p_1})) \\ ((b_{q_1}; \beta_{q_1}, B_{q_1})), [\beta-\sigma+n, -\sigma-\alpha-n-1; (\mu,\nu)] \end{matrix}\right] \quad \ldots(3.1)$$

Provided that $(\mu, \nu) \geqslant 0$, $(\text{Re}\,\alpha, \text{Re}\,\beta, \text{Re}\,(\sigma + \mu d_i/\delta_i), \text{Re}\,(\sigma + \nu f_j/F_j) > -1$ $(i = 1, \ldots, m_2; j = 1, \ldots, m_3)$, $|\arg y| < \frac{1}{2}\mu_1\pi$, $|\arg z| < \frac{1}{2}\mu_2\pi$.

Note : In (3.1) by suitably adjusting the parameters and putting

$$\alpha_{p_1} = A_{p_1} = \beta_{q_1} = B_{q_1} = \gamma_{p_2} = \delta_{q_2} = E_{p_3} = F_{q_3} = 1,$$

we obtain a result due to Choubisa [1973, p. 220, (3.4.1)].

(ii) In (2.1) putting $\mu = \nu = 0$, $\sigma = \beta$, $h = k$ and on the right-hand side substituting from (1.1), interchanging the order of summation and integration, evaluating the inner series with the help of

$${}_3F_2\left[\begin{matrix} -n, \alpha, \beta; 1 \\ \gamma, \alpha+\beta-\gamma-n+1 \end{matrix}\right] = \frac{(\gamma-\alpha)_n (\gamma-\beta)_n}{(\gamma)_n (\gamma-\alpha-\beta)_n} \quad \ldots(3.2)$$

then using Sneddon [1956, p. 18, (9, iii)]

$$\frac{\Gamma(n+1)}{\Gamma(n-s+1)} = (-1)^s (-n) \quad \ldots(3.3)$$

and (1.1), we obtain a result recently established by Shah [1973, p. 53, (2.3)].

(iii) In (2.1) setting $h = \mu$, $k = \nu$, we have

$$\int_{-1}^{1} (1-x)^{\rho} (1+x)^{\sigma} P_n^{(\alpha,\beta)}(x)\, H[y(1-x^2)^{\mu}, z(1-x^2)^{\nu}]\, dx$$

$$= \frac{2^{\rho+\sigma+1}\Gamma(n+\alpha+1)}{\Gamma(n+1)\,\Gamma(\alpha+1)} \sum_{m=0}^{n} \frac{(-n)_m (\alpha+\beta+n+1)_m}{m!\,(\alpha+1)_m}$$

$$\times H^{0,n_1+2}_{p_1+2,q_1+1}\left[\begin{matrix} 2^{2\mu}y \\ 2^{2\nu}z \end{matrix}\middle| \begin{matrix} [-\rho-m, -\sigma; (\mu,\nu)], ((a_{p_1}; \alpha_{p_1}, A_{p_1})) \\ ((b_{q_1}; \beta_{q_1}, B_{q_1})), (-\rho-\sigma-m-1; 2\mu. 2\nu) \end{matrix}\right] \quad \ldots(3.4)$$

provided that $(\mu, \nu) \geqslant 0$, $(\text{Re}\,\beta$, $\text{Re}\,(\rho + \mu d_i/\delta_i)$, $\text{Re}\,(\sigma + \mu d_i/\delta_i)$, $\text{Re}\,(\rho + \nu f_j/F_j)$, $\text{Re}\,(\sigma + \nu f_j/F_j)) \geqslant -1$ $(i = 1, \ldots, m_2;\ j = 1, \ldots, m_3)$, $|\arg y| < \frac{1}{2}\mu_1\pi$,

$|\arg z| < \frac{1}{2}\mu_2\pi$.

(iv) In (2.1) taking $n_1 = p_1 = q_1 = 0$, we get

$$\int_{-1}^{1} (1-x)^\rho (1+x)^\sigma P_n^{(\alpha,\beta)}(x)\, H_{p_2,q_2}^{m_2,n_2}\left[y(1-x)^h (1+x)^\mu \left| \begin{matrix} ((c_{p_2}, \gamma_{p_2})) \\ ((d_{q_2}, \delta_{q_2})) \end{matrix} \right.\right]$$

$$\times H_{p_3,q_3}^{m_3,n_3}\left[z(1-x)^k (1+x)^\nu \left| \begin{matrix} ((e_{p_3}, E_{p_3})) \\ ((f_{q_3}, F_{q_3})) \end{matrix} \right.\right] dx$$

$$= \frac{2^{\rho+\sigma+1}\Gamma(n+\alpha+1)}{\Gamma(n+1)\,\Gamma(\alpha+1)} \sum_{m=0}^{n} \frac{(-n)_m\,(\alpha+\beta+n+1)_m}{m!\,(\alpha+1)_m}$$

$$\times H_{2,1:p_2,q_2;p_3,q_3}^{0,2:m_2,n_2;m_3,n_3}\left[\begin{matrix} 2^{h+\mu}y \\ 2^{k+\nu}z \end{matrix} \left| \begin{matrix} (-\rho-m;h,k),(-\sigma;\mu,\nu): \\ (-\rho-\sigma-m-1;h+\mu,k+\nu): \end{matrix} \right. \begin{matrix} ((c_{p_2}, \gamma_{p_2}));((e_{p_3}, E_{p_3})) \\ ((d_{q_2}, \delta_{q_2}));((f_{q_3}, F_{q_3})) \end{matrix} \right] \qquad \ldots(3.5)$$

provided that $(h, k, \mu, \nu) \geqslant 0$, $(\text{Re}\,\beta$, $\text{Re}\,(\rho + hd_i/\delta_i)$, $\text{Re}\,(\sigma + \mu d_i/\delta_i)$,

$(\text{Re}\,\rho + kf_j/F_j)$, $(\text{Re}\,\rho + \nu f_j/F_j) - 1 (i = 1, \ldots, m_2; j = 1, \ldots, m_3)$,

$|\arg y| < \frac{1}{2}\mu_1\pi$, $|\arg z| < \frac{1}{2}\mu_2\pi$.

where

$$\lambda_1 = (\sum_{1}^{n_2} \gamma_j - \sum_{n_2+1}^{p_2} \gamma_j + \sum_{1}^{m_2} \delta_j - \sum_{m_2+1}^{q_2} \delta_j) > 0$$

and

$$\lambda_2 = (\sum_{1}^{n_3} E_j - \sum_{n_3+1}^{p_3} E_j + \sum_{1}^{m_3} F_j - \sum_{m_3+1}^{q_3} F_j) > 0.$$

(v) In (3.5) putting $k = \nu = 0$ and replacing h by δ, we get a result due to Anandani [1976, p. 183, (2.1)].

4. Expansion

In this section we obtain the following expansion for the H-function of two variables in series involving the Jacobi polynomial and the H-function of two variables by using the integral (2.1).

$$(1-x)^{\rho}(1+x)^{\sigma} H[y(1-x)^{h}(1+x)^{\mu}, z(1-x)^{k}(1+x)^{\nu}]$$

$$= 2^{\rho+\sigma} \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-r)_m (\alpha+\beta+2r+1)\,\Gamma(\alpha+\beta+r+m+1)}{m!\,\Gamma(\alpha+m+1)\,\Gamma(\beta+r+1)} P_r^{(\alpha,\beta)}(x)$$

$$\times H^{0,n_1+2}_{p_1+2,q_1+2}\left[\begin{array}{c|c} 2^{h+\mu}y & (-\rho-\alpha-m;h,k),(-\sigma-\beta;\mu,\nu),((a_{p_1};\alpha_{p_1},A_{p_1})) \\ 2^{k+\nu}z & ((b_{q_1};\beta_{q_1},B_{q_1})),(-\rho-\sigma-\alpha-\beta-m-1;h+\mu,k+\nu) \end{array}\right] \quad \ldots(4.1)$$

under similar conditions as given in (2.1).

PROOF : To obtain (4.1), let

$$(1-x)^{\rho}(1+x)^{\sigma} H[y(1-x)^{h}(1-x)^{h}(1+x)^{\mu}, z(1-x)^{k}(1+x)^{\nu}]$$

$$= \sum_{r=0}^{\infty} M_r P_r^{(\alpha,\beta)}(x). \quad \ldots(4.2)$$

Equation (4.2) is valid, since the expression on the left-hand side is continuous and is of bounded variation in the open interval $(-1, 1)$.

Multiplying both the sides of (4.2) by $(1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x)$, integrating w.r.t. x between the limits -1 to $+1$; on the right-hand side using the orthogonality property for the Jacobi polynomials [Erdélyi *et al.* 1954, p. 285, (5) & (9)] and evaluating the left-hand side with the help of (2.1), we get

$$M_n \frac{2^{\rho+\sigma}\Gamma(\alpha+\beta+n+1)(\alpha+\beta+2n+1)}{\Gamma(\alpha+1)\,\Gamma(\beta+n+1)} \sum_{m=0}^{n} \frac{(-n)_m (\alpha+\beta+n+1)_m}{m!\,(\alpha+1)_m}$$

$$\times H^{0,n_1+2}_{p_1+2,q_1+1}\left[\begin{array}{c|c} 2^{h+\mu}y & (-\rho-\alpha-m;h,k),(-\sigma-\beta;\mu,\nu),((a_{p_1};\alpha_{p_1},A_{p_1})) \\ 2^{k+\nu}z & ((b_{q_1};\beta_{q_1},B_{q_1})),(-\rho-\sigma-\alpha-\beta-m-1;h+\mu,k+\nu) \end{array}\right] \quad \ldots(4.3)$$

Substituting the value of M_r from (4.3) in (4.2), we get the result (4.1).

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