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ON A SUBCLASS OF p-VALENT FUNCTIONS

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In this paper the author considers the integrals of certain p-valent functions in the unit disc $E = \{z : |z| < 1\}$. Some known results of Bernardi and Goel are generalized.

Introduction

Let $A(p)$ denote the class of functions $f(z) = z^p + a_{p+1}z^{p+1} + \ldots$, p a positive integer, which are regular in $E = \{z : |z| < 1\}$. Recently Goel and Sohi (1980) studied the classes K_{n+p-1} of functions $f(z) \in A(p)$ and satisfying one of the conditions

$$\text{Re}\left[\frac{(z^n f(z))^{(n+p)}}{(z^{n-1} f(z))^{(n+p-1)}}\right] > \frac{n+p}{2}, \quad z \in E \qquad \ldots(1)$$

n any integer greater than $-p$. It was proved that each function on K_{n+p-1} is p-valent and $K_{n+p} \subset K_{n+p-1}$. It was also shown that a function f in $A(p)$ belongs to K_{n+p-1} if and only if

$$\text{Re}\left[\frac{D^{n+p} f(z)}{D^{n+p-1} f(z)}\right] > \tfrac{1}{2}, \quad z \in E \qquad \ldots(2)$$

where $D^{n+p-1} f(z) = \dfrac{z^p}{(1-z)^{n+p}} * f(z)$. (The operation $*$ stands for the convolution of two functions.)

In this paper we consider the classes of functions $f(z) \in A(p)$ which satisfy the condition

$$\text{Re}\left\{\frac{z(D^{n+p-1} f(z))'}{D^{n+p-1} f(z)}\right\} > 0, \quad z \in E. \qquad \ldots(3)$$

Using the identity

$$z(D^{n+p-1} f(z))' = (n+p)\, D^{n+p} f(z) - n D^{n+p-1} f(z) \qquad \ldots(4)$$

condition (3) can be re-written as

$$\text{Re}\left[\frac{D^{n+p} f(z)}{D^{n+p-1} f(z)}\right] > \frac{n}{n+p}, \quad n > -p, \quad z \in E. \qquad \ldots(5)$$

We denote these classes by T_{n+p-1}. It is obvious that $T_{n+p-1} \subset K_{n+p-1}$ for $n \geqslant p$. One can easily prove that $T_{n+p} \subset T_{n+p-1}$. Since T_0 is the class of functions which satisfy the condition $\text{Re}\left[\frac{zf'(z)}{f(z)}\right] > 0$ and we know (Umezawa 1957) that such functions are p-valent, the p-valence of functions in T_{n+p-1} follows from the inclusion relation.

Theorem 1 — Let $f \in A(p)$ satisfy the condition

$$\text{Re}\left\{\frac{z(D^{n+p-1}f(x))'}{D^{n+p-1}f(z)}\right\} > -\frac{1}{2c}, \quad c > 0 \qquad \ldots(6)$$

then

$$F(z) = \frac{p+c}{z^c}\int_0^z t^{c-1}f(t)\,dt \qquad \ldots(7)$$

belongs to T_{n+p-1}.

PROOF : Let $w(z)$ be a regular function in E, $w(0) = 0$, $w(z) \neq -1$ defined by

$$\frac{z(D^{n+p-1}F(z))'}{D^{n+p-1}F(z)} = p\,\frac{1-w(z)}{1+w(z)}. \qquad \ldots(8)$$

Using the identity

$$z(D^{n+p-1}F(z))' = (c+p)\,D^{n+p-1}f(z) - cD^{n+p-1}F(z) \qquad \ldots(9)$$

(8) can be rewritten as

$$\frac{D^{n+p-1}f(z)}{D^{n+p-1}F(z)} = \frac{1+\{(c-p)/(c+p)\}\,w(z)}{1+w(z)}. \qquad \ldots(10)$$

Taking logarithmic differentiation of (10), we get after a simple computation

$$\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} = p\left\{\frac{1-w(z)}{1+w(z)} - \frac{2}{c+p}\,\frac{zw'(z)}{(1+\{(c-p)/(c+p)\}\,w(z))(1+w(z))}\right\}. \qquad \ldots(11)$$

Now we claim that $|w(z)| < 1$ for $z \in E$, for otherwise by Jack's Lemma (1971), there exists a z_0, $z_0 \in E$, such that

$$z_0 w'(z_0) = kw(z_0) \qquad (12)$$

with $|w(z_0)| = 1$ and $k \geqslant 1$.

From (11) and (12) we have

$$z_0\,\frac{(D^{n+p-1}f(z_0))'}{D^{n+p-1}f(z_0)} = p\left[\frac{1-w(z_0)}{1+w(z_0)} - \frac{2}{c+p}\,\frac{kw(z_0)}{(1+\{(c-p)/(c+p)\}\,w(z_0))\,(1+w(z_0))}\right]. \qquad \ldots(13)$$

Since $\text{Re}\left[\frac{1 - w(z_0)}{1 + w(z_0)}\right] = 0$, $\text{Re}\left\{\frac{w(z_0)}{(1 + \{(c - p)/(c + p)\} w(z_0))(1 + w(z_0))}\right\} \geqslant \frac{c + p}{4c}$ and $k \geqslant 1$, it follows from (13) that

$$\text{Re}\left\{\frac{z_0(D^{n+p-1} f(z_0))'}{D^{n+p-1} f(z_0)}\right\} \leqslant -\frac{1}{2c},$$

which contradicts (6). Hence $| w(z_0) | < 1$ and by (8) $F \in T_{n+p-1}$. This completes the proof of Theorem 1.

Putting $p = 1$ and taking $n = 0$ and $n = 1$ in Theorem 1, we obtain the following extensions of the earlier results of Bernardi (1969):

Corollary 1 — If f is starlike of order $-\frac{1}{2c}$, then the function

$$F(z) = \frac{1 + c}{z^c} \int_0^z t^{c-1} f(t)\, dt$$

is starlike.

Corollary 2 — If f is convex of order $-\frac{1}{2c}$, then F is convex.

Theorem 2 — If $F \in T_{n+p-1}$ and f is defined by (7), then $f \in T_{n+p-1}$ for

$$| z | < \frac{c + p}{(2p + 1 + c^2)^{1/2} + p + 1}.$$

The result is sharp.

PROOF : Since $F \in T_{n+p-1}$, therefore we can write

$$\frac{z(D^{n+p-1} F(z))'}{D^{n+p-1} F(z)} = pu(z) \qquad \text{...(14)}$$

where $u(z)$ is regular in E and satisfies the conditions $\text{Re}\, u(z) > 0$, $u(0) = 1$.

Using (4), (14) becomes

$$\frac{D^{n+p} F(z)}{D^{n+p-1} F(z)} = \frac{1}{n + p} [n + pu(z)]. \qquad \text{...(15)}$$

Taking logarithmic differentiation of (15) and using the identity (9), we get after a simple computation

$$\frac{z(D^{n+p-1} f(z))'}{D^{n+p-1} f(z)} = p\left(u(z) + \frac{zu'(z)}{c + pu(z)}\right). \qquad \text{...(16)}$$

It is well known that for $|z| = r < 1$,

$$|zu'(z)| \leqslant \frac{2r}{1-r^2} \operatorname{Re} u(z). \qquad \ldots(17)$$

Thus from (16) and (17) we have for $|z| = r < 1$,

$$\operatorname{Re}\left\{\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)}\right\} \geqslant p\left(1 - \frac{2r}{(1-r)(c+p+(c-p)r)}\right) \operatorname{Re} u(z) \qquad \ldots(18)$$

Thus $$\operatorname{Re}\left\{\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)}\right\} > 0 \text{ if } |z| < \frac{c+p}{(2p+1+c^2)^{1/2}+p+1}.$$

The result is sharp for the function

$$f(z) = \frac{z^{1-c}}{p+c}(z^c F(z))'$$

where $F(z)$ is given by

$$\frac{z(D^{n+p-1}F(z))'}{D^{n+p-1}F(z)} = p\frac{1-z}{1+z}.$$

Corollary 3 — Putting $n + p = 1 = c$ in the above theorem we get the result obtained by Goel (1972).

Corollary 4 — For $p = 1$ and $n = 0$ we see that if $F \in S^*$ (the class of starlike functions) then $f \in S^*$ for $|z| < \frac{c+1}{(3+c^2)^{1/2}+2}$. This is a result due to Bernardi (1970).

Theorem 3 — If $f \in T_{n+p-1}$ and F is defined by

$$F(z) = \frac{n+p}{z^n}\int_0^z t^{n-1}f(t)\,dt, \qquad \ldots(19)$$

then $F \in T_{n+p}$.

PROOF : From (19) we have

$$nF(z) + zF'(z) = (n+p)f(z).$$

Therefore

$$nD^{n+p-1}F(z) + D^{n+p-1}(zF'(z)) = (n+p)\,D^{n+p-1}f(z)$$

or

$$nD^{n+p-1}F(z) + z(D^{n+p-1}F(z))' = (n+p)\,D^{n+p-1}f(z). \qquad \ldots(20)$$

Using (4), we conclude from (20) that

$$D^{n+p}F(z) = D^{n+p-1}f(z). \qquad ...(21)$$

Taking logarithmic differentiation of (21) and using the fact that $f \in T_{n+p-1}$ we have

$$\text{Re}\left\{\frac{z(D^{n+p}F(z))'}{D^{n+p}F(z)}\right\} = \text{Re}\left\{\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)}\right\} > 0, \quad z \in E.$$

Hence $F \in T_{n+p}$.

Remark : This theorem also holds if we replace T_{n+p-1} by K_{n+p-1} and T_{n+p} by K_{n+p}.

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