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ON THE MIXED BOUNDARY VALUE PROBLEM FOR THE POISSON EQUATION IN DOMAINS WITH CORNERS

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This paper concerns mixed boundary value problem for the Poisson equation in a plane domain Ω with corners on its boundary. Conditions sufficient for the solution to belong to $C^{\nu}(\bar{\Omega})$ are given.

1. INTRODUCTION

In this paper we study the mixed boundary value problem for the Poisson equation

$$\triangle u = f(x, y) \quad \ldots(1.1)$$

in a domain Ω with boundary $\partial\Omega$. If $\partial\Omega$ is smooth, then the smoothness (up to the boundary) of the solution improves with the improvement of the smoothness properties of f in (1.1) and the boundary data (cf. Agmon *et al.* 1959). If $\partial\Omega$ contains corners, this may not be true. We study here the problem when $\partial\Omega$ contains corners and for simplicity we assume that there is only one corner point on $\partial\Omega$. Recently, we have studied the Dirichlet problem for general second order linear elliptic equations (Azzam 1979). In particular let f in (1.1) be of class $C^{m+\alpha}(\bar{\Omega})$, $m \geqslant 0$ an integer and $\alpha \in (0, 1)$. Let $\partial\Omega$ be of class $C^{m+2+\alpha}$ except at a point 0 where it has a corner of interior angle ω, $0 < \omega < 2\pi$. If the boundary value of u is of class $C^{m+2+\alpha}(\partial\Omega\setminus\{0\}) \cap C^{0}(\partial\Omega)$, then it follows from Azzam (1979) that $u \in C^{\nu}(\bar{\Omega})$, where $\nu = \min(m + 2 + \alpha, \pi/\omega - \epsilon)$, with $\epsilon > 0$ arbitrarily small (see also Azzam and Kreyszig 1980). In Azzam (1980a, b) the Dirichlet problem in n-dimensional domains with edges is studied.

Asymptotic expansions of the solution of mixed boundary value problems for (1.1) near a corner were given by Wasow (1957), Lehman (1959) and Wigley (1964). Using these expansions, Wigley (1970) proved that $u \in C^{1}(\bar{\Omega}_1)$ if $f \in C^{\alpha}(\bar{\Omega}_1)$ and $\omega < \pi/2$, where $\Omega_1 \subset \Omega$ is a neighbourhood of the corner point. This result will be improved here and the relation between the smoothness of the solution and the value of ω will be given. Since our results are of local nature, then we shall make no assumptions concerning the boundary or the boundary conditions outside a neighbourhood of the corner point. The arrangement of the paper is as follows. In section 2 we state Theorem 1, the main result of the paper. The proof of this theorem will follow from two lemmas given in sections 3 and 4. Section 4 also contains some remarks.

2. Principal Results

Consider a sector Ω_{r_0} bounded by $\Gamma_1 : \theta = 0$, $\Gamma_2 : \theta = \omega$ and by $r = r_0$, where (r, θ) are the polar coordinates of (x, y). In Ω_{r_0} we consider the mixed boundary value problem

$$\triangle u = f \qquad \ldots(2.1)$$

$$u \mid_{\Gamma_1} = 0 \qquad \ldots(2.2a)$$

$$u_n \mid_{\Gamma_2} = 0 \qquad \ldots(2.2b)$$

where $u_n = \frac{\partial u}{\partial n}$ is the outward normal derivative. We now state our main result.

Theorem 1 — Let u be a bounded solution of (2.1), (2.2). Let $f \in C^{m+\alpha}(\bar{\Omega}_{r_0})$, $m \geqslant 0$ an integer and $0 < \alpha < 1$. If $\omega < \pi/2(m + 2 + \alpha)$ then $u \in C^{m+2+\alpha}(\bar{\Omega}_{r_1})$, where $0 < r_1 < r_0$.

We have introduced a method (see Azzam 1979) for proving smoothness statements, near corners, for solutions of the Dirichlet problem for second order linear elliptic equations. The method consists of the following three steps:

1. In a sector $\bar{\Omega}_{3r_1}$, $0 < 3r_1 \leqslant r_0$ we first find a bound of the form

$$| u(x, y) | \leqslant Mr^{\nu}, \nu \leqslant m + 2 + \alpha, \qquad \ldots(2.3)$$

2. Using this bound we then obtain in $\bar{\Omega}_{2r_1}$ bounds for the derivatives of u;

$$\left| \frac{\partial^k u(x, y)}{\partial x^{k_1} \partial y^{k-k_1}} \right| \leqslant M_k r^{\nu-k}, \; k = 1, \ldots, m + 2. \qquad \ldots(2.4)$$

3. Then we obtain the smoothness statement

$$u \in C^{\nu}(\bar{\Omega}_{r_1}). \qquad \ldots(2.5)$$

In proving (2.4) and (2.5) use has been made of Schauder estimates of the form

$$\| u \|^{\Omega'}_{m+2+\alpha} \leqslant \eta \, [\| u \|^{\Omega}_0 + \| f \|^{\Omega}_{m+\alpha} + \| \phi \|^{\Gamma}_{m+2+\alpha}] \qquad \ldots(2.6)$$

where $\Omega' \subset \Omega$, $\Gamma = \partial\Omega \cap \partial\Omega' \in C^{m+2+\alpha}$ (cf. Agmon *et al.* 1959). We apply this method to prove Theorem 1 and once we obtain a bound of the form (2.3) for the solution of our problem, we can apply steps 2 and 3. The only modification is that for our problem the third term on the right-hand side of (2.6) is omitted [cf. (2.2)]. Thus to prove Theorem 1 it is sufficient to find, for the solution u of (2.1) – (2.2), a bound of the form

$$| u(x, y) | \leqslant Mr^{m+2+\alpha}.$$

This will be proved in Sec. 4 under the following additional condition

$$f_{(0)}^{(k_1, k-k_1)} = \left.\frac{\partial^k f(x, y)}{\partial x^{k_1} \partial y^{k-k_1}}\right|_{x=y=0} = 0 \qquad \ldots(2.7)$$

$$k_1 = 0, \ldots, k;\ k = 0, \ldots, m.$$

In Lemma 1 of the next section it will be shown that condition (2.7) is not a restriction.

3. One Lemma

In this section we shall show that there exists a polynomial $w(x, y)$ such that the function $v = u - w$ satisfies $\triangle v = g$, with g satisfying (2.7).

Lemma 1 — There exists a polynomial $w(x, y)$ such that $w\,|_{\Gamma_1} = w_n\,|_{\Gamma_2} = 0$. Furthermore the function $g = f - \triangle w$ satisfies $g_{(0)}^{(k_1, k-k_1)} = 0$, $k = 0, \ldots, m$ [cf. (2.1), (2.2) and (2.7)].

Proof : We write w in the form

$$w = \sum_{p=0}^{m} w_p$$

where w_p is a polynomial of the form

$$w_p = y \sum_{i+j=p+1} b_{ij} x^i y^j.$$

The coefficients $\{b_{ij}\}$ will be defined from the condition

$$\frac{\partial w_p}{\partial n} = 0 \quad \text{on} \quad \Gamma_2 \qquad \ldots(3.1)$$

and by comparing the coefficients of the different powers on both sides of the relation

$$\triangle w_p = \sum_{q=0}^{p} \frac{1}{p!} \binom{p}{q} x^q y^{p-q} f_{(0)}^{(q, p-q)}. \qquad \ldots(3.2)$$

Thus we obtain a system of $p + 2$ linear equations $AB = C$, where $B = \{b_{ij}\}$. If $C = 0$ we take $B = 0$. If $C \neq 0$, then B will be determined if A is nonsingular.

We now show that this is the case. Suppose not. Then we find a nontrivial solution $B^0 = \{b^0_{ij}\}$ of $AB = 0$. The polynomial $w^0_p = y \sum_{i+j=p+1} b^0_{ij} x^i y^j$ is harmonic, vanishes on $y = 0$ and $\frac{\partial w^0_p}{\partial n} = 0$ on Γ_2. Thus from the principal of symmetry it follows that w^0_p vanishes on all the rays $y = x \tan 2k\omega$, $k = 0, 1, 2, \ldots$. The number of these rays is infinite if $\frac{\pi}{2\omega}$ is irrational. If $\frac{\pi}{2\omega} = \frac{N}{D}$, a reduced fraction, $D \geqslant 1$, then the number of the rays on which w^0_p vanishes is N and $N \geqslant \frac{N}{D} = \frac{\pi}{2\omega} > m+2+\alpha$. In both cases we reach a contradiction since w^0_p is a polynomial of degree $p + 2$ and $p \leqslant m$. The lemma is proved.

In the following section we estimate the solution u of (2.1) – (2.2) under the conditions of Theorem 1 and assuming (2.7) to be satisfied. To obtain this bound we use the technique of the barrier functions.

4. Bound for the Solution

As it was mentioned in section 2, the assertion of Theorem 1 will follow from the following lemma.

Lemma 2 — Suppose that the assumptions of Theorem 1 as well as (2.7) are satisfied. Then in $\bar{\Omega}_{r_0}$ we have

$$| u(x, y) | \leqslant M r^{m+2+\alpha}, \quad M > 0 \text{ a constant.} \qquad \ldots(4.1)$$

Proof : From (2.7) and the assumption $f \in C^{m+\alpha}(\bar{\Omega}_{r_0})$ it follows that in $\bar{\Omega}_{r_0}$

$$| f(x, y) | \leqslant R r^{m+\alpha}. \qquad \ldots(4.2)$$

Consider in Ω_{r_0} the function $V(x, y) = -M r^{m+2+\alpha} \cos \lambda(\omega - \theta)$, where $M > 0$ is a constant to be specified later and $\lambda = \frac{\pi - 2\delta}{2\omega} > m + 2 + \alpha = \nu$, $\delta > 0$. Since $\triangle V = (\lambda^2 - \nu^2) M r^{m+\alpha} \cos \lambda(\omega - \theta)$, and $\cos \lambda(\omega - \theta) \geqslant \sin \delta > 0$ for $0 < \theta < \omega$, then choosing M sufficiently large we obtain in Ω_{r_0}

$$\triangle V \geqslant R r^{m+\alpha} \geqslant f(x, y) \quad \text{[cf. (4.2)]}.$$

The function $W = u - V$ satisfies $\triangle W \leqslant 0$ in Ω_{r_0} and $W_n = 0$ on Γ_2. Thus W attains its minimum either on Γ_1 or on $r = r_0$ (cf. Miranda 1970). On Γ_1 we have $W \geqslant 0$ and since u is bounded on $r = r_0$ then W can be made nonnegative on $r = r_0$ by taking M sufficiently large. Thus in $\bar{\Omega}_{r_0}$ we have $W \geqslant 0$. That is

$$u \geqslant -Mr^{m+2+\alpha} \cos \lambda (\omega - \theta) \geqslant -Mr^{m+2+\alpha}.$$

The other part of (4.1) may be similarly established. The proof of Lemma 2 is now complete and Theorem 1 is proved.

We conclude this section by two remarks.

Remark 1 : If in Theorem 1 the condition $\pi/2\omega > m + 2 + \alpha$ is not satisfied, then the corresponding result will be $u \in C^{(\pi/2\omega)-\epsilon}(\bar{\Omega}_{r_1})$, with $\epsilon > 0$ arbitrarily small, The proof is the same as before with the following modification. If $\pi/2\omega \leqslant 2$, then condition (2.7) may be replaced by $|f| \leqslant R$, Lemma 1 is not needed and (in Lemma 2) λ and ν will be given by $\lambda = \frac{\pi - 2\delta}{2\omega} > \frac{\pi}{2\omega} - \epsilon = \nu$. If $\pi/2\omega > 2$, then always we can find m_1 and α_1; $m_1 \geqslant 0$ an integer and $\alpha_1 \in (0, 1)$ such that

$$\pi/2\omega > m_1 + 2 + \alpha_1 = (\pi/2\omega) - \epsilon, \epsilon > 0$$

arbitrarily small. m and α in (2.7) and in Lemmas 1 and 2 are then replaced by m_1 and α_1.

Remark 2 : Let Ω be a polygon with straight sides $\Gamma_1, \Gamma_2, \ldots, \Gamma_p$. Let u be a solution of the boundary value problem

$$\triangle u = f \quad \text{in } \Omega$$

$$\alpha_i u + (1 - \alpha_i)\, u_n = 0 \quad \text{on} \quad \Gamma_i,\ i = 1, \ldots, p$$

where α_i is 0 or 1 and $\alpha_i + \alpha_{i+1} \neq 0, \alpha_{p+1} = \alpha_1$. Let $f \in C^{m+\alpha}(\bar{\Omega})$ and let the interior angle at $0_i = \Gamma_i \cap \Gamma_{i+1}$ be $\omega_i = \frac{1}{2}(\alpha_i + \alpha_{i+1})\, \beta_i$. Then $u \in C^{m+2+\alpha}(\bar{\Omega}_0)$, where Ω_0 is a compact subregion of $\bar{\Omega}$ with positive distance from the corner points. In the neighbourhood of the corner point 0_i $u \in C^{\nu_i}$, where $\nu_i = \min (m + 2 + \alpha, \pi/\beta_i - \epsilon)$. $\epsilon > 0$ is arbitrarily small. This follows from Theorem 1 and Remark 1 $(\alpha_i + \alpha_{i+1} = 1)$ and from Azzam (1979) $(\alpha_i + \alpha_{i+1} = 2)$.

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