Indian J. pure appl. Math., **11**(11) : 1515–1529, November 1980

HALL CURRENT EFFECTS ON HARTMANN-STOKES LAYERS

K. N. VENKATASIVA MURTHY

Department of Mathematics, Post Graduate Centre (SVU), Anantapur (A.P.) 515003

(*Received* 5 *May* 1979; *after revision* 2 *November* 1979)

The hydromagnetic unsteady flow of a viscous, incompressible electrically conducting fluid induced by torsional oscillations of a conducting disk, superimposed on a state of steady rotation is studied in the presence of a uniform transverse magnetic field and Hall currents. The generation and propagation of wave due to viscous-Coriolis-Lorentz force interactions during the transient evolution is studied in the boundary layer and the flow outside. In comparison with the existing theories of hydromagnetic spin-up, it is shown that the Hall currents and the forcing frequency of oscillation of the disk are both capable of inducing several interesting additional features.

1. INTRODUCTION

The simple problem concerning the manner in which a rotating fluid bounded by one or two disks adjusts from one state of rigid rotation to another provides much valuable insight into the dynamics of homogeneous rotating fluids and is a model of several complicated problems of practical interest (Howard *et al.* 1967). Models of fluid bounded by one or two infinite disks oscillating about a state of rigid rotation are also analysed by several authors (Jones 1969, Claire Jacobs 1971, Venkatasiva Murthy 1979). The transient motion and the depth of penetration of the boundary layers are discussed.

A body of fluid in steady rotation is able to sustain a wave motion propogating along the axis of rotation with axial symmetry. The elasticity which rotation confers on fluid and which provides the restoring mechanism makes possible the propagation of waves. It is also known that magnetic field provides a source of diffusivity, resulting in the propagation of waves in an electrically conducting fluid. These two concepts led Chawla (1972) to consider a problem of hydromagnetic spin-up wherein he studied the generation and propagation of hydromagnetic waves during spin-up of an electrically conducting, viscous, incompressible fluid, in the presence of a transverse magnetic field. The wave motion in the boundary layer and in the inviscid interior and the harmonic waves supported by the fluid region are studied for several cases of interest.

It is the aim of the present work to study the hydromagnetic wave motion in the unsteady flow created by small amplitude torsional oscillations of a single infinite disk, superimposed on a basic state of rigid rotation in a semi-infinite viscous incompressible electrically conducting fluid. The whole system is under the influence of a magnetic field parallel to the axis of rotation. In the presence of a strong magnetic, field, as the motion of the charged particles across the magnetic field is hindered, there arise currents which are in a direction perpendicular to both electric and magnetic fields, known as Hall currents. These Hall currents have physical relevance to several Astrophysical situations and in the MHD power generation. Taking the Hall currents also into consideration, a study of the hydromagnetic wave propagation resulting out of the time dependent interaction of viscous, Coriolis and electromagnetic forces is made. The structure of the double-decker boundary layers on the disk is determined in terms of various parameters. The Hall currents and the forcing frequency of the disk introduce several interesting and useful results.

2. Formulation of the Problem

A homogeneous, viscous, incompressible, electrically conducting fluid fills the space between two pole pieces $z = 0, h$ of a magnet of strength H_0 which is a perfect conductor. Prior to time $t = 0$, the whole system is in a state of rigid rotation at angular speed Ω about the z-axis. At time $t = 0$, the boundary angular speed is impulsively accelerated to $\Omega(1 + \epsilon e^{i\omega t})$ and the applied field is not changed. The parameter ϵ, the Rossby number, is of small magnitude compared to unity. In this paper the pole pieces are assumed to be infinitely separated, that is, $h = \infty$. The cylindrical polar coordinates (r, θ, z) are chosen with accompanying fluid velocity $\mathbf{v}$ the magnetic field $\mathbf{H}$ and hydromagnetic pressure p. Taking into account the Hall currents also, the fundamental MHD equations are

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}\mathbf{v}^2\right) + (\nabla \times \mathbf{v}) \times \mathbf{v}\right] = -\nabla p + \mu \mathbf{J} \times \mathbf{H} - \mu_f \nabla \times (\nabla \times \mathbf{v}) \qquad \ldots(1)$$

$$\mathbf{J} + \frac{\omega_e \tau_e}{H_0} \mathbf{J} \times \mathbf{H} = \sigma\left(\mathbf{E} + \mu \mathbf{v} \times \mathbf{H} + \frac{1}{e n_e} \nabla p_e\right) \qquad \ldots(2)$$

$$\mathbf{J} = \nabla \times \mathbf{H} \qquad \ldots(3)$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \mathbf{B} = \mu \mathbf{H} \qquad \ldots(4)$$

$$\nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{H} = 0 \qquad \ldots(5)$$

where $\rho, \sigma, \mu, \omega_e, \tau_e, e, n_e, p_e$ are the fluid density, electrical conductivity, magnetic permeability, the cyclotron frequency, the electron collision time, the electric charge, the number density of electron and electron pressure. The usual assumption that the electron pressure term can be neglected is made. The following transformations which are consistent with axial symmetry and continuity will be taken.

$$\mathbf{V} = r\Omega\bar{\theta} + \epsilon\Omega\,[rF_z\mathbf{r} + rG\bar{\theta} - 2F\mathbf{z}] \quad \text{...(6)}$$

$$\mathbf{H} = H_0\mathbf{z} + \epsilon\Omega\,[-rN_z\mathbf{r} - rM\bar{\theta} + 2N\mathbf{z}] \quad \text{...(7)}$$

$$p = \tfrac{1}{2}\rho r^2\Omega^2 + \epsilon\Omega p'. \quad \text{...(8)}$$

Here F, G, N, M and p' are functions of z and t and $\mathbf{r}$, $\bar{\theta}$ and z are unit vectors in r, θ and z directions respectively. Using these substitutions (6) – (8) for $\mathbf{V}$, $\mathbf{H}$ and p in eqns. (1) – (5) and neglecting terms quadratic in ϵ, the following equations are obtained:

$$\nu F_{zzz} - F_{zt} + 2\Omega G = (\mu H_0/\rho)\, N_{zz} \quad \text{...(9)}$$

$$\nu G_{zz} - G_t - 2\Omega F_z = (\mu H_0/\rho)\, M_z \quad \text{...(10)}$$

$$\eta N_{zzz} - N_{zt} + h\eta M_{zz} = H_0 F_{zz} \quad \text{...(11)}$$

$$\eta M_{zz} - M_t - h\eta N_{zzz} = H_0 G_z \quad \text{...(12)}$$

where ν, η and $h(=\omega_e\tau_e)$ are the kinematic viscosity, magnetic diffusivity and the Hall parameter respectively. The boundary in contact with the fluid is a perfect conductor and hence the tangential current must be zero on $z = 0$. The appropriate initial and boundary conditions are:

At $t = 0$,

$$F = F_z = G = M = N = N_z = 0 \quad \text{for} \quad z \geqslant 0$$

for $t \geqslant 0$

$$F = F_z = 0,\ G = e^{i\omega t},\ M_z = N_{zz} = N = 0 \quad \text{on} \quad z = 0$$

$$F_z,\ G,\ M,\ N_z \rightarrow 0 \quad \text{as} \quad z \rightarrow \infty.$$

Defining the functions P and Q, for the sake of convenience, as

$$P = F_z + iG,\ Q = N_z + iM$$

we obtain the equations governing P and Q as

$$\nu P_{zz} - P_t - 2i\Omega P = (\mu H_0/\rho)\, Q_z$$

$$\eta Q_{zz} - Q_t - ih\eta Q_{zz} = H_0 P_z.$$

Using the Laplace transformation, the transformed variables $\bar{P}$ and $\bar{Q}$ satisfy the following differential equations and boundary conditions:

$$\nu \mathbf{P}_{zz} = (s + 2i\Omega)\, \mathbf{P} = H_0 \mathbf{Q}_z \quad \text{...(13)}$$

$$\eta(1 - ih)\, \mathbf{Q}_{zz} - s\mathbf{Q} = H_0 \mathbf{P}_z \quad \text{...(14)}$$

$$\mathbf{P}(0) = i/(s - i\omega),\ \mathbf{Q}_z = 0,\ \mathbf{P}(\infty) = \mathbf{Q}(\infty) = 0. \qquad ...(15)$$

The solution of eqns. (13) and (14) satisfying the boundary conditions (15) is

$$\mathbf{P} = Ae^{-m_1 z} + Be^{-m_2 z},\ \mathbf{Q} = Ce^{-m_1 z} + De^{-m_2 z} \qquad ...(16)$$

where

$$m_1, m_2 = \frac{1}{2(\nu\eta(1 - ih))^{1/2}} (\{A_0^2 + [(\eta(s - ihs + 2i\Omega))^{1/2} + (\nu s)^{1/2}]^2\}^{1/2}$$
$$\pm \{A_0^2 + [(\eta(s - ihs + 2i\Omega))^{1/2} - (\nu s)^{1/2}]^2\}^{1/2}) \qquad ...(17)$$

$$A = \frac{i(s + 2i\Omega - \nu m_2^2)}{\nu(s - i\omega)(m_1^2 - m_2^2)},\ B = \frac{i(s + 2i\Omega - \nu m_1^2)}{\nu(s - i\omega)(m_2^2 - m_1^2)} \qquad ...(18)$$

$$m_1 C = - m_2 D = \frac{-iH_0(s + 2i\Omega)}{\nu\eta(1 - ih)(s - i\omega)(m_1^2 - m_2^2)} \qquad ...(19)$$

and $A_0 = (\mu H_0^2/\rho)^{1/2}$ is the Alfvèn velocity.

3. Boundary Layers and the Flow Outside

This section is devoted to the mathematical analysis and study of the flow within the boundary layer and outside the boundary layer. To obtain the boundary layer solution, we take $\eta \neq 0$, ν and z tend to zero while $z/\nu^{1/2}$ remains finite. We obtain

$$\nu^{1/2} m_1 = [f(s)]^{1/2} \qquad ...(20)$$

$$\nu^{1/2}\eta^{*1/2} m_1 m_2 = [s(s + 2i\Omega)]^{1/2} \qquad ...(21)$$

where $f(s) = A_0^2 \eta^{*-1} + s + 2i\Omega$ and $\eta^* = \eta(1 - ih)$. In the calculations $(\nu s)^{1/2}$ has been omitted in the limit $\nu \to 0$. Thus the results of this section will not be valid for initial times of the development of the flow. The Laplace transformed velocity function $\mathbf{P}$ is given by

$$\mathbf{P} = \frac{i}{s - i\omega}\left\{\frac{s + 2i\Omega}{f(s)} \exp\left[-z\left(\frac{f(s)}{\nu}\right)^{1/2}\right] + \frac{A_0^2}{f(s)\,\eta^*}\right\}. \qquad ...(22)$$

Evaluating the Laplace inversion integral we obtain the velocity function P as

$$P = \frac{iA_0^2}{\eta^* f(i\omega)} \{\exp(i\omega t) - \exp[-f(0)\,t] \operatorname{erf}[z/2(\nu t)^{1/2}]\}$$
$$- \frac{2\Omega + \omega}{2\eta^* f(i\omega)} \exp(i\omega t) \Big\{\exp[-z(f(i\omega)/\nu)^{1/2}]$$
$$\times \operatorname{erfc}\left[\frac{z}{2(\nu t)^{1/2}} - (f(i\omega)\,t)^{1/2}\right] + \exp[z(f(i\omega)/\nu)^{1/2}]$$
$$\times \operatorname{erfc}\left[\frac{z}{2(\nu t)^{1/2}} + (f(i\omega)\,t)^{1/2}\right]\Big\}. \qquad ...(23)$$

A result due to Strand (1965) which is shown in the 'Appendix' will be used to expand the error functions. The second term in the curly brackets takes the form

$$\{\,\} = \exp\left(-2i\Omega t - \frac{iA_0^2 ht}{\eta^2(1+h^2)}\right)$$
$$\times\left[\exp(zC_1/\nu)\,\phi\left(\frac{z+2C_1t}{2(\nu t)^{1/2}},\,(t/\nu)^{1/2}C_2\right)\right.$$
$$\left.-\exp(-zC_1/\nu)\,\phi\left(\frac{|z-2C_1t|}{2(\nu t)^{1/2}},\,(t/\nu)^{1/2}C_2\right)\right]$$
$$+\,2\exp(i\omega t)\exp(-zC_1/\nu)\,[\cos(zC_2/\nu) - i\sin(zC_2/\nu)].$$
$$\text{for } z < 2C_1t \qquad \text{...(24)}$$

$$= \exp\left(-2i\Omega t - \frac{iA_0^2 ht}{\eta^2(1+h^2)}\right)$$
$$\times\left[\exp(zC_1/\nu)\,\phi\left(\frac{z+2C_1t}{2(\nu t)^{1/2}},\,(t/\nu)^{1/2}C_2\right)\right.$$
$$\left.+\exp(-zC_1/\nu)\,\bar{\phi}\left(\frac{|z-2C_1t|}{2(\nu t)^{1/2}},\,(t/\nu)^{1/2}C_2\right)\right],\ \text{for } z > 2C_1t \ \text{...(25)}$$

where

$$R^2 = \frac{A_0^4}{\eta^2(1+h^2)^2} + \left(2\Omega + \omega + \frac{A_0^2 h}{\eta^2(1+h^2)}\right)^2 \qquad \text{...(26)}$$

$$\tan 2\theta = \left(2\Omega + \omega + \frac{A_0^2 h}{\eta^2(1+h^2)}\right)\Big/\left(\frac{A_0^2}{\eta(1+h^2)}\right) \qquad \text{...(27)}$$

and $\qquad C_1, C_2 = (\nu R)^{1/2}(\cos\theta, \sin\theta).$

When $\omega = 0$ and $h = 0$, these results correspond to the results of Chawla (1972). The diffusing boundary layer interacts with the inertial oscillations, Coriolis and electromagnetic forces resulting in the propagation of diffusive waves (diffusivity ν) travelling away from the pole-piece $z = 0$ with velocity $2(\nu R)^{1/2}\cos\theta$. The decay length of the wave is of order $(\nu/R)^{1/2}\sec\theta$ so that the wave takes a damping time of order $(1/2R)\sec^2\theta$. The various interactions that result in the formation of modified Stokes-Hartmann layer on the pole-piece $z = 0$ reduce, under the absence of any magnetic field, to the modified Stokes layer (Claire Jacobs 1971, Jones 1969) of thickness order $(\nu/|2\Omega + \omega|)^{1/2}$. Analysing this thickness we find that if $\omega \gg \Omega$, when the viscous force would balance the acceleration, the depth of penetration (decay length) is a Stokes layer $O(\nu/\omega)$ and if $\Omega \gg \omega$, when the viscous force would balance the Coriolis force, the depth of penetration is an Ekman layer $O(\nu/\Omega)^{1/2}$. We note that the frequency $\omega = -2\Omega$ is a resonant frequency in the non-magnetic case and the presence of the magnetic field thus eliminates the occurrence of resonance in the corresponding non-magnetic case.

When the frequency of oscillation ω of the pole-piece increases, the boundary layer thickness decreases and for large values of the forcing frequency ω

$$\left(\omega \gg 2\Omega + \frac{A_0^2 h}{\eta^2(1 + h^2)}, \frac{A_0^2}{\eta(1 + h^2)}\right),$$

there results a thin boundary layer of thickness of the order of the Stokes layer. For small values of ω and h the thickness of the boundary layer corresponds to the Hartmann-Ekman depth. It can also be observed from the expressions (24) – (27) that as the Hall parameter increases, the wave velocity decreases and the decay length increases. If

$$\left| 2\Omega + \omega + \frac{A_0^2 h}{\eta^2(1 + h^2)} \right| \leqslant \frac{A_0^2}{\eta(1 + h^2)},$$

the wave front moves with the velocity $2A_0\,[\nu/\eta(1 + h^2)]^{1/2}$ and the diffused wave damps out at distances from the pole-piece, of order $[\rho\nu\eta(1 + h^2)/\mu H_0^2]^{1/2}$. This is the Hartmann depth modified in the presence of the Hall currents.

It may also be noticed from (23) that the second term is negligible at distances of order $(\nu/R)^{1/2}\sec\theta$ (the boundary layer thickness) and the first term in (23) provides an indication of the flow outside the boundary layer. In this term the exponential decays at times of order $|\,\eta(1 - ih)/A_0^2\,|$. The inviscid flow outside the boundary layer is obtained by keeping η and z fixed and letting $\nu \to 0$.

$$\mathbf{P} = \frac{iA_0^2}{(s - i\omega)\,\eta^* f(s)} \exp\left[-z\left(\frac{s(s + 2i\Omega)}{\eta^* f(s)}\right)^{1/2}\right]. \qquad \ldots(28)$$

The inversion will be evaluated for two cases of interest

$$|\,A_0^2/\eta(1 - ih)\,| \ll |\,s + 2i\Omega\,|$$

and

$$|\,A_0^2/\eta(1 - ih)\,| \gg |\,s + 2i\Omega\,|\,.$$

These cases are of interest in view of the two facts:

(i) the expression on the left-hand side of these inequalities provides a measure of the time for the transient motion to decay

(ii) The first inequality is satisfied by fluids of small electrical conductivity in particular and the second inequality is satisfied by large conductivity fluids for times which are moderate or large.

For the first case, the inversion of (28) gives

$$P \simeq \frac{A_0^2}{2\eta^*(2\Omega + \omega)} \Big[\exp(i\omega t) \Big\{ \exp(\pm z(-i\omega/\eta^*)^{1/2})$$

$$\times \operatorname{erfc}\left(\frac{z}{2(\eta^* t)^{1/2}} \pm i(-i\omega t)^{1/2}\right)\Big\}$$

$$- \exp(-2i\Omega t) \Big\{ \exp(\pm z(-2i\Omega/\eta^*)^{1/2})$$

$$\times \operatorname{erfc}\left(\frac{z}{2(\eta^* t)^{1/2}} \pm (-2i\Omega t)^{1/2}\right)\Big\}\Big] \qquad \ldots(29)$$

where $\eta^* = \eta(1 - ih)$ and the summation is done over both signs in each bracket (see eqn. 23). Expanding the function erfc using Strand series we obtain

$$P \simeq \frac{A_0^2}{\eta^*(2\Omega + \omega)} \Big[\Big\{ \pm \tfrac{1}{2} \exp\left(\pm zC_3(\tfrac{1}{2}\omega)^{1/2} - \frac{iz^2C_7}{2t} \right)$$

$$\times \phi\left(\left| \frac{zC_5}{2t^{1/2}} \pm (\tfrac{1}{2}\omega t)^{1/2} \right|, \left| \frac{zC_6}{2t^{1/2}} \pm (\tfrac{1}{2}\omega t)^{1/2} \right| \right) \Big\}$$

$$+ 2\exp(i\omega t)\exp[-z(\tfrac{1}{2}\omega)^{1/2}(C_3 + iC_4)]$$

$$- \Big\{ \pm \tfrac{1}{2} \exp\left(\pm z\Omega^{1/2}C_4 - \frac{iz^2C_7}{2t} \right)$$

$$\times \bar{\phi}\left(\left| \frac{zC_5}{2t^{1/2}} \pm (\Omega t)^{1/2} \right|, \left| \frac{zC_6}{2t^{1/2}} \mp (\Omega t)^{1/2} \right| \right) \Big\}$$

$$- \exp(-2i\Omega t)\exp[-z\Omega^{1/2}(C_4 + iC_3)] \Big]$$

in the case when

$$z < C_5^{-1}(2\omega)^{1/2}t < C_6^{-1}(2\omega)^{1/2}t$$

and

$$z < 2C_5^{-1}\Omega^{1/2}t < 2C_6^{-1}\Omega^{1/2}t \qquad \ldots(30)$$

where

$$C_3, C_4 = C_5 \mp C_6$$

$$C_5, C_6 = (\eta_1, h_1)/\eta(1 + h^2)^{1/2}, \; C_7 = C_5 \,.\, C_6$$

and

$$\eta^{*1/2} = \eta_1 - ih_1.$$

The solution for any one case reveals the properties of wave motion and hence it is unnecessary to present the lengthy expressions for the other cases. The first two terms of the solution correspond to diffusion from a source of diffusivity $(\eta^2 + h^2)/\eta_1^2$ moving with velocities $(2\omega)^{1/2}/C_5$ and $(2\omega)^{1/2}/C_6$.

The fluid region also supports harmonic waves whose velocity of propagation is $(2\omega)^{1/2}/C_4$ and wave number $(\omega/2)^{1/2}C_4$. Since $\eta_1 + h_1 > \eta_1 > h_1$, the wave front with velocity $(2\omega)^{1/2}/C_6$ moves faster than the other two. Comparing the solution with that in the non-oscillatory case (Chawla 1972), we find that all these waves occur only in the presence of the forcing frequency ω of oscillation of the pole-piece. The depth of penetration of these waves is $(2/\omega)^{1/2}/C_3$ and again we find that the depth of penetration is effectively reduced when the forcing frequency ω is large. When $h \ll 1$, the depth is of order $(\eta/\omega)^{1/2}$. As h increases the depth of penetration increases.

The remaining terms in the equation represent diffusion from a source of diffusivity $(\eta^2 + h^2)/\eta_1^2$ moving with velocities $v_1 = 2\Omega^{1/2}/C_5$, $v_2 = 2\Omega^{1/2}/C_6$. The fluid region also supports harmonic electromagnetic waves propagating with velocity $v_3 = 2\Omega^{1/2}/C_3$. Noting that $\eta_1 > h_1$ and $\eta_1 - h_1 \gtrless h_1$ according as $h \lessgtr 2/\sqrt{3}$, we find that if $h < 2/\sqrt{3}$, the wave front $z = v_2 t$ travels faster than the others and the region behind this wave front and the edge of the boundary layer supports the other waves. In this case, if further $h \ll 1$, we find that the velocities v_1 and v_3 are of order $(\Omega\eta)^{1/2}$ whereas v_2 is of order $(\Omega\eta)^{1/2}/h$. Thus the fast travelling front is much ahead of the other wave fronts. The harmonic travelling waves arise from the steady parts and hence they will not be seen in the solution in certain cases, say for example, when the symbols $<$ are replaced by $>$ in eqns. (30).

The decay length of these waves is of order $\eta\,[(1 + h^2)/\Omega(\eta_1 + h_1)]^{1/2}$ which reduce to $(\eta/\Omega)^{1/2}\,(1 + \frac{1}{2}h)$ if $h \ll 1$ and as h increases the decay length increases. The Hall current effects are responsible for:

(i) the occurrence of travelling waves with velocity v_2. In the absence of any Hall currents we find only the diffusion from a source of diffusivity η moving with velocity $2(\eta\Omega)^{1/2}$ through a depth $(\eta/\Omega)^{1/2}$ and the region behind supports harmonic waves;

(ii) the waves travelling with velocities v_1 and v_3 are also influenced by the time independent harmonic waves, in comparison with the case $h = 0$, in which these harmonic waves are absent.

Thus the oscillations of the pole-piece and the Hall currents not only modify the properties of waves occurring in the corresponding non-oscillatory case without Hall currents (i.e., hydromagnetic spin-up), but also introduce new waves travelling with different velocities. The Coriolis-Lorentz force balance splits the waves.

When $\quad | A_0^2/\eta(1 - ih) | \gg | s + 2i\Omega |$,

$$\mathbf{P} \simeq \frac{i}{s - i\omega} \exp\left\{-z\left[\frac{s(s + 2i\Omega)}{A_0^2}\right]^{1/2}\right\}$$

Evaluation of the Laplace inversion integral gives

$$P = i\Bigg[H\left(t - \frac{z}{A_0}\right)\exp\left(i\omega t - i\omega\frac{z}{A_0} - i\Omega\frac{z}{A_0}\right)$$

$$- \Omega z \int_0^t H(u - z/A_0)\exp(i\omega t - i\Omega u - i\omega u)\frac{J_1(\Omega(u^2A_0^2 - z^2)^{1/2}/A_0)}{(u^2A_0^2 - z^2)^{1/2}}\,du\Bigg] \quad ...(31)$$

$$= i\Bigg\{\exp(i\omega t - iz(\omega(\omega + 2\Omega))^{1/2}/A_0)$$

$$+ \Omega z \int_t^\infty \exp(i\omega t - i\Omega u - i\omega u)\frac{J_1(\Omega(u^2A_0^2 - z^2)^{1/2}/A_0)}{(u^2A_0^2 - z^2)^{1/2}}\,du\Bigg\} \quad ...(32)$$

where H is a Heaviside function.

This shows that

$$P \to i\exp\left\{i\omega t - \frac{iz\,[\omega(\omega + 2\Omega)]^{1/2}}{A_0}\right\}$$

as $t \to \infty$.

By integrating the expression for P in eqn. (31) we can obtain an expression in a form similar to eqn. (4.3) of Chawla (1972) which again does not give an estimate of the steady axial flow. For this reason, the expression for P is reconsidered and integrated to find F and then inverted to obtain

$$F = \frac{A_0\exp(i\omega t)}{[\omega(\omega + 2\Omega)]^{1/2}}[1 - \exp(-iz\,[\omega(2\Omega + \omega)]^{1/2}/A_0)]$$

$$+ \frac{z^2}{\omega}\left[\frac{iA_0\Omega}{2\pi t^3}\right]^{1/2}\exp(-z^2 i\Omega/2A_0^2 t) \quad \text{if} \quad \omega \neq 0. \quad ...(33)$$

When $\omega = 0$, F is given by eqn. (4.5) of Chawla (1972). The expression (31) for P shows that the wave front $z = A_0 t$ moves with the Alfven velocity and the region ahead of the front $z = A_0 t$ oscillates purely vertically. The region $z \leqslant A_0 t$ supports harmonic waves of a wave number Ω/A_0 travelling with velocity $\omega A_0/\Omega$. If $\omega \ll \Omega$, the Alfven front is much ahead of the front $z = A_0\omega t/\Omega$ whereas if $\omega \gg \Omega$ the whole region $z \leqslant A_0 t$ supports the harmonic waves. Thus the existence of the forcing frequency ω of the pole-piece creates a progressive harmonic wave, which, in the non-oscillatory case reduces to a stationary wave in the region $z \leqslant A_0 t$. The harmonic wave is absent when there is no Coriolis force ($F = 0$) and the expression (31) represents Alfven waves in a non-rotating medium. The Coriolis-Lorentz force balance leads to the splitting of the Alfvèn waves.

The disturbances created by the impulsive oscillations of the pole-piece move far away from the pole-piece and the wave pattern in the region beyond the boundary layer collapses for large time due to the continuous distortion by vortex lines and magnetic field lines. The wave pattern diffuses with induced diffusivity A_0^2/Ω as seen from eqn. (33).

The inviscid solution and the viscous boundary layer solution are matched at the edge of the boundary layer to obtain the axial flow as

$$\mathbf{F} = \frac{i(s + 2i\Omega)\,\nu^{1/2}}{(s - i\omega)\,[f(s)]^{3/2}} \{1 - \exp[-z(f(s)/\nu)^{1/2}]\}$$

$$+ \frac{iA_0^2}{(s - i\omega)\,[s(s + 2i\Omega)\,\eta^* f(s)]^{1/2}} \left\{1 - \exp\left[-z\left(\frac{s + 2i\Omega}{\eta^* f(s)}\right)^{1/2}\right]\right\}.$$

Evaluating the Laplace inversion integral, in the limit $s \to 0$, we obtain the steady solution for $\omega \neq 0$ as

$$F = \frac{-(2\Omega + \omega)\,\nu^{1/2} \exp(i\omega t)}{[f(i\omega)]^{3/2}} \left\{1 - \exp\left[-z\left(\frac{f(s)}{\nu}\right)^{1/2}\right]\right\}$$

$$+ \frac{A_0 \exp(i\omega t)}{[\omega(\omega + 2\Omega)\,\eta^* f(i\omega)]^{1/2}} \left\{1 - \exp\left[-z\left(\frac{-\omega(\omega + 2\Omega)}{\eta^* f(i\omega)}\right)^{1/2}\right]\right\}$$

$$+ \frac{(A_0 z)^2\,(2i\Omega/\pi)^{1/2}}{2\omega\,[\eta^* f(0)\,t]^{3/2}} \exp\left(\frac{-i\Omega z^2}{2t\eta^* f(0)}\right). \quad \ldots(34)$$

If $\omega = 0$, the expression is given by (5.10) of Chawla (1972) with $\eta(1 - ih)$ replaced for η.

The first term shows that immediately adjacent to the pole-piece, there is a boundary layer which acts as a transition region for the change in velocity field required by the boundary conditions. It is the only region of the flow in which viscosity is crucial and magnetic diffusivity also acts therein. It grows initially by viscous diffusion (see solution for small times), eventually slows down and reaches a steady state due to the distortion of vortex lines and magnetic field lines in the region exterior to it. The second term represents an outer layer in which the viscosity is relatively unimportant and the distances at which the exponential term decays is a measure of the depth of penetration of the waves induced by magnetic diffusion. This outer layer is a magnetic diffusion region the establishment of which is an important feature of the hydromagnetic interaction with rotational forces. The magnetic diffusion region arises to satisfy the exterior boundary conditions which the inner boundary layer (modified Stokes-Hartmann layer) cannot satisfy and the thickness of this magnetic diffusion region is much greater than the modified Stokes-Hartmann layer for all realistic values of the parameters.

Thus the flow field function governed by (34) shows the transient evaluation of the double-decker boundary layers through two different spatial scales (depths of penetration). These depths are such that as the strength of the magnetic field increases the thickness of the inner layer decreases while that of the outer layer increases. In view of the matching of the inner and outer solutions, the edge of the inner layer provides a transition from the inner to the outer layer.

The third term in (34) shows that the outermost region diffuses parabolically with a characteristic diffusion constant $\eta(1 + h^2)\, R/\Omega(1 + \sin 2\theta)$ where R, θ are given by (26) and (27) with $\omega = 0$.

4. Behaviour at Small Times

It has been remarked already that the analysis developed in section 3 is not valid for small times i.e., initial stages of motion. A separate mathematical analysis for small times can be developed on the lines of section 6 of Chawla (1972) based on the assumption that small time behaviour corresponds to large values of s. The lengthy mathematical details are spared to the reader and the sequence of process in the early stages of the development of the flow is described below.

Immediately following the impulsive, a simple Rayleigh shear layer developes in azimuthal flow and starts thickening due to viscous diffusion. The effect of the Coriolis force is to induce inertial oscillations as seen from one of the terms in the solution

$$\exp\left[i\omega t - z\left(\frac{2i\Omega + i\omega}{\nu}\right)^{1/2}\right] \operatorname{erfc}\left[\frac{z}{2(\nu t)^{1/2}} - ((2\Omega + \omega)\, it)^{1/2}\right]$$

and results in propagating a diffusive source with the velocity $[2(2\Omega + \omega)\,\nu]^{1/2}$ through a depth $[\nu/(2\Omega + \omega)]^{1/2}$. Also, the Rayleigh shear layer interacts with the electromagnetic body force which results in the propagation of electromagnetic waves with velocities

$$[2(2\Omega + \omega)\,(1 + h^2)]^{1/2}\, \eta\eta_1^{-1} \quad \text{and} \quad [2(2\Omega + \omega)\,(1 + h^2)]^{1/2}\, \eta h_1^{-1}$$

represented mathematically by a term in the solution obtained from (35) by replacing η^* for ν. The wave front travelling with the latter velocity is solely due to the presence of the Hall currents.

5. Behaviour at Large Times

The study of the behaviour of the magnetic field function Q and velocity field function P will now be made. The dominant contributions to these functions are associated with the singularities $s = i\omega$, $s = 0$ and $s = -2i\Omega$. Further, the ultimate behaviour corresponds to the regions of the complex plane near the singularity $s = 0$. To study the approach to the ultimate state, the functions of s are expanded in ascending powers of $|s|$ taking $|s|$ and $|s + 2i\Omega|$ to be small.

$$m_1 = (\eta^* f(0)/\nu\eta)^{1/2} (1 + ks - \lambda s^2 + \dots)$$

where $f(s) = A_0^2 \eta^{*-1} + 2i\Omega + s$

$$m_2 = [s(s + 2i\Omega)/\eta^* f(0)]^{1/2} (1 - ks + \dots)$$

$$C \simeq \frac{2\Omega H_0 \nu^{1/2}}{\eta^*(s - i\omega)\, [f(0)]^{3/2}}$$

$$D \simeq \frac{iH_0}{s - i\omega} \left[\frac{s + 2i\Omega}{s\eta^* f(0)}\right]^{1/2}$$

where

$$k = [(\nu + \eta^*)\, A_0^2\, \eta^{*-2} + 2i\Omega]/2\, [f(0)]^2 \qquad \dots(36)$$

$$\lambda = [\beta^2\eta^{*2} + 6A_0^2\, \nu\eta^*\beta + \nu^2 A_0^2\, (A_0^2 - 8i\Omega\eta^*)]/8\beta^4 \qquad \dots(37)$$

$$\beta = A_0^2 + 2i\Omega\eta^*.$$

The inversion of $\mathbf{Q} = Ce^{-m_1 z} + De^{-m_1 z}$ gives

$$\begin{aligned} Q = {} & \frac{2H_0\Omega(\nu\eta^*)^{1/2}}{\beta^{3/2}} \exp\left(-z\left(\frac{\beta}{\nu\eta^*}\right)^{1/2}\right) \exp(i\omega t) \\ & \times \exp\left[-z\omega(\lambda\omega + ik)\left(\frac{\beta}{\nu\eta^*}\right)^{1/2}\right] \\ & \times \left[1 - \tfrac{1}{2}\operatorname{erfc}\left(\frac{t - z(k + 2i\omega\lambda)\,(\beta/\nu\eta^*)^{1/2}}{2(z\lambda)^{1/2}\,(\beta\nu\eta^*)^{1/4}}\right)\right] \\ & - \frac{H_0 \exp(-i\Omega t)}{\pi\beta^{1/2}} [1 - \eta^*\Omega\nu(2\Omega + \omega)\,\beta^{-2}]\, K_0\,[-i\Omega(t^2 - z^2/\beta)^{1/2}] \\ & + \frac{2H_0\nu\Omega\eta^*}{\beta^{5/2}} \left\{\frac{\Omega t \exp(-i\Omega t)\, K_1\,[-i\Omega(t^2 - z^2/\beta)^{1/2}]}{\pi(t^2 - z^2/\beta)^{1/2}}\right. \\ & \left. - (\Omega/\pi) \exp(-i\Omega t)\, K_0\,[-i\Omega(t^2 - z^2/\beta)^{1/2}]\right\} \\ & + \left[\frac{2\nu\eta^*\Omega\omega}{\beta^2} - 1\right] \frac{iH_0(2\Omega + \omega)}{\pi\beta^{1/2}} \\ & \times \int_0^t \exp[i\omega(t - u) - iu\Omega]\, K_0\,[-i\Omega(t^2 - z^2/\beta)^{1/2}]\, du \end{aligned} \qquad \dots(38)$$

where K_0 and K_1 are the modified Bessel functions of the second kind. The first term in (38) represents a diffused hydromagnetic wave propagating through the hydromagnetic boundary layer with velocity which is the real part of $(\nu\eta^*)^{1/2}/\beta^{1/2}(k + 2i\omega\lambda)$

where k and λ are given by (36) and (37). When ν/η is small, the velocity of propagation is $4(\nu R)^{1/2}(2\cos\theta + \omega\sin\theta)/(4+\omega^2)$ which decreases as ω increases. When $A_0^2 \gg |2i\Omega\eta^*|$, the velocity is the real part of

$$\frac{4A_0^3(\nu\eta^*)^{1/2}}{2A_0^2(\eta+\nu-ih) - i\omega\,[\eta^{*2} + 6\nu\eta^* + \nu^2]}$$

which reduces to $2A_0(\nu\eta)^{1/2}/(\eta+\nu)$ when $h = 0$ and $\omega = 0$. It may be noted that as h increases the velocity decreases. The region behind the wave front supports travelling harmonic wave, which when $\omega = 0$, reduces to a stationary wave. The behaviour of the other terms in the solution can be studied by making an asymptotic expansion of the functions K_0 and K_1 for large times. Writing $\delta^2 = (A_0^2 + 2h\eta\Omega)^2 + 4\eta^2\Omega^2$, farther away from the pole-piece we take $t^2 - z^2(A_0^2 + 2h\eta\Omega)/\delta^2$ to be small and $2z^2\eta\Omega/\delta^2$ large so that

$$\exp(-i\Omega t)\,K_0[-i\Omega(t^2 - z^2/\beta)^{1/2}] \simeq \frac{(\pi i)^{1/2}\exp[-z\Omega(\eta\Omega)^{1/2}/\delta]}{(2\Omega)^{1/2}(t^2 - z^2/\beta)^{1/4}}$$

$$\times \exp[i\Omega(\eta\Omega)^{1/2}\delta^{-1}z - i\Omega t]$$

$$\times \exp\left[\frac{i\delta\Omega}{4z(\eta\Omega)^{1/2}}(t^2 - z^2(A_0^2 + 2h\eta\Omega)\,\delta^{-2})\right].$$

The expression corresponds to a diffused hydrodynamic wave packet consisting of dispersive harmonic waves with wave velocity $(\delta^2/\eta\Omega)$ and group velocity

$$\delta^2/(A_0^2 + 2h\eta\Omega).$$

We note that these expressions coincide with those obtained by Chawla in the corresponding case ($h = 0 = \omega$). The decay length of the wave is $\delta/\eta^{1/2}\Omega^{3/2}$ which increases as h increases. When the magnetic field is strong ($A_0^2 \gg 2h\eta\Omega, 2\eta\Omega$), the velocity of the wave is of order $A_0^2/(\eta\Omega)^{1/2}$ and the group velocity is of the order of the Alfven velocity. The electromagnetic Coriolis force balance splits the Alfven waves.

From the quasi-steady solution for the velocity field equation (34), when $\omega = 0$, the axial flow at infinity is given by $F(\infty) = -2\Omega\nu^{1/2}(\cos 3\theta)/R^{3/2}$ where R, θ are given by (26) and (27) with $\omega = 0$. If $\eta \to \infty$, we have $F(\infty) = \frac{1}{2}(\nu/\Omega)^{1/2}$. As the strength of the magnetic field increases, the suction velocity decreases and becomes zero when $A_0^2(\sqrt{3} - h)/\eta(1 + h^2) = 2\Omega$. Thus, if $h > \sqrt{3}$, the suction velocity never becomes zero. The presence of moderate Hall current altogether eliminates the property observed in the corresponding case $h = 0$ that the suction velocity becomes zero as the magnetic field increases and there is a transition from suction to injection at infinity. When $A_0^2(\sqrt{3} - h)/\eta(1 + h^2) > 2\Omega$, there is an outflow at infinity which reaches a maximum when

$$\cot[\tfrac{2}{3}\cot^{-1}(-h)] = A_0^2\eta/[A_0^2 h + 2\Omega\eta^2(1 + h^2)]$$

and then starts decreasing. For large values of the magnetic field i.e.,

$$A_0^2 h/\eta(1 + h^2) \gg 2\Omega,$$

the axial flow decreases according as $F(\infty) = - 2\Omega\nu^{1/2}\eta^{3/2}(1 + h)^{3/4} (\cos 3\theta)/A_0^3$ where $\tan 2\theta \simeq h$. When $\omega > 0$,

$$F(\infty) = \frac{-(2\Omega + \omega)\nu^{1/2}\cos(\omega t - 3\theta)}{R^{3/2}} + \frac{A_0^2 R^* \cos(\omega t - \theta + \alpha)}{\eta(1 + h^2)^{1/2}[\omega R(2\Omega + \omega)]^{1/2}}$$

where

$$R^* = (\eta_1^2 + h_1^2)^{1/2}, \tan\alpha = h_1/\eta_1. \qquad ...(39)$$

From (39), it is evident that the axial velocity increases with the magnetic field. This is natural in view of the observation that the depth of penetration of the outer diffusion region increases with the magnetic field and more fluid is required to meet the outward radial flow in the diffusion region.

5. Conclusions

It may be concluded that the Hall currents produce new interesting features, like introducing wave modes in addition to the wave modes observed when there are no Hall currents, eliminating transition from suction to injection at infinity. The forcing frequency of oscillations ω of the pole-piece introduces double-decker boundary layer structure which also suggests that a non-linear analysis can be developed using the method of matched asymptotic expansions. This is the subject of the subsequent study.

Acknowledgement

The author is thankful to Prof. S. K. Trehan for the suggestions towards the improvement of the manuscript.

References

Chawla, S. S. (1972). On hydromagnetic spin-up. *J. Fluid Mech.*, **53**, 545.

Claire Jacobs (1971). Transient motions produced by disks oscillating about a state of rigid rotation. *Q. Jl Mech. appl. Math.*, **24**, 221.

Howard, L. N., Moore, D. W., and Spiegel, E. A. (1967). Solar spin-down problem. *Nature, Lond.*, **214**, 1297.

Jones, A. F. (1969). The model of a single infinite disk oscillating about a state of steady rotation. *J. Fluid Mech.*, **39**, 257.

Strand, O. N. (1965). A method of computation of the error function of a complex variable. *Math. Comp.*, **19**, 127.

Venkatasiva Murthy, K. N. (1979). MHD viscous flow between torsionally oscillating disks. *Appl. Sci. Res.*, **35**, 111.

Appendix

Strand (1965) has shown that, for $x > 0, y \geqslant 0$,

$$\operatorname{erfc}(x + iy) = e^{-2ixy}\phi(x, y) \qquad \text{...(I)}$$

where

$$\left.\begin{aligned}
\phi(x, y) &= \sum_{0}^{\infty} (xy)^{2n} [\gamma_n(x) - ixy(n+1)\gamma_{n+1}(x)], \\
\gamma_{n+1}(x) &= \frac{2}{(2n+1)\sqrt{\pi}}\left[\frac{e^{-x^2}}{(n+1)!\,x^{2n+1}} - \frac{\sqrt{\pi}\,\gamma_n(x)}{(n+1)}\right] \\
&\qquad\qquad (n = 0, 1, 2, \ldots) \\
\gamma_0(x) &= \operatorname{erfc} x.
\end{aligned}\right\} \qquad \text{...(II)}$$

Since $\operatorname{erf}(-x-iy) = -\operatorname{erf}(x+iy)$ and $\operatorname{erf}(x-iy) = \overline{\operatorname{erf}(x+iy)}$, these cases are also covered by (I) and (II) above, but the case $x = 0$ is not. $\phi(x, y)$ is a complex function which tends to zero as $x \to \infty$.