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A NUMERICAL STUDY OF THE THICKENING OF SHOCK WAVES

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A numerical technique based on linearization and finite difference method is presented to solve the nonlinear partial differential equations governing the propagation of shock waves. The process of thickening and steepening of the shock is studied by solving test examples. On taking different forms of the initial shock, it is found that the asymptotic form of the shock remains unaltered. In another example, it has been shown that the thickness of the shock increases due to its interaction with a wave representing disturbance. It is also found that the effect of the disturbance wave on the shock wave almost vanishes shortly after their interaction.

1. INTRODUCTION

It has been noted that a characteristic 'N' shaped wave is formed at or near the ground level below the flight path of a flying supersonic aircraft at high altitude. The mechanism which governs the thickening of the compressive part of these waves is not in conformity with the theory that there is a balance between the nonlinear steepening effect and the thickening effect due to the diffusion; Ffowcs Williams and Howe (1973) have concluded that a wave may be thickened by a factor of 2 at the most by examining theories based on wave-scattering mechanism. Bauer (1971) has experimentally demonstrated that thickening of the shock may arise due to the atmospheric turbulence. The practical significance of shock thickening is that it reduces the power of annoyance resulting from the sonic boom (Rice 1972). Hence one would like to examine the factors which give rise to the shock thickening.

In this paper, we present a numerical method based on linearization for the solution of the equation governing the propagation of shock waves to study the structure of the shock and the shock thickening on passing through a disturbance wave.

2. FORMULATION OF THE PROBLEM

For examining the phenomenon of shock thickening when a shock passes through disturbance, we have considered one dimensional 'piston problem'. Although this is a simplified model, the essential features of steepening and thickening of the shock on interaction with a disturbance wave should be present in it (Beasley 1974). We consider a long tube of length L containing a fluid. At one end of the tube, there is a piston moving with uniform velocity and sending a shock wave down the tube.

At the other end of the tube, a second piston is moving in an oscillatory manner and sending a series of compression and expansion waves in opposite directions to meet and pass through the oncoming shock wave. For producing a wave resembling disturbance, the motion of the oscillating piston can be adjusted to have relatively large amplitudes and small gradients compared to the shock wave. A mathematical model for this problem can be formulated as follows.

The basic equations of plane sound waves of finite amplitude with linearized diffusion terms have been given by Lighthill (1956):

$$v_t + vv_x + \frac{2}{\gamma - 1} aa_x = \delta v_{xx} \qquad \text{...(1)}$$

$$a_t + va_x + \frac{\gamma - 1}{2} av_x = 0 \qquad \text{...(2)}$$

where t is time, a the local speed of sound, v the particle velocity, γ the ratio of specific heats and δ the combination of different diffusivities which affect attenuation of sound waves, is called the diffusivity of sound.

It is more convenient to put the variables in eqns. (1) and (2) in non-dimensional form with respect to the speed of sound a_0 in the undisturbed fluid and to the time t_0 defined below:

$$v = a_0 V,\ a_0 = a_0 (1 + A),\ t = t_0 T,\ x = a_0 t_0 X \text{ and } \delta = a_0^2\, t_0 D.$$

The non-dimensional forms of eqns. (1) and (2) are

$$V_T + VV_X + \frac{2}{\gamma - 1} (1 + A) A_X = DV_{XX} \qquad \text{...(3)}$$

and

$$A_T + VA_X + \frac{\gamma - 1}{2} (1 + A) V_X = 0. \qquad \text{...(4)}$$

We take t_0 such that $D = 1$.

Let $A^{(n)}$ and $V^{(n)}$ denote the numerical values of A and V at the n-th iteration and $A^{(0)}$ and $V^{(0)}$ be the initial guess. We consider the following quasilinearized forms for eqns. (4) and (3) :

$$A_T^{(n+1)} + \frac{\gamma - 1}{2} V_X^{(n+1)} + V^{(n)} A_X^{(n)} + \frac{\gamma - 1}{2} A^{(n)} V_X^{(n)} = 0,$$
$$n = 0, 1, 2, \ldots \qquad \text{...(5)}$$

and

$$V_T^{(n+1)} + \frac{2}{\gamma - 1} A_X^{(n+1)} + V^{(n)} V_X^{(n)} + \frac{2}{1 - \gamma} A^{(n+1)} A_X^{(n+1)} = V_{XX}^{(n+1)},$$
$$n = 0, 1, 2, \ldots . \qquad \text{...(6)}$$

The sequence of the linear problems given by eqns. (5) and (6) along with the given initial and boundary conditions is solved by using the finite difference method.

3. Finite Difference Equations

The domain in the X-T plane is discretized by a grid with step length $\triangle X = h$ and time step $\triangle T = k$. The numerical values of A and V at the grid point $(X, T) \equiv (rh, sk)$ are denoted by $A_{r,s}$ and $V_{r,s}$, respectively, $r = 0, 1, 2, \ldots, N$; $s = 0, 1, 2, \ldots$ and $L = Nh$. We have taken the pistons as being beyond the length L of the tube, this enables us to do the numerical work over a constant length L of the tube. We take

$$A_T \mid_{(2r+1)/2,\ (2s+1)/2} = \frac{1}{2k}\left(A_{r+1,s+1} - A_{r+1,s} + A_{r,s+1} - A_{r,s}\right) \qquad \ldots(7)$$

$$A_X \mid_{(2r+1)/2,\ (2s+1)/2} = \frac{1}{2h}\left(A_{r+1,s+1} - A_{r,s+1} + A_{r+1,s} - A_{r,s}\right) \qquad \ldots(8)$$

$$VA_X \mid_{(2r+1)/2,\ (2s+1)/2} = \frac{1}{2h} V_{r,s+1}\left(A_{r+1,s+1} - A_{r,s+1}\right) + \frac{1}{2h} V_{r+1,s}\left(A_{r+1,s} - A_{r,s}\right) \qquad \ldots(9)$$

and

$$AV_X \mid_{(2r+1)/2,\ (2s+1)/2} = \frac{1}{2h} A_{r,s+1}\left(V_{r+1,s+1} - V_{r,s+1}\right) + \frac{1}{2h} A_{r+1,s}\left(V_{r+1,s} - V_{r,s}\right). \qquad \ldots(10)$$

Substituting eqns. (7) - (10) in eqn. (5), the resulting algebraic system of equations takes the form

$$\frac{1}{2k}\left(A_{r+1} - C_{r+1} + A_r - C_r\right) + \frac{\Upsilon - 1}{4h}\left(W_{r+1} - W_r + Z_{r+1} - Z_r\right)$$
$$+ \frac{1}{2h} W_r\left(B_{r+1} - B_r\right) + \frac{1}{2h} Z_{r+1}\left(C_{r+1} - C_r\right)$$
$$+ \frac{\Upsilon - 1}{4h} B_r\left(W_{r+1} - W_r\right) + \frac{\Upsilon - 1}{4h} C_{r+1}\left(Z_{r+1} - Z_r\right) = 0;$$
$$r = 0, 1, 2, \ldots, N-1;\ s = 0, 1, 2, \ldots;\ n = 0, 1, 2, \ldots \qquad \ldots(11)$$

where

$$A^{(n+1)}_{r+1,s+1} \equiv A_{r+1},\ A^{(n)}_{r,s+1} \equiv B_r,\ A^{(n_s)}_{r,s} \equiv C_r \qquad \ldots(12)$$

$$V^{(n+1)}_{r+1,s+1} \equiv V_{r+1},\ V^{(n)}_{r,s+1} \equiv W_r,\ V^{(n_s)}_{r,s} \equiv Z_r \qquad \ldots(13)$$

and the superscript n_s denotes the final number of iterations required to obtain an acceptable approximation to the value of $A_{r,s}$ at the grid points on the line $T = sk$ subject to the criterion

$$\max_r \left| A_{r,s}^{(n+1)} - A_{r,s}^{(n)} \right| \leqslant 10^{-5}, \quad 1 \leqslant r \leqslant N-1. \qquad \ldots(14)$$

The difference scheme (11) can be written in a simpler form as follows:

$$A_r = kF - A_{r+1}, \quad r = N-1, N-2, \ldots, 1, 0 \qquad \ldots(15)$$

where

$$F = \frac{1}{k}(C_{r+1} + C_r) - \frac{\gamma - 1}{2h}(W_{r+1} - W_r + Z_{r+1} - Z_r)$$

$$- \frac{1}{h} W_r(B_{r+1} - B_r) - \frac{1}{h} Z_{r+1}(C_{r+1} - C_r)$$

$$+ \frac{\gamma - 1}{2h} B_r(W_{r+1} - W_r) - \frac{\gamma - 1}{2h} C_{r+1}(Z_{r+1} - Z_r). \qquad \ldots(16)$$

For eqn. (6), we use the following finite difference approximations:

$$V_{XX}\,|_{\,r,(2s+1)/2} = \frac{1}{2h^2}(V_{r+1,s+1} - 2V_{r,s+1} + V_{r-1,s+1} + V_{r+1,s}$$

$$- 2V_{r,s} + V_{r-1,s}) \qquad \ldots(17)$$

$$V_T\,|_{\,r,(2s+1)/2} = \frac{1}{k}(V_{r,s+1} - V_{r,s}) \qquad \ldots(18)$$

$$V_X\,|_{\,r,(2s+)1/2} = \frac{1}{4h}(V_{r+1,s+1} - V_{r-1,s+1} + V_{r+1,s} - V_{r-1,s}) \qquad \ldots(19)$$

$$VV_X\,|_{\,r,(2s+1)/2} = \frac{1}{4h} V_{r,s+1}(V_{r+1,s+1} - V_{r-1,s+1})$$

$$+ \frac{1}{4h} V_{r,s}(V_{r+1,s} - V_{r-1,s}) \qquad \ldots(20)$$

$$AA_X\,|_{\,r,(2s+1)/2} = \frac{1}{4h} A_{r,s+1}(A_{r+1,s+1} - A_{r-1,s+1})$$

$$+ \frac{1}{4h} A_{r,s}(A_{r+1,s} - A_{r-1,s}). \qquad \ldots(21)$$

Substituting eqns. (17) – (21) in eqn. (6), the resulting algebraic system of equations takes the form

$$\frac{1}{k}(V_r - Z_r) + \frac{1}{2h(1-\gamma)}(A_{r+1} - A_{r-1} + C_{r+1} - C_{r-1})$$
$$+ \frac{1}{4h} W_r(W_{r+1} - W_{r-1}) + \frac{1}{4h} Z_r(Z_{r+1} - Z_{r-1})$$
$$+ \frac{1}{2h(1-\gamma)} A_r(A_{r+1} - A_{r-1}) + \frac{1}{2h(1-\gamma)} C_r(C_{r+1} - C_{r-1})$$
$$= \frac{1}{2h^2}(V_{r+1} - 2V_r + V_{r-1} + Z_{r+1} - 2Z_r + Z_{r-1});$$
$$r = 1, 2, ..., N-1; \; s = 0, 1, 2, ...; \; n = 0, 1, 2, ... \qquad ...(22)$$

The difference scheme (21) can be written as

$$-\alpha V_{r+1} + \beta V_r - \eta V_{r-1} = G, \; r = 1, 2, ..., N-1, \qquad ...(23)$$

where

$$\alpha = \frac{1}{2h^2}, \quad \beta = \frac{1}{h^2} + \frac{1}{k}, \quad \eta = \frac{1}{2h^2} \qquad ...(24)$$

and

$$G = \frac{1}{k} Z_r - \frac{1}{2h(1-\gamma)}(A_{r+1} - A_{r-1} + C_{r+1} - C_{r-1})$$
$$- \frac{1}{4h} W_r(W_{r+1} - W_{r-1}) - \frac{1}{4h} Z_r(Z_{r+1} - Z_{r-1})$$
$$- \frac{1}{4h} Z_r(Z_{r+1} - Z_{r-1}) - \frac{1}{2h(1-\gamma)} A_r(A_{r+1} - A_{r-1})$$
$$- \frac{1}{2h(1-\gamma)} C_r(C_{r+1} - C_{r-1}) + \frac{1}{2h^2}(Z_{r+1} - 2Z_r + Z_{r-1}). \qquad ...(25)$$

The local truncation error of schemes (11) and (22) is found to be $O(k^2 + h^2)$.

4. Numerical Method

The initial condition on V is

$$V(X, 0) = f(X), \; X \in [0, L]. \qquad ...(26)$$

Equation for the local acoustic speed in the absence of the diffusion is

$$a = a_0 + \tfrac{1}{2}(\gamma - 1)\, v. \qquad ...(27)$$

Its non-dimensional form is

$$A(X, T) = \tfrac{1}{2}(\gamma - 1)\, V(X, T). \qquad ...(28)$$

The initial condition on A is given by

$$A(X, 0) = \tfrac{1}{2}(\gamma - 1) f(X), \quad X \in [0, L]. \qquad ...(29)$$

Given the boundary conditions $V(0, T)$ and $V(L, T)$; $A(L, T)$ is computed by using eqn. (28).

The numerical method for solving eqns. (15) and (23) subject to the given initial and boundary conditions is as follows:

Put $B_r = C_r$ and $W_r = Z_r$ for all r. Calculate A_r from eqn. (15) by the backward sweep method by using $A_N \equiv A(L, T)$. Using these values of A_r, calculate the values of V_r from eqn. (23) by solving the tri-diagonal system of equations.

Take $B_r = A_r$, $W_r = V_r$ and find the improved values of A_r by using eqn. (15). By using the improved values of A_r, calculate improved values of V_r. Repeat the iterative procedure till the successive approximations to A_r and V_r differ within the prescribed tolerance error 10^{-5}. Repeat the procedure at further times. We have solved the tri-diagonal system of linear equations at each time step by Thomas algorithm (Ames 1977), thereby eliminating the matrix operations.

At each time level, we use Shuman operator (Vliegenthart 1970):

$$\bar{A}_r = \tfrac{1}{4}(A_{r+1} + 2A_r + A_{r-1}), \quad r = 1, 2, \ldots, N-1, \qquad \ldots(30)$$

and

$$\bar{V}_r = \tfrac{1}{4}(V_{r+1} + 2V_r + V_{r-1}), \quad r = 1, 2, \ldots, N-1. \qquad \ldots(31)$$

It results in removing the overshooting of the shock front and the nonlinear instability in the scheme. We solve test examples to illustrate this numerical method.

5. Numerical Examples and Results

The Asymptotic Shock

We study the asymptotic shock in the absence of disturbance to determine its thickness and the shape. There is no precise definition of the thickness of the shock, but one can introduce various measures of the scale, such as the length over which 90% of the change occurs (Whitham 1974). Lighthill (1956) defined the shock wave thickness as the distance h_1 in which v changes from $0.05\ v_0$ to $0.95\ v_0$, where v_0 is the particle velocity on the compressive side of the shock. Then the shock thickness is given by

$$h_1 = \frac{12\delta}{(1+\gamma)\, v_0}. \qquad \ldots(32)$$

The non-dimensional shock thickness is

$$H = \frac{12}{(1+\gamma)\, V_0} \qquad \ldots(33)$$

where V_0 is the non-dimensional particle velocity corresponding to v_0. Equation (33) shows that a shock thickness of approximately 50 units in X may be expected for a value of V_0 of 0.1. We have used Lighthill's definition of the shock thickness for the computational work.

Example 1 — Consider the difference equations (15) and (23) with the initial conditions

$$\begin{aligned} V_{r,0} &= 0.1, \quad 0 \leqslant r \leqslant N_1 \\ &= 0.1\,\frac{r - N_2}{N_1 - N_2}, \quad N_1 < r < N_2 \\ &= 0, \quad N_2 \leqslant r \leqslant N \end{aligned} \qquad \ldots(34)$$

where $N_1 < N_2 < N$, and

$$A_{r,0} = \tfrac{1}{2}(\gamma - 1)\, V_{r,0}. \qquad \ldots(35)$$

The boundary conditions are

$$V_{0,s} = 0.1, \quad V_{N,s} = 0 \qquad \ldots(36)$$

and

$$A_{N,s} = 0. \qquad \ldots(37)$$

For the computational work, we have taken $N = 400$, $h = 2$ and $k = 0.1$; these values have been chosen after doing some numerical experiments.

The results in Fig. 1 show that the shock becomes thicker due to the diffusion effect up to a certain time. Then the non-linearities cause the shock to steepen and the shock thickness decreases. This is in conformity with the result that the two opposing effects of thickening and steepening operate at different rates; the variation due to the diffusion depends on the square root of the time while that due to the nonlinear steepening depends directly on time. Thus, for small intervals of time, the shock thickening effect will dominate the shock steepening effect, so that the shock will become progressively thicker. Different values of N_1 and N_2 have been used corresponding to the initial shock thickness H equal to 45, 50.4 and 54 units in X, respectively. In all these cases, the same asymptotic shock structure has been obtained irrespective of the initial shock forms which may be 'too steep' or 'too thick'. These results confirm that the asymptotic shape of the shock is independent of the initial shock form. A typical shape of the asymptotic shock is shown in Fig. 2. The numerical results are in conformity with the results obtained by Beasley (1974). It may be remarked that the numerical method used by Beasley involved the solution of nonconstant tri-diagonal system of equations at each time level.

Example 2 — We present another case to show that the asymptotic shape of the shock is independent of the initial shock form. We take the initial shock form given by

$$V_{r,0} = 0.05\left(1 - \tanh\frac{rh}{d}\right),\ 0 \leqslant r \leqslant N \qquad \ldots(38)$$

where d is a constant specifying the initial shock thickness.

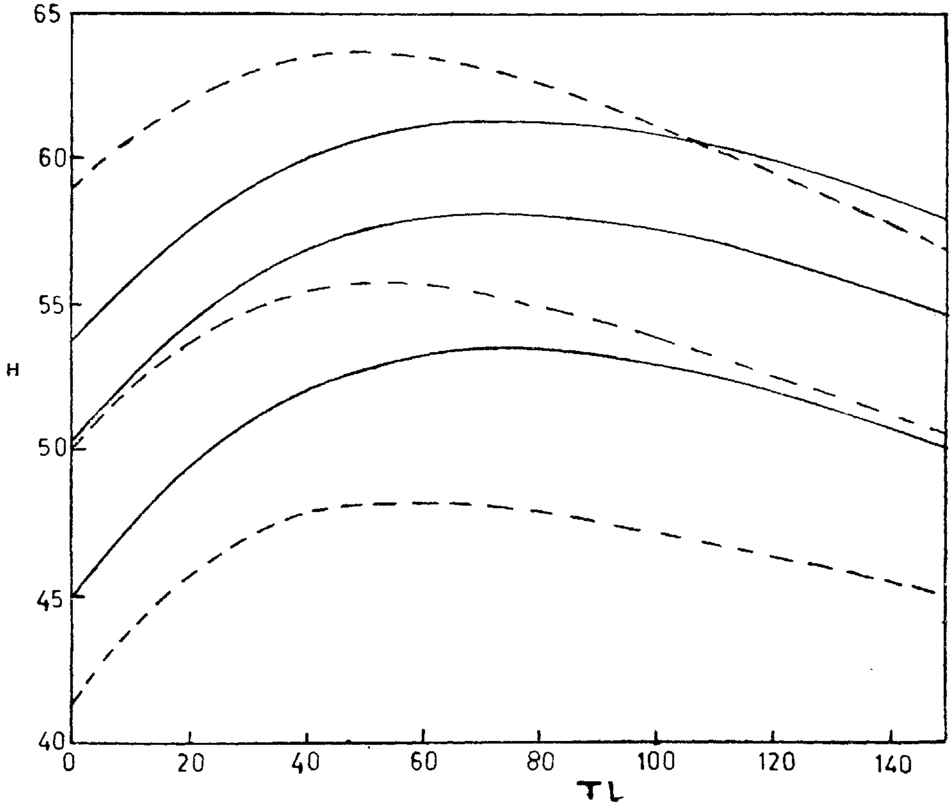

FIG. 1. H non-dimensional shock thickness, TL time level, —— Numerical solution of Example 1, - - - - Numerical solution of Example 2.

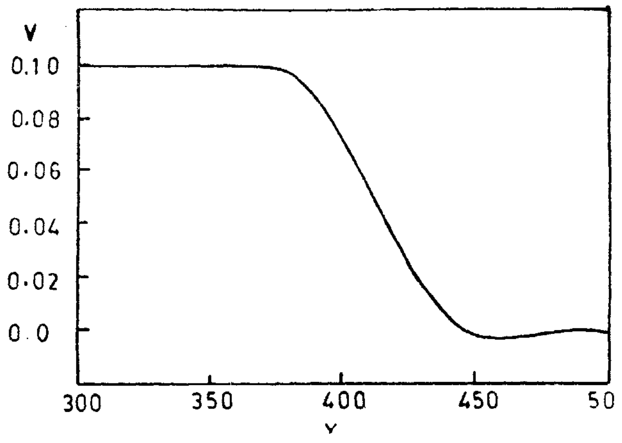

FIG. 2. Typical asymptotic shock front form.

The boundary conditions are given by eqns. (36) and (37). We have used d equal to 14, 17 and 20 corresponding to the initial shock thickness of 41.2, 50.1 and 59 units in X, respectively. In all these cases, the same asymptotic shock form has been obtained. The results in Fig. 1 show that the diffusion effects are now effective for shorter time; the steepening effects take place much earlier in these cases as compared to those in Example 1. This phenomenon is due to the fact that the initial shock shape for this example is close to the asymptotic shock form.

5.2 *Interaction between Shock Wave and Disturbance Wave*

In the following example, we investigate the effect of the interaction of a disturbance wave upon a shock having a strong tendency to preserve its asymptotic form.

Example 3 — Consider the difference eqns. (15) and (23) with the initial conditions (34) and (35). The boundary conditions are

$$\left.\begin{aligned} V_{0,s} &= 0.1 \\ V_{N,s} &= q[1 - \cos(p\,\pi\,t)] \\ \text{and} \quad A_{N,s} &= 0.5(\gamma - 1)\, V_{N,s} \end{aligned}\right\} \qquad \ldots(39)$$

where q and p are real parameters.

Because of the numerical constraints, a single disturbance wave has been used to demonstrate the nonlinear interaction between the shock wave and the disturbance wave.

Case I — We take $q = 0.6$ and $p = 0.8$. To calculate the increase in the shock thickness H due to the interaction of the disturbance wave with the shock wave, we require the value of H corresponding to the instant T at which the interaction of the two waves is just over. We compare this value of H with the shock thickness obtained in the calculation of the asymptotic shock starting from the same initial shape and at the same T in the absence of the disturbance wave. It is found that the presence of the disturbance wave increases the thickness of the shock by 3.67 units in X. On continuing the computations beyond time T, the numerical results show that the effect of the disturbance wave on the shock wave almost vanishes and the shock wave turns back to its asymptotic form with a few ripples on both sides of the shock.

Case II — In this case, we have reduced the amplitude of the disturbance wave to half compared to that of Case I by taking $q = 0.3$ and $p = 0.8$.

The final structure of the shock obtained in Case I is repeated in this case also, but the shock thickness is now increased by 1.29 units in X.

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