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ON BOUNDARY LAYER FLOW OF A POWER LAW LIQUID PAST A WEDGE*

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A boundary layer flow of power law liquid past a symmetrical wedge is studied in the neighbourhood of the stagnation point. Since the potential flow velocity in this region is proportional to the arc length raised to a power, the similarity transformation is successfully applied. The asymptotic, two-point boundary value problem, thus obtained, is further reduced to an initial value problem by applying the Nachtsheim-Swigert iterative scheme, which later invokes the fourth-order Runge-Kutta scheme for its solution. The FORTRAN program for the problem was run over Burroughs 6700 computer for different values of the power law index n. The expressions for the local non-dimensional skin friction coefficient, displacement thickness, momentum thickness and kinetic energy thickness are also found out.

STATEMENT AND FORMULATION OF THE PROBLEM

A boundary layer flow of a power law liquid past a wedge, whose included angle is $\pi\gamma$ as shown in Fig. 1, is a generalization of the like problem for ordinary viscous liquid. To construct the system of equations let the origin be taken at the stagnation point, the x-axis along the wall while the y-axis perpendicular to it. Thus we have

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + k\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)^n \qquad \ldots(1)$$

$$\frac{\partial p}{\partial y} = 0 \qquad \ldots(2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \ldots(3)$$

the boundary conditions being

$$\left.\begin{aligned} u = v = 0;\ y = 0 \\ u = U(x)\ ;\ y = \infty \end{aligned}\right\} \qquad \ldots(4)$$

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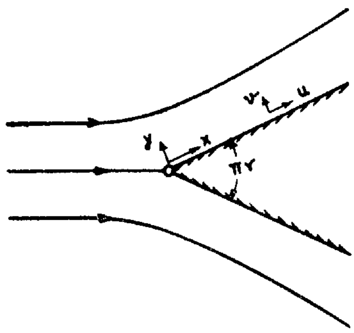

FIG. 1. Schematic representation of a flow past a wedge.

where k is the kinematic viscosity for the power law liquid, n is the power law index and $U(x)$ is the local potential flow velocity.

Here we can assume

$$U\frac{dU}{dx} = -\frac{1}{\rho}\frac{dp}{dx}$$

which reduces eqn. (1) to

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + k\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)^n. \qquad \ldots(5)$$

Again introducing the stream function ψ which satisfies the continuity equation automatically we get from eqn. (5)

$$\psi_y\psi_{yx} - \psi_x\psi_{yy} = U\frac{dU}{dx} + k\frac{\partial}{\partial y}(\psi_{yy})^n \qquad \ldots(6)$$

with the boundary conditions

$$\left.\begin{aligned} \psi_y &= \psi_x = 0; \; y = 0 \\ \psi_y &= U(x) \quad ; \; y = \infty. \end{aligned}\right\} \qquad \ldots(7)$$

In the neighbourhood of the stagnation point the potential flow velocity U may be written as

$$U = cx^m, \; m \geqslant 0 \quad \text{with} \quad m = \frac{\gamma}{2-\gamma}$$

where x is the arc length measured from the stagnation point.

Hence application of the similarity transformation with the following dimensionless variables

$$f(\eta) = \frac{\psi(x, y)}{\left(\frac{n+1}{(2n-1)\,m+1}\right)^{1/(n+1)} c^{(2n-1)/(n+1)} K^{1/(n+1)} x^{((2n-1)m+1)/(n+1)}}$$

$$\eta = \frac{y\left(\frac{(2n-1)\,m+1}{n+1}\right)^{1/(n+1)} c^{(2-n)/(n+1)} x^{((2-n)m-1)/(n+1)}}{k^{1/(n+1)}}$$

turns eqn. (6) to

$$n(f'')^{n-1} f''' + ff'' + \beta(1 - f'^2) = 0 \qquad \text{...(8)}$$

where $\beta = \frac{m(n+1)}{(2n-1)\,m+1}$. The dash denotes the differentiation with respect to η.

The corresponding boundary conditions (7) assume the forms

$$f' = f = 0;\ \eta = 0 \quad \text{and} \quad f' \to 1;\ \eta \to \infty. \qquad \text{...(9)}$$

For $n = 1$ eqn. (8) and its associated boundary conditions lead to Falkner-Skan equation (see Schlichting 1968).

Method of Solution

Equation (8) along with the boundary conditions (9) form an asymptotic two-point boundary value problem. Asymptotic convergence $f' \to 1.0$ as $\eta \to \infty$ is taken to be equivalent to the conditions $f'' = 0$ as well as $f' = 1.0$ at $\eta = \infty$. A Nachtsheim-Swigert iterative scheme (see Nachtsheim and Swigert 1965) has been used to solve this problem. The procedure is to estimate the unknown value of $f''(0)$ by iteration satisfying the conditions,

$$\left.\begin{aligned} f'(\eta_{max}) &= 1 + \delta_1 \\ f''(\eta_{max}) &= \delta_2 \end{aligned}\right\} \qquad \text{...(10)}$$

where δ_1 and δ_2 are very small quantities and η_{max} is the numerical equivalent to infinity as specified in the boundary conditions.

For a preassigned value of η_{max} if the initial choice of $f''(0)$ is within the range of convergence the value of $f''(0)$ can be improved by iteration using the relation

$$\triangle f''(0) = \frac{f'_x (1 - f'_c) - f''_x f''_c}{f'^2_x + f''^2_x} \qquad \text{...(11)}$$

where f'_c = the calculated value of $f'(\eta_{max})$ for assumed value of $f''(0)$.

f'_x = the change rate of $f'(\eta_{max})$ with respect to $f''(0)$, determined by the relation

$$f'_x = \frac{f'_2(\eta_{max}) - f'_1(\eta_{max})}{f''_2(0) - f''_1(0)}$$

$f'_1(\eta_{max})$ = the value of $f'(\eta_{max})$ obtained by integration of eqn. (8) with $f''_1(0)$, an assumed value of $f''(0)$.

$f'_2(\eta_{max})$ = the value of $f'(\eta_{max})$ obtained by integration of eqn. (8) with $f''_2(0)$, a changed value of $f''(0)$, $(f''_2(0) = f''_1(0) + \epsilon)$.

Similarly f''_c and f''_x are also defined.

To integrate the third order nonlinear ordinary differential eqn. (8), it is first resolved into three equivalent first order equations

$$\frac{df}{d\eta} = f', \frac{df'}{d\eta} = f'' \text{ and } \frac{df''}{d\eta} = -(ff'' + \beta(1 - f'^2))/n(f'')^{n-1} \quad \text{...(12)}$$

and then the fourth-order Runge-Kutta formulae (Adams and Rogers 1973) are used to integrate eqns. (12) over $(0, \eta_{max})$, divided into a number of subintervals of equal length.

Numerical Results

A Fortran program was made with the input data given in Table I, and carried over a Burroughs 6700 electronic computer.

Table I

n	Initial estimate of $f''(0)$	Initial estimate of η_{max}
0.8	0.80	2.0
0.9	0.70	2.0
1.0	0.60	5.0
1.1	0.71	5.0
1.2	0.77	5.0

The values of γ, δ_1 and δ_2 were 0.25, 5×10^{-6} and 5×10^{-6} respectively.

For the sake of brevity the output for the power law indexes $n = 0.9$ and 1.1 only are given in Table II and Table III respectively. The same for $n = 1$ are given in Adams and Rogers (1973).

TABLE II

η	f''	f'	f
0.0	0.797188	0.000000	0.000000
0.3	0.712209	0.226540	0.034617
0.6	0.619059	0.426491	0.133270
0.9	0.515212	0.596888	0.287556
1.2	0.403787	0.734843	0.488152
1.5	0.292867	0.839200	0.725092
1.8	0.192939	0.911668	0.988473
2.1	0.113035	0.956984	1.269371
2.4	0.057394	0.981936	1.560626
2.7	0.024912	0.993701	1.857218
3.0	0.008285	0.998273	2.156135
3.3	0.002081	0.999659	2.455870
3.6	0.000341	0.999959	2.755826
3.9	0.000029	1.000000	3.055822
4.2	0.000001	1.000003	3.355823
4.5	0.000000	1.000003	3.655824

Results after each 30 subintervals.

SOME RELEVANT FORMULATIONS

The local non-dimensional skin-friction coefficient c_{fx} is

$$c_{fx} = \frac{\tau(x, 0)}{\frac{1}{2}\rho U^2}$$

$$= 2\left(\frac{(2n-1)\,m+1}{n+1}\right)^{n/(n+1)} \mathrm{Re}_x^{-1/(1+n)}\,(f''(0))^n$$

where Re_x is the local Reynolds number, defined by

$$\mathrm{Re}_x = \frac{U^{2-n}x^n}{k}.$$

The displacement thickness δ^* is

$$\delta^* = \int_0^{y_{max}} \left(1 - \frac{u}{U}\right) dy$$

$$= \left[\frac{\left(\frac{n+1}{(2n-1)\,m+1}\right) kx}{U^{2-n}}\right]^{1/(n+1)} \int_0^{\eta_{max}} (1 - f')\, d\eta. \qquad \text{...(13)}$$

TABLE III

η	f''	f'	f
0.0	0.668183	0.000000	0.000000
0.4	0.577690	0.249292	0.051063
0.8	0.481341	0.461334	0.194473
1.2	0.379554	0.633609	0.414821
1.6	0.279326	0.765173	0.695916
2.0	0.190274	0.858572	1.021854
2.4	0.119725	0.919885	1.378487
2.8	0.069834	0.957129	1.754554
3.2	0.038052	0.978177	2.142038
3.6	0.019568	0.989343	2.535788
4.0	0.009603	0.994962	2.932781
4.4	0.004547	0.997675	3.331376
4.8	0.002099	0.998944	3.730732
5.2	0.000952	0.999525	4.130441
5.6	0.000428	0.999787	4.530311
6.0	0.000192	0.999905	4.930252
6.4	0.000086	0.999958	5.330227
6.8	0.000039	0.999982	5.730216
7.2	0.000018	0.999993	6.130211
7.6	0.000008	0.999998	6.530210
8.0	0.000004	1.000001	6.930210

Results after each 40 subintervals.

The momentum thickness θ is

$$\theta = \int_0^{y_{max}} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

$$= \left[\frac{\left(\frac{n+1}{(2n-1)\,m+1}\right)kx}{U^{2-n}}\right]^{1/(n+1)} \int_0^{\eta_{max}} f'(1-f')\, d\eta \qquad \ldots(14)$$

And the kinetic energy thickness δ^{**} is

$$\delta^{**} = \int_0^{y_{max}} \frac{u}{U}\left(1 - \frac{u^2}{U^2}\right) dy$$

(*equation continued on p.* 1546)

$$= \left[\frac{\left(\dfrac{n+1}{(2n-1)\,m+1} \right) kx}{U^{2-n}} \right]^{1/(n+1)} \int_{0}^{\eta_{max}} f'(1 - f'^2)\, d\eta. \qquad \ldots(15)$$

The integrals on the right-hand sides of eqns. (13) to (15) can be easily solved by Simpson's $\frac{1}{3}$ rule.

Discussion

By trial and error an initial estimate of the unknown $f''(0)$ is made for a pre-assigned η_{max}. Following the *R-K* scheme eqns. (12) are now integrated using the given values $f(0)$ and $f'(0)$, together with this initial estimate of $f''(0)$. The values of $f'(\eta_{max})$ and $f''(\eta_{max})$, thus obtained, are compared with the desired approximations (10). If they do not tally we increase the value of $f''(0)$ by $\epsilon (= 0.001)$ and again calculate and compare their values with (10). This time also if we fail to achieve the desired approximations, value of $f''(0)$ is improved by *N-S* iteration using the relation (11) until the convergence is assured.

If this initial estimate is not within the range of convergence, the attempted solution will diverge. A procedure which helps a divergent problem is to choose a value of η_{max} which is smaller than the desired value and get the corresponding improved value of $f''(0)$ following *N-S* iteration. Now this improved value of $f''(0)$ is used for further *N-S* iteration with increased η_{max}, and so on.

Figure 2 shows the deviation of the velocity profile $f'(\eta)$ for varying power law index n. Here as n increases $f'(\eta)$ assumes the value 1.0, i.e. the velocity component u attains the outer flow velocity U, at an increasing η_{max}. Thus we see that the boundary layer thickness for the considered type of flow of power law fluid differs from that of the ordinary viscous fluid (in case $n = 1.0$) and this thickness is either greater or smaller according as $n >$ or < 1.0.

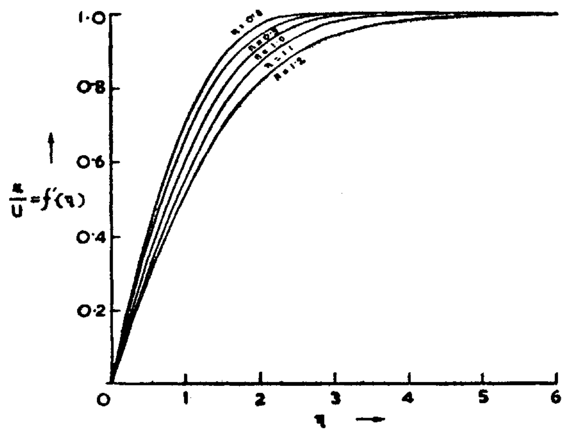

FIG. 2. Velocity profiles for $n = 0.8, 0.9, 1.0, 1.1$ and 1.2.

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