SOME NECESSARY CONDITIONS FOR ABSOLUTE MATRIX SUMMABILITY FACTORS

B. E. RHoades

Department of Mathematics, Indiana University, Bloomington, IN 47405-7106, U.S.A.

AND

EKREM SAVAŞ

Department of Mathematics, Yüzüncü Yıl University, Van, Turkey

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We obtain necessary conditions for a lower triangular matrix to have the property that \( \sum a_n \Lambda_n \) is summable \( \left\| T \right\|_k \) whenever the series \( \sum a_n \) is bounded \( \left\| T \right\|_k \).

Key Words: Absolute Matrix Summability; Lower Triangular Matrix

The concept of absolute summability of order \( k \) was defined by Flett as follows. Let \( \sum a_n \) be a given infinite series with partial sums \( s_n \) and let \( \sigma_n^\alpha \) denote the \( n \)th Cesáro means of order \( \alpha, \alpha > -1 \), of the sequence \( \{s_n\} \). The series \( \sum a_n \) is said to be summable \( \left\| C, \alpha \right\|_k, k \geq 1, \alpha > -1 \) if

\[
\sum_{n=1}^{\infty} n^{k-1} \left| \Delta \sigma_{n-1}^\alpha \right|^k < \infty, \tag{1}
\]

where, for any sequence \( \{b_n\} \), \( \Delta b_n := b_n - b_{n+1} \).

In defining absolute summability of order \( k \) for weighted mean methods Bor and others used the definition

\[
\sum_{n=1}^{\infty} \left( \frac{p_n}{p_n} \right)^{k-1} \left| \Delta u_{n-1} \right|^k < \infty, \tag{2}
\]

where \( u_n := \sum_{v=0}^{n} p_v s_v \).
In using (2) as the definition, it was apparently assumed that the $n$ in (1) represented the reciprocal of the $n$th main diagonal term of $(C, 1)$. But this interpretation cannot be correct. For, if it were, then the Cesáro methods $(C, \alpha)$, for $\alpha \neq 1$ would have to satisfy the condition

$$\sum_{n=1}^{\infty} |a_n^{\alpha} - 1| A_{n-1}^\alpha |^k < \infty.$$ 

However, Flett\textsuperscript{4} stays with $n$ for all values of $\alpha > -1$.

Let $A$ be a triangle; i.e., $A$ is a lower triangular matrix with nonzero diagonal entries. Let

$$A_n = \sum_{\nu=0}^{n} a_{n \nu} s_{\nu}$$

As noted in\textsuperscript{7}, we shall say that a series $\Sigma a_n$ is summable $|A|_k$, $k \geq 1$ if

$$\sum_{n=1}^{\infty} n^{k-1} \sum_{\nu=0}^{n} |A_{n-1}|^k < \infty. \quad \ldots \quad (3)$$

There is a fairly large literature dealing with questions related to absolute summability of order $k \geq 1$. For example, using (3), Bor\textsuperscript{2} obtained necessary and sufficient conditions for $|\bar{N}, p_n|_k$ to imply $|\bar{N}, q_n|_k$, and Orhan and Cakar\textsuperscript{6} obtained necessary and sufficient conditions for $|\bar{N}, p_n|_k$ to imply $|\bar{N}, q_n|_k$, where $A$ is a lower triangular matrix. However, many of the papers use (2). See also [5] and [8]. For additional references the interested reader is encouraged to consult Math. Reviews, or the AMS Math Sci Net Search engine on the internet.

Recently, Bor and Kuttner\textsuperscript{3} obtained necessary conditions for absolute weighted mean summability factors using (2).

In this paper we use (3) to obtain a generalization of the Theorem of\textsuperscript{3}.

Let $T$ be a triangle with positive entries and row sums one. Let

$$\bar{t}_{n \nu} := \sum_{i=\nu}^{n} t_{ni} \quad \text{and} \quad \hat{t}_{n \nu} := t_{n \nu} - \hat{t}_{n-1, \nu}$$

**Theorem 1** — Let $\Sigma a_n$ be bounded $|T|_k$, $k \geq 1$. If $\Sigma a_n \lambda_n$ is summable $|T|_k$, then the following conditions are necessary:

(i) $\lambda_{\nu} = O((\nu t_{\nu \nu})^{1/k - 1})$,

(ii) $\sum_{n=\nu+1}^{\infty} n^{k-1}|\bar{t}_{n \nu} \lambda_{\nu}|^k = O(1)$,

(iii) $\sum_{n=\nu+1}^{\infty} n^{k-1}|\Delta(\hat{t}_{n \nu} \lambda_{\nu})|^k = O(t_{\nu \nu})$. 

The phrase \( \Sigma a_n \) is bounded \( \| T_n \|_k \) means

\[
\sum_{v=1}^{\infty} t_{n,v} s_{n,v}^k = O(1).
\]

... (4)

**Proof**: Writing

\[
T_n = \sum_{v=0}^{n} \tilde{t}_{n,v} a_v \lambda_v
\]

we define

\[
Y_n = T_n - T_{n-1} = \sum_{v=1}^{n} \tilde{t}_{n,v} a_v \lambda_v, n \geq 1, Y_0 = 0, a_0.
\]

... (5)

Since the series \( \Sigma \lambda_v a_n \) is summable \( \| T_n \|_k, k \geq 1 \), we have

\[
\sum_{n=1}^{\infty} n^{k-1} |Y_n|^k < \infty.
\]

... (6)

The space of sequences \( \{ Y_n \} \) satisfying (6) is a Banach space, if normed by

\[
\| Y \| = \left( \sum_{n=1}^{\infty} n^{k-1} |Y_n|^k \right)^{1/k}.
\]

... (7)

The space of sequences \( \{ s_n \} \) satisfying (4) is also a Banach space with norm

\[
\| s \| = \sup_{n \geq 0} \left( \sum_{v=0}^{n} t_{n,v} s_{n,v}^k \right)^{1/k}.
\]

... (8)

Note that (5) transforms the space of sequences satisfying (4) into the space of sequences satisfying (6). Applying the Banach-Steinhaus theorem, there exists a positive constant \( M \) such that

\[
\| Y \| \leq M \| s \|
\]

... (9)

for all sequences satisfying (4).

For any fixed \( v \geq 1 \) we apply (9) with

\[
a_v = 1, a_{v+1} = -1, a_n = 0, n \neq v, v + 1.
\]

Then

\[
s_v = 1 \) and \( s_n = 0, n \neq v.
\]
\[
\sum_{v=0}^{n} t_{n,v} \hat{s}_v k = \begin{cases} 
0, & \text{if } n < v \\
t_{v,v} & \text{if } n \geq v
\end{cases}
\]

and
\[
Y_n = \begin{cases} 
0, & \text{if } n < v, \\
\hat{t}_{v,v} \hat{\lambda}_v & \text{if } n = v, \\
\Delta (\hat{t}_{n,v} \hat{\lambda}_v) & \text{if } m \geq v.
\end{cases}
\]

Noting that \(\hat{t}_{v,v} = t_{v,v}\) it follows from (9) that
\[
\left( t_{v,v}^{k-1} t_{v,v} \lambda_v^k + \sum_{n = v+1}^{\infty} n^{k-1} \Delta (\hat{t}_{n,v} \lambda_v)^k \right)^{1/k} = O (t_{v,v}^{1/k}).
\]

Since the sum is \(O (t_{v,v})\), each term must be \(O (t_{v,v})\). We then have
\[
t_{v,v}^{k-1} t_{v,v} \lambda_v^k = O (t_{v,v}) \quad \text{... (11)}
\]

and
\[
\sum_{n = v+1}^{\infty} n^{k-1} \Delta (\hat{t}_{n,v} \lambda_v)^k = O (t_{v,v}) \quad \text{... (12)}
\]

Eq. (11) becomes
\[
(t_{v,v})^{k-1} t_{v,v} \lambda_v^k = O (1),
\]

which is equivalent to (i).

Condition (12) is (iii).

We now apply (9) with
\[
a_v = 1, a_n = 0, n \neq v. \quad \text{... (13)}
\]

\[
s_n = \begin{cases} 
0, & \text{if } n < v, \\
1, & \text{if } n \geq v.
\end{cases}
\]

Thus
\[
\sum_{v=0}^{n} t_{n,v} s_v k = \begin{cases} 
0, & \text{if } n < v, \\
\hat{t}_{n,v} \hat{\lambda}_v & \text{if } n \geq v.
\end{cases}
\]

Since \(T\) has row sums 1, \(\bar{t}_{n,v} \leq 1\) for each \(n \geq v\), and the limit is 1.

Therefore (8) implies that \(\| s \| = 1\).

Using (13) in (5) yields
\[
Y_n = \begin{cases} 
0, & \text{if } m < v, \\
t_{v,v} \lambda_v & \text{if } n = v, \\
\hat{t}_{n,v} \lambda_v & \text{if } n > v.
\end{cases}
\]
Substituting into (9) we have
\[ \left( \sum_{n = \nu}^{\infty} n^{k-1} \hat{t}_n \lambda_n^k \right)^{1/k} \leq M, \]
or, equivalently,
\[ \nu^{k-1} |t_{\nu} \lambda_\nu|^k + \sum_{n = \nu + 1}^{\infty} n^{k-1} \hat{t}_n \lambda_n^k = O(1). \] ... (14)

It then follows that
\[ \nu^{k-1} |t_{\nu} \lambda_\nu|^k = O(1) \] ... (15)
and
\[ \sum_{n = \nu + 1}^{\infty} n^{k-1} \hat{t}_n \lambda_n^k = O(1). \]

From (15),
\[ \nu^{k-1} |t_{\nu} \lambda_\nu|^k = (\nu t_{\nu})^{k-1} |t_{\nu} \lambda_\nu|^k \leq (\nu t_{\nu})^{k-1} |\lambda_\nu|^k = O(1) \]
from condition (i), so (15) is automatically satisfied. Condition (16) is (ii).

**Corollary 1** — Let \( \Sigma \lambda_n \) be bounded \( |N, p_n|_k \). If \( \Sigma \lambda_n a_n \) is summable \( |N, p_n|_k \), then the following conditions are necessary.

\( (i) \lambda_\nu = O\left(\left(\frac{p_{\nu}}{p_\nu}\right)^{1-1/k}\right) \)
\[ (ii) |\lambda_\nu p_{\nu-1}|^k \sum_{n = \nu + 1}^{\infty} n^{k-1} \left(\frac{p_n}{p_n p_{n-1}}\right)^k = O(1), \]
\[ (iii) |\Delta (p_{\nu-1}) \lambda_\nu|^k \sum_{n = \nu + 1}^{\infty} n^{k-1} \left(\frac{p_n}{p_n p_{n-1}}\right)^k = O\left(\frac{p_\nu}{p_\nu}\right). \]

**Proof**: With \( T = (N, p_n), t_{\nu} = \frac{p_\nu}{p_\nu} \), (i) follows immediately from condition (i) of Theorem 1.

From the definition of \( \hat{t}_{\nu} \) with \( t_{\nu} = \frac{p_\nu}{p_\nu} \),
\[ \hat{t}_{\nu} = \hat{t}_\nu - \hat{t}_{\nu-1}, \nu = \sum_{i = \nu}^{n} t_{ni} - \sum_{i = \nu - 1}^{n} t_{n-1, i} \]
\[
= \frac{1}{p_n} \sum_{i=v}^{n} p_i - \frac{1}{p_{n-1}} \sum_{i=v}^{n-1} p_i \\
= \frac{1}{p_n} (p_n - p_{v-1}) - \frac{1}{p_{n-1}} (p_{n-1} - p_{v-1}) \\
= -\frac{p_n p_{v-1}}{p_n p_{n-1}}.
\]

Substituting into conditions (ii) and (iii) of Theorem 1 yields (ii) and (iii) of Corollary 1.

**Corollary 2** — Let \( \Sigma a_n \) be bounded \( (C, 1)_{k} \). If \( \Sigma \lambda_n a_n \) is summable \( (C, 1)_{k} \), then the following conditions are necessary.

(i) \( \lambda_v = O(1) \),

(ii) \( \Delta \lambda_v = O(v^{-1/k}) \).

**Proof:** For \((C, 1)\), \( p_n = 1 \) for all \( n \). Condition (i) follows immediately from condition (i) of Corollary 1.

Condition (ii) of Corollary 1 becomes

\[
|v \lambda_v|^k \sum_{n = v+1}^{\infty} n^{k-1} \left( \frac{1}{(n+1)n} \right)^k = O(1).
\]  

... (17)

\[
\sum_{n = v+1}^{\infty} \frac{1}{n (n+1)^k} > \sum_{n = v+1}^{\infty} \frac{1}{(n+1)^{k+1}} \\
> \int_{v+1}^{\infty} \frac{1}{(x+1)^{k+1}} dx = \frac{1}{k (v+1)^k}. 
\]  

... (18)

Substituting into (17) yields \( |v \lambda_v|^k = O(1) \), which is equivalent to (i). Using (18), condition (iii) of Corollary 1 implies that

\[
\frac{|\Delta (v \lambda_v)|^k}{v^k} = O \left( \frac{1}{v} \right),
\]

which implies that \( \Delta (v \lambda_v) = O(v^{1-k}) \).

Thus \( v \Delta \lambda_v = \lambda_v - \Delta (v \lambda_v) = O(v^{1-1/k}) \)

or \( v \Delta \lambda_v = \lambda_v + O(v^{1-1/k}) = O(v^{1-1/k}) \).
Therefore $\Delta \lambda_v = O(v^{-1/k})$, which is condition (ii).

**Remark**: The following example shows that the conditions of Theorem 1 are not sufficient. Let $T$ be the identity matrix and set $k = 1$. Then condition (i) becomes $\lambda_v = O(1)$, so we shall set $\lambda_v = 1$ for all $v$. For $n > v$, $t_{n,v} = \bar{t}_{n,v} - \bar{t}_{n-1,v} = 1 - 1 = 0$, so conditions (ii) and (iii) are automatically satisfied. For the identity matrix, $T_n = t_{nn}$, so the condition $\sum a_n \lambda_n$ summable $|T|$ reduces to

$$
\sum_{n=1}^{\infty} |a_{n-1} - a_n| < \infty
$$

... (19)

The condition that $\sum a_n$ is bounded $|T|$, see (4), becomes $t_{nn} |s_{n-1}| = |s_n| = O(1)$. Now set $a_n = (-1)^n/(n + 1)$. Then $s_n = \log 2$ for large $n$, and $s_n = O(1)$. But

$$
\sum_{n=1}^{\infty} |a_{n-1} - a_n| = \sum_{n=1}^{\infty} \left( \frac{1}{n} + \frac{1}{n+1} \right) = \infty,
$$

contradicting (19).

**REFERENCES**