FAULT-TOLERANT FIXED ROUTING IN HYPERCUBE GENERALIZATIONS

ANNELI LANKINEN, JUHANI NIEMINEN, MATTI PELTOLA AND PASI RUOTSALAINEN

Department of Mathematics, University of Oulu, P. O. Box 4500, 90014, Oulu University, Finland
(E-mail address: anneli.lankinen@ee.oulu.fi; juhani.nieminen@oulu.fi; mpa@ee.oulu.fi; past.ruotsalainen@ee.oulu.fi)

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A new class of highly fault tolerant communication graphs is constructed. The class is based on hypercubes and a new graph product. It is proved that the communication graphs of the new class have as efficient shortest path routings as hypercubes.

Key Words: Hypercubes; Fault-Tolerance; Shortest Path Routing; Surviving Graph

Interconnection networks are usually modeled by graphs in which the points represent processors and the lines communication links. The message delivery system must find a route along which to send each message to its destination, where a route is a path from one point to another. The problem is greatly simplified if one chooses a route in advance for each source/destination pair and uses that route for all messages. Such choice of routes is called routing, and if the routing is computed only once for a given graph, considerable effort can be put into its computation.

For a graph $G = (V(G), X(G))$ a routing $\rho$ is a function which assigns to each pair $x, y \in V(G), x \neq y$, a fixed $x - y$ path. The routing $\rho$ is called a shortest path routing/geodetic routing, if its each path/route is a shortest path/geodesic in $G$. The route graph $R(G, \rho)$ of $G$ has the same point set as $G$, and two points $x$ and $y$ of $R(G, \rho)$ are adjacent, if there is an assigned $x - y$ route/path $\rho(x, y)$ in the routing $\rho$. A fault in $G$ is either a point or a line in $G$. Let $F$ be a set of faults in $G$. An $x - y$ route $\rho(x, y)$ is said to avoid $F$ if no fault is contained in it. Given a set $F$ of faults in $G$, the fault free routing $\rho/F$ is defined to be a reduction of $\rho$ to fault free routes. The fault free routing $\rho/F$ induces the surviving route graph $R(G, \rho)/F = (V/F, \text{dom}(\rho/F))$, where $V/F$ is the set of all nonfaulty points of $G$. Usually the routes of this paper are undirected, i.e. $\rho(x, y) = \rho(y, x)$ for each pair $x, y \in V(G), x \neq y$ but there is also a special routing $\lambda$ which is directed.

The efficiency of fault tolerance of a fixed routing $\rho$ in a graph $G$ is measured by the diameter $\text{diam} (R(G, \rho) / F)$ of the surviving graph $R(G, \rho)/F$. This concept was introduced by Dolev et al.\textsuperscript{2} We say that $\rho$ is an $(k, m)$-routing in $G$ if $\text{diam}(R(G, \rho)/F) \leq k$ for any set of faults $F$ with $|F| \leq m$. A communication graph is highly fault-tolerant, if there is an $(k, m)$-routing with great values of $m$ and small values of $k$. It has been shown\textsuperscript{2} that an arbitrary shortest path
routing/geoetic routing $\rho$ in an $n$-cube $Q_n$ is a $(3, n - 1)$-routing. Moreover, one can construct a special shortest path $(2, n - 1)$-routing $\lambda$ in $Q_n$ as done in the paper\textsuperscript{2}. As the numbers $(3, n - 1)$ and $(2, n - 1)$ show, $n$-cubes are good examples of highly fault-tolerant communication graphs. Other highly fault-tolerant graphs are constructed further in the papers\textsuperscript{1, 5} and recently in the paper\textsuperscript{6}. The purpose of this note is to present a new class of highly fault-tolerant graphs. We follow terms and notations introduced by Harary\textsuperscript{3}, where the reader is referred to in order to find the basic definitions not given in this paper.

Almost always a highly fault-tolerant graph looses its high fault tolerance when a point/line is removed/added and thus the construction of highly fault-tolerant graphs requires an idea of regrouping graphs. A good regrouping idea is the cartesian product of graphs, the good properties of which are proved in\textsuperscript{1}. Since by\textsuperscript{1, 2} each hypercube is a highly fault-tolerant graph, the newly constructed highly fault-tolerant graphs must be as fault tolerant as hypercubes. The purpose of this paper is to find a new broad class of highly fault-tolerant graphs, which constitutes an alternative for the class of hypercubes. The construction idea we shall use is called a dot product. As well known, $Q_{n+2} = Q_{n+1} \times K_2$ but one can easily see that $Q_{n+2} = Q_n \times C_4$ where $K_2$ is the complete graph of two points and $C_4$ is the cycle of four points. The product $Q_{n+2} = Q_n \times C_4$ is the idea we shall modify and which implies the dot product $Q_n \odot C_4$.

The dot product $Q_n \odot C_4$ contains four hypercubes $Q_n^i$, $i = 1, 2, 3, 4$ as (induced) sub-graphs. The points of a hypercube can be divided into two maximum independent pointsets $V_1$ and $V_2 : V(Q_n) = V_1 \cup V_2$. Let the point of $Q_n^i$, corresponding to a point $x \in Q_n$ be denoted by $x^i$. The lines joining the subgraphs $Q_n^i$, $i = 1, 2, 3, 4$ in the dot product $Q_n \odot C_4$ are constructed as follows:

The points of $V_1^1 \subset V(Q_n^1)$ are joined to the corresponding points of $V_1^2 \subset V(Q_n^2)$, the points of

![Diagram](https://via.placeholder.com/150)

**Fig. 1a.** $Q_2 \odot C_4$
$V_2^2 \subset V(Q_n^2)$ to the corresponding points of $V_2^3 \subset V(Q_n^3)$, the points of $V_1^3 \subset V(Q_n^3)$ to the corresponding points of $V_1^4 \subset V(Q_n^4)$, and, finally, the points of $V_2^4 \subset V(Q_n^4)$ are joined to the corresponding points of $V_2^1 \subset V(Q_n^1)$. By the definition, $Q_n \circ C_4$ contains $4 \times 2^n = 2^{n+2}$ points, $(n+1)2^{n+1}$ lines, and the degree $\text{deg}(x)$ of each point $x$ is $n + 1$. The $n$-cube $Q_{n+2}$ contains $2^{n+2}$ points, $(n+2)2^{n+1}$ lines, and the degree of each point $x$ is $\text{deg}(x) = n + 2$. The differences between numbers of lines and the degrees show that the hypercube $Q_{n+2}$ and the graph $Q_n \circ C_4$ are nonisomorphic graphs. Figure 1a illustrates the graph $Q_2 \circ C_4$ and Figure 1b the graph $Q_3 \circ C_4$.

![Diagram](image.png)

**Fig. 1b. $Q_3 \circ C_4$**

When considering faults in the graph $Q_n \circ C_4$ we try to use the results of Dolev et al.\(^2\) by projecting the new graph on the hypercube $Q_n$. Especially we project all the faults of $Q_n \circ C_4$ onto the hypercube denoted by $Q_n^0$ as follows:

**Definition 1** — When projecting the graph $Q_n \circ C_4$ onto the graph $Q_n$, the point $x^i$ is mapped onto the point $x^0$ such that

1) if the point $x^i$ is faulty, then denote the point $x^0 \in V(Q_n^0)$ faulty;
2) if the line $x^i u^j (x_i \neq u_j)$ is faulty, then denote the line $x^0 u^0 \in X(Q_n^0)$ faulty; and

3) if the line $x^i x^{i+1}$ is faulty, then denote the point $x^0 \in V(Q_n^0)$ faulty.

After projecting the faults onto $Q_n^0$, there are at most as many faults in $Q_n^0$ as there are in the graph $Q_n \odot C_4$. Denote by $C_1 (x, y)$ the set $\{z \mid z$ is a point or line on some shortest $x - y$ path/on some $x - y$ geodesic$\}$. The convex $C(x, y)$ is the point set $\{z \mid z$ is a point on the $u - v$ geodesic with $u, v \in C(x, y)\}$, and thus in each graph for each pair $x, y$ of points $V(C_1 (x, y)) \subseteq C(x, y)$ but only in special cases $C(x, y) \subseteq V(C_1 (x, y))$ (e.g. in hypercubes but not in dotproducts $Q_n \odot C_4$). If $Q_n^0$ contains at most $n - 1$ faults, there is by Theorem 1 for any pair $x^0, y^0$ of points a three stage $x^0 - y^0$ connection through the points $u^0$ and $v^0$ such that the sets $C_1 (x^0, u^0), C_1 (u^0, v^0)$ and $C_1 (v^0, y^0)$ are fault free (i.e. for any geodetic routing $\rho$ there are three fault free $\rho$-geodesics $x^0 - u^0, u^0 - v^0$ and $v^0 - y^0$). Note that the points $u^0, v^0$ and $y^0$ may coincide.

When proving the fault tolerance efficiency of the graphs $Q_n \odot C_4$ we frequently need the following lemma.

Lemma 1 — Let $x^0 \neq y^0$. If the subgraph induced by the convex $C(x^0 - y^0)$ in the projection $Q_n^0$ is fault free, then the set $C_1 (x^1 - y^2) = \{z \mid z$ is a point or line on some $x^1 - y^2$ geodesic$\}$ of the dot product $Q_n \odot C_4$ is fault free.

PROOF : Clearly an $x^1 - y^2$ geodesic can contain exactly one line $w^1 w^2$ joining the subhypercubes $Q_n^1$ and $Q_n^2$. On the other hand, since the point set in each subhypercube $Q_n^i$ is divided into two maximum independent sets $V_1^i$ and $V_2^i$, every two points of any path in $Q_n^i$ are from the set $V_1^i$. Thus each $x^1 - y^1$ path with $x^1 \neq y^1$ contains at least one point $w^1$ adjacent to a point in $Q_n^2$. Each (sub) path of a geodesic must be a geodesic, and thus $x^1 - w^1$ and $w^2 - y^2$ paths are geodesics. Clearly also the chain $w^1 - y^1$ consisting of points corresponding to the points of the $w^2 - y^2$ geodesic must be a geodesic. Thus each $x^0 - y^0$ geodesic induces an $x^1 - y^2$ geodesic and vice versa, and if the subgraph induced by the convex $C(x^0, y^0)$ is fault free, then the set $C_1 (x^1, y^2)$ is fault free, too, by the definition of projecting faults onto the hypercube $Q_n^0$. This completes the proof.

Theorem 1 — In the graph $Q_n \odot C_4$ every geodetic routing $\rho$ is a (3, n)-routing i.e. if the number $|F|$ of faults $F$ is at most $n$ in the graph $Q_n \odot C_4$, then every two points $x$ and $y$ can always be connected by using at most three fault free $\rho$-geodesic.

PROOF : Consider the connection between the points $x^i$ and $y^i$ with $i = 1, 2, 3, 4$. All other cases are similar and hence omitted. Note that the points of any $\rho$-route between two points $u$ and
are contained in the set \( C_1(u,v) \) and thus, if the set \( C_1(u,v) \) is fault free, then also \( u-v \) geodesic of any \( \rho \)-routing is fault free.

Assume first that \( x^0 \neq y^0 \), these points are fault free and there is a fault free \( x^0-y^0 \) connection through the points \( u^0 \) and \( v^0 \) such that the subgraphs induced by the convexes \( C(x^0,u^0), C(u^0,v^0) \) are fault free. If \( y^1 = y^1 \), the sets \( C_1(x^1,u^1), C_1(u^1,v^1) \) and \( C_1(v^1,y^1) \) are clearly fault free and the assertion follows. Note that the points \( u^1, v^1 \) and \( y^1 \) may coincide. If \( y^1 = y^2 \), then the sets \( C_1(x^1,u^2), C_1(u^2,v^2) \) and \( C_1(v^2,y^2) \) are fault free, and we have an \( x^1-y^2 \) connection of the theorem (note that the points \( u^2, v^2 \) and \( y^2 \) may coincide). If \( y^1 = y^3 \), the sets \( C_1(x^1,u^3), C_1(u^2,v^3) \) are by Lemma 1 fault free and the subgraph induced by the convex \( C(v^3,y^3) \) is fault free because subgraph induced by the convex \( C(v^3,y^0) \) is fault free (note that the points \( u^3 \) and \( y^3 \) may coincide), and the assertion follows. The case \( y^1 = y^4 \) is similar to the case \( y^1 = y^2 \). The other part of the proof for \( x^0 \neq y^0 \) (\( x^0 \) and \( y^0 \) are fault free) we divide into several subcases. 1) \( y^1 = y^1 \) then 1a) \( x^1, y^1 \in V_1 \) (the case \( x^1, y^1 \in V_1 \) is analogous) and 1b) \( x^1 \in V_1 \) and \( y^1 \in V_2 \) (the case \( x^1 \in V_1 \) and \( y^1 \in V_2 \) is analogous). 2) \( y^1 = y^2 \) and 2a) \( x^1 \in V_1 \) and \( y^2 \in V_2 \) and 2b) \( x^1 \in V_2 \) and \( y^2 \in V_2 \). 2c) \( x^1 \in V_1 \) and \( y^2 \in V_2 \), and 2d) \( x^1 \in V_2 \) and \( y^2 \in V_1 \). 3) \( y^1 = y^3 \) and 3a) \( x^1 \in V_1 \) and \( y^3 \in V_3 \) (the case \( x^1 \in V_1 \) and \( y^3 \in V_3 \) is analogous) and 3b) \( x^1 \in V_1 \) and \( y^3 \in V_3 \) (the case \( x^1 \in V_1 \) and \( y^3 \in V_3 \) is analogous).

1) If the subhypercube \( Q_n^1 \) contains at most \( n-1 \) faults, then there is the asserted fault free \( x^1-y^1 \) connection by Theorem 1. Thus only only the case where \( Q_n^1 \) contains all \( n \) faults needs a separate proof. In the case 1a) the asserted connection consists of the fault free sets \( C_1(x^1,y^2), C_1(x^2,y^2) \) and \( C_1(y^2,x^1) \). In the case 1b) let the points of an \( x^1-y^1 \) geodesics be \( x^1, u_1, u_2, u_3, \ldots, u_t, y^1 \). Note that the points \( x^1 \) and \( y^1 \) may be adjacent and then \( x^1 = u_1 \) and \( u_1 = y^1 \). If \( x^1 \) and \( y^1 \) are not adjacent, then, because \( x^1 \in V_1 \) and \( y^1 \in V_2 \), each \( x^1-y^1 \) geodesics contains at least two points between \( x^1 \) and \( y^1 \), and hence \( u_1 \neq u_t \) and \( u_1 \in V_2, u_t \in V_1 \). The asserted \( x^1-y^1 \) consists now of the fault free sets \( C_1(x^1,u^2_1), C_1(u^2_1,u^4_t) \) and \( C_1(u^4_t,y^1) \). Note that the shortest \( u_1^2-u_t^4 \) path through the points of \( V(Q_n^3) \) is always shorter that the shortest \( u_1^2-u_t^4 \) path through the points of \( V(Q_n^1) \).

2) If the subgraph induced by the point set \( V(Q_n^1) \cup V(Q_n^2) \) contains at most \( n-1 \) faults,
then the projection $Q_n^{00}$ of this induced subgraph also contains at most $n - 1$ faults and thus there is a fault free $x^{00} - y^{00}$ connection in $Q_n^{00}$. This connection implies the asserted $x^1 - y^2$ connection as shown in the beginning of the proof of Theorem 1. Thus only he case where the subgraph induced by $V(Q_n^1) \cup V(Q_n^2)$ contains all $n$ faults requires further considerations. 2b) Because all faults are in the subgraph induced by $V(Q_n^1) \cup V(Q_n^2)$, the fault free sets $C_1(x^1, x^4), C_1(x^4, y^3), C_1(y^3, y^2)$ constitute the asserted $x^1 - y^2$ connection. 2a) Denote by $N(x)$ neighbourhoud of a point $x$ i.e. $N(x)$ consists of all points which are adjacent to $x$. If there are fault free points $a\in N(x^1)$ and $b\in N(y^2)$ with fault free lines $x^1 a^1$ and $b^2 y^2$, then the sets $C_1(x^1, a^4), C_1(a^4, b^3), C_1(b^3, y^2)$ imply the asserted $x^1 - y^2$ connection. If any fault free point $a^1$ (or any fault free point $b^1$) with corresponding fault free line $x^1 a^1$ (with fault free line $b^2 y^2$) does not exist, then all $n$ faults are in the subhypercube $Q_n^1$ (in the subhypercube $Q_n^2$), and the sets $C_1(x^1, x^2), C_1(x^2, y^2)$ (the sets $C_1(x^1, y^1), C_1(y^1, y^2)$) imply the asserted solution. 2c) If there is at least one fault free point $a^1 \in N(x^1)$ with the fault free line $x^1 a^1$, then the sets $C_1(x^1, a^4), C_1(a^4, y^3), C_1(y^3, y^2)$ imply the asserted solution. If such a point $a^1$ with the fault free line $x^1 a^1$ does not exist, then all $n$ faults are in the subhypercube $Q_n^1$ and the asserted geodesics are contained in the fault free sets $C_1(x^1, x^2)$ and $C_1(x^2, y^2)$. 2d) If there is at least one fault free point $b^2 \in N(y^2)$ with the fault free line $b^2 y^2$, then the asserted geodesics are contained in the sets $C_1(x^1, x^4), C_1(x^4, b^3), C_1(b^3, y^2)$. If such a point $b^2$ with the fault free line $b^2 y^2$ does not exist, then all $n$ faults are in subhypercube $Q_n^2$, and the sets $C_1(x^1, y^1), C_1(y^1, y^2)$ imply the asserted solution.

3) If the subgraph induced by the point set $V(Q_n^1) \cup V(Q_n^2) \cup V(Q_n^3)$ (or by $V(Q_n^1) \cup V(Q_n^4) \cup V(Q_n^3)$) contains at most $n - 1$ faults, then there also is in the projection $Q_n^{000}$ at most $n - 1$ faults and thus a fault free $x^{000} - y^{000}$ connection which implies the asserted $x^1 - y^3$ connection. Thus only the case where both induced subgraphs contain all $n$ faults needs further consideration. It is possible that both induced subgraphs contain all $n$ faults only if the subhypercubes $Q_n^1$ and $Q_n^3$ contain all $n$ faults. This means that the subhypercubes $Q_n^2$ and $Q_n^4$ as well as all the lines into and from $Q_n^2$ and $Q_n^4$ are fault free. 3a) Because $Q_n^2$ and the lines into and from $Q_n^2$ are fault free, the asserted $x^1 - y^3$ connection consists of the sets $C_1(x^1, u^4), C_1(u^4, y^4)$ and $C_1(y^4, y^3)$. 3b) If there is at least one fault free point $u^1 \in N(x^1)$ with
the fault free line \( x^1 u^1 \), the asserted \( x^1 - y^3 \) connection is contained in the fault free sets \( C_1 (x^1, x^2) C_1 (u^4, y^4) \) and \( C_1 (y^4, y^3) \). If all points \( u^1 \in N(x^1) \) (or the corresponding line \( x^1 u^1 \)) are faulty, then all \( n \) faults are in the subhypercube \( Q_n^1 \) and the asserted \( x^1 - y^3 \) connection consists of the sets \( C_1 (x^1, x^2), C_1 (x^2, b^2) \) and \( C_1 (b^2, y^3) \) where \( b^2 \) is a point adjacent to \( y^2 \). This completes the proof of the case \( x^0 \neq y^0 \) and \( x^0, y^0 \) are fault free.

Consider now the case, where \( x^0 = y^0 \) and the point \( x^0 \) is fault free. If \( y^0 = y^2 = x^2 \), and if \( x^1 \in V_1^1 \) then the line \( x^1 x^2 = C (x^1, x^2) \) satisfies the assertion of the theorem, because the point \( x^0 \) is fault free. In the following we also use the projection of faults of the subgraph induced by the point set \( V(Q_n^1) \cup V(Q_n^2) \) on the graph \( Q_n^{00} \). If \( x^1 \in V_2^1 \) and there is at least one fault free point \( a^{00} \in N(x^{00}) \) with the fault free line \( x^{00} a^{00} \), then the subgraphs induced by the convexes \( C (x^1, a^1), C (a^1, x^2) \) and \( C (a^2, x^2) \) imply the desired fault free connection of the theorem. If in \( Q_n^0 \) each point \( a^{00} \in N(x^{00}) \) (or the corresponding line \( x^{00} a^{00} \)) is faulty, there cannot be in the graph \( Q_n \cap C_4 \) any other faults than those in the subgraph induced by the point set \( V(Q_n^1) \cup V(Q_n^2) \), because \( deg (x^{00}) = n \) and the maximum number of faults is \( n \). Let \( a^{00} \in N(x^{00}) \). Then for example the sets \( C_1 (x^1, a^4), C_1 (a^4, x^3) \) and \( C_1 (x^3, x^2) \) contain the fault free connection of the theorem. If \( y^1 = y^3 \) (\( = x^3 \)) and there is least one fault free point \( a^0 \) with a fault free line \( x^0 a^0 \) in \( Q_n^0 \), the desired fault free connection of the theorem is obtained through the sets \( C_1 (x^1, a^2) \) and \( C_1 (a^2, x^3) \) (or through the sets \( C_1 (x^1, a^4) \) and \( C_1 (a^4, x^3) \)). If such a point \( a^0 \) does not exist, we first assume, as above, that \( x^1 \in V_1^1 \), and because \( x^0 \) is fault free, the lines \( x^1 x^2 \) and \( x^4 x^3 \) are fault free. If there is a fault free point \( a^2 \in N(x^2) \) with fault free lines \( x^2 a^2 \) and \( a^3 x^3 \), then the fault free sets \( C_1 (x^1, a^2) \) and \( C_1 (a^2, x^3) \) imply the assertion of the theorem. If such a point \( a^2 \in N(x^2) \) does not exist, there is a fault free point \( a^1 \in N(x^1) \) with fault free lines \( x^1 a^1 \) and \( a^4 x^4 \), since there are at most \( n \) faults in \( Q_n \cap C_4 \). Hence in this case the sets \( C_1 (x^1, a^4) \) and \( C_1 (a^4, x^3) \) contain the desired fault free connection of the theorem. All other cases are symmetrical and thus omitted.

Consider now the cases where \( x^0 \) or \( y^0 \) or both are faulty and \( x^0 \neq y^0 \). This means that at least one line \( x^i x^{i+1} \) or/and at least one line \( y^j y^{j+1} \) is faulty or/and at least one point \( x^i \neq x^1 \) or/and at least one point \( y^j \neq y^1 \) is faulty. Since \( x^0 \) or \( y^0 \) or both are faulty in \( Q_n^0 \), we have at most \( n - 1 \) other faults in \( Q_n^0 \) and thus, by Theorem 1, there is in \( Q_n^0 \) at least one fault free (except points \( x^0 \) and \( y^0 \)) system of subgraphs induced by the convexes \( C (x^0, u^0), C (u^0, v^0), C (v^0, y^0) \) connecting
the points $x^0$ and $y^0$. Note that the points $u^0, v^0$ and $y^0$ may coincide. Thus we have subcases as follows: A) There are four disjoint points $x^0, u^0, v^0, y^0$ such that the fault free convexes $C(x^0, u^0), C(u^0, v^0), C(v^0, y^0)$ connect the points $x^0$ and $y^0$. B) $x^0$ is adjacent to $u^0$ which is adjacent to $v^0$. C) $x^0$ and $y^0$ are adjacent.

A) If $y^i = y^2$, then the sets $C_1(x^1, y^1), C_1(u^1, v^2), C_1(v^2, y^2)$ contain the fault free geodesics of the theorem. If $y^i = y^3$ and $x^1 \in V^1_2$, then the point $x^1$ is not adjacent to any point of $Q_n^2$, and $C_1(x^1, u^2)$ is fault free as well as $C_1(u^2, v^3)$. Hence the sets $C_1(x^1, u^2), C_1(u^2, v^3), C_1(v^3, y^3)$ satisfy the demands of the theorem. If $x^1 \in V^1_1$, then the fault free sets $C_1(x^1, u^4), C_1(u^4, v^3), C_1(v^3, y^3)$ prove the assertion of the theorem. Note that the proof of this case implies that in the case C) the system of fault free sets is $C_1(x^0, y^0) = \{x^0, y^0, x^0, y^0\}$ and in the case B) $C_1(x^0, u^0) = \{x^0, u^0, x^0, u^0\}, C_1(u^0, y^0) = \{u^0, y^0, u^0, y^0\}$.

B) Let $y^i = y^2$. If $x^1, x^1 \in V^1_2$, then $x^2, y^2 \in V^2_2$. Since the point $u^0$ is fault free, the points $u^1$ and $u^2$ as well as the line $u^1 u^2$ are fault free, and hence the sets $C_1(x^1, u^1), C_1(u^1, u^2)$ and $C_1(u^2, y^2)$ contain the set of geodesics of the theorem. If $x^1, y^1 \in V^1_1$, then $x^2, y^2 \in V^2_1$. If one of the lines $x^1 x^2, y^1 y^2$ (with its points) is fault free, it is easy to construct the set of fault free geodesics (if $x^1 x^2, x^1$ and $x^2$ are fault free, then the sets $C_1(x^1, x^2), C_1(x^2, u^2)$ and $C_1(u^2, y^2)$ contain the geodesics of the theorem) and thus we assume that the lines $x^1 x^2, y^1 y^2$ or the corresponding points are faulty. Thus there are at most $n - 2$ other faults in $Q_n^0$. Because $Q_n^0$ is a hypercube, there is exactly one point $v^0$ such that $C(x^0, y^0) = \{x^0, u^0, v^0, y^0\}$. If the subgraph induced by $C(x^0, y^0)$ and the line $y^4 y^3$ with the points $y^3, y^4$ are fault free, then the fault free sets $C_1(x^1, y^4), C_1(y^4, u^3)$ and $C_1(u^3, y^2)$ imply the solution. This solution does not exist (we assume so) if at least one of elements $x^0 v^0, v^0 y^0, y^4 y^3, y^4, y^3$ is faulty. Thus there are at most $n - 3$ other faults in $Q_n^0$. There are $n$ points adjacent to $u^0$, two of which are $x^0$ and $y^0$. Thus there must be a point $z^0 \in N(u^0)$ such that the subgraph induced by the convex $C(x^0, z^0) = \{x^0, u^0, z^0, w^0\}$ is fault free. Now the fault free sets $C_1(x^1, z^4), C_1(z^4, u^3)$ and $C_1(u^3, y^2)$ satisfy the demands of the theorem.

Let, finally, $y^i = y^3$. The construction of the route satisfying the demands of the theorem is easy if one can use at least one of the lines $x^1 x^2$ and $y^4 y^3$ (then, for example, the sets $C_1(x^1, u^2), C_1(u^2, y^3)$ contain the necessary fault free geodesics) and thus we assume that these lines cannot be used, which means at least 2 faulty elements and hence there are at most $n - 2$ other
faults in $Q_n^0$. As above, because $Q_n^0$ is hypercube, there is exactly one point $v^0$ such that $C(x^0, y^0) = \{x^0, u^0, v^0, y^0\}$. If the subgraph induced by $C(x^0, y^0)$ and the line $y^1, y^2$ with the points $y^1, y^2$ are fault free, then the fault free sets $C_1(x^1, y^1), C_1(y^1, u^2)$ and $C_1(u^2, y^3)$ imply the solution. This solution does not exist (we assume so) if at least one of elements $x^0, v^0, y^0, y^4, y^3, y^4, y^3$ is faulty. Thus there are at most $n - 3$ other faults in the hypercube $Q_n^0$. As above, we find a point $z^0 \in N(u^0)$ such that the subgraph induced by the convex $C(x^0, z^0) = \{x^0, u^0, z^0, w^0\}$ is fault free. Now the fault free sets $C_1(x^1, z^4), C_1(z^4, u^3)$ and $C_1(u^3, y^3)$ satisfy the demands of the theorem.

C) Let $y^j = y^2, x^1 \in V_1^1$ and $y^2 \in V_2^2$, since $x^0$ and $y^0$ are adjacent. If the line $x^1 x^2$ is fault free, the geodesics of the fault free sets $C_1(x^1, x^2)$ and $C_1(x^2, y^2)$ solve the problem. Thus we assume that the line $x^1 x^2$ or the point $x^2$ is faulty, and so there are at most $n - 1$ other faults in the graph $Q_n^0$. One possible fault free route is now $C_1(x^1, y^4), C_1(y^4, x^3), C_1(x^3, y^2)$, if the lines $y^1 y^4, x^4 x^3, y^3 y^2$ and the corresponding points are fault free, and hence we assume that at least one of them is faulty, and thus we have at most $n - 2$ other faults in $Q_n^0$. Because $Q_n^0$ is $n$-cube, there are $n$ adjacent points $Z_0$ such that $C(x^0, z^0) = \{x^0, y^0, z^0, u^0\}$, and because there are at most $n - 2$ other faults, there is at least one point $z^0$ with $C(x^0, z^0) = \{x^0, y^0, z^0, u^0\}$ where the points $u^0$ and $z^0$ as well as the lines $x^0 u^0, u^0 z^0$ and $z^0 y^0$ are fault free (note that the point $y^1$ may be faulty). Thus we have the fault free route through the sets $C_1(x^1, u^1), C_1(u^1, z^2), C_1(z^2, y^2)$ which proves the assertion of the theorem. Let $x^1 \in V_2^1$ and $y^2 \in V_1^2$. If the line $y^1 y^2$ and the point $y^1$ are fault free, the sets $C_1(x^1, y^1)$ and $C_1(y^1, y^2)$ prove the assertion of the theorem. Thus we assume that the line $y^1 y^2$ or the point $y^1$ (or both) is faulty, and thus we have at most $n - 1$ other faults in $Q_n^0$. One possible fault free route goes through the sets $C_1(x^1, x^4), C_1(x^4, y^3), C_1(y^3, y^2)$. Thus we assume that this route is not fault free, whence there are at most $n - 2$ other faults in $Q_n^0$, and, as above, we can find at least one point $z^0$ such that $C(x^0, z^0) = \{x^0, y^0, z^0, u^0\}$ where the points $u^0$ and $z^0$ as well as the lines $x^0 u^0, u^0 z^0$ and $z^0 y^0$ are fault free (note that the point $x^2$ may be faulty). But then we have the asserted system of fault free geodesics through the sets $C_1(x^1, u^2), C_1(u^2, z^2)$ and $C_1(z^2, y^2)$. Let, finally, $y^j = y^3, x^1 \in V_1^1$ and $y^3 \in V_2^2$. If the lines $x^1 x^2, x^2 y^2, y^2 y^3$ and the points $x^2, y^2$ are fault free, we have the solution $C_1(x^1, x^2), C_1(x^2, y^2), C_1(y^2, y^3)$ of fault free sets. A similar easy solution is found also then when the points $y^1, y^4, x^4$ and the lines $y^1 y^4, y^4 x^4, x^4 x^3$ are fault free. Hence we assume that these routes are faulty, and thus
there are at most \( n - 2 \) other faults in \( Q_n^0 \). This means, as above, that there is at least one point \( z^0 \) such that \( C(x^0, z^0) = \{x^0, y^0, z^0, u^0\} \) where the points \( u^0 \) and \( z^0 \) as well as the lines \( x^0, u^0, \overline{u^0 z^0} \) and \( z^0, y^0 \) are fault free. Then the sets \( C_1(x^1, u^4), C_1(u^4, z^3), C_1(z^3, y^3) \).

Consider finally the case, where \( x^0 = y^0 \) and \( x^0 \) is faulty. Note that \( x^1 \) and \( y^1 \) are not faulty; if at least one of them is faulty, then the whole problem of a (fault free) \( x^1 - y^1 \) connection is absurd. Since \( x^0 \) is faulty, at least one point \( x^i (\neq x^1, y^1) \) or one line \( x^i x^{i+1} \) is faulty. Note that the faulty line \( x^i x^{i+1} \) may be incident to \( x^1 \) or to \( y^1 = \overline{y} \). Thus there are at most \( n - 1 \) other faults in \( Q_n \cap C_4 \). Assume first that \( x^1 \in V_2^1 \) and \( y^1 = x^2 \). Because \( \deg(x^1) = n + 1 \) and \( x^1 \) is not adjacent to \( x^2 \), there is at least one point \( w^1 \in V_1^1 \) such that \( w^1 \) and \( x^1 w^1 \) are fault free. If the corresponding point \( w^2 \) and the lines \( w^1 w^2, w^2, x^2 \) are fault free, the fault free sets \( C_1(x^1, w^1), C_1(w^1, w^2) \) and \( C_1(w^2, x^2) \) contain the desired geodesics of the theorem. If \( w^2 \) or at least one of the lines \( w^1 w^2 \) and \( w^2 x^2 \) is faulty, there is another fault free, we are done, and if not, there is another fault free point \( v^1 \in N(x^1) \) with the fault free line \( x^1 v^1 \) (because there are at most \( n - 1 \) other faults in \( Q_n \cap C_4 \) and \( n \) points adjacent to \( x^1 \) in \( Q_n^1 \)). Thus we can check another corresponding point \( v^2 \) with lines \( v^1 v^2 \) and \( v^2 x^2 \). If the point \( v^2 \) and the lines \( v^1 v^2, v^2 x^2 \) are fault free, we are done, and if not, there is another fault free point \( z^1 \in N(x^1) \) with the fault free line \( x^1 z^1 \) (because there are at most \( n - 1 \) other faults in \( Q_n \cap C_4 \) and \( n \) points adjacent to \( x^1 \) in \( Q_n^1 \)). By continuing this process, we see that there is a \( x^1 - x^2 \) connection of the theorem.

Assume now that \( x^1 \in V_1^1 \) and \( y^1 = y^2 = x^2 \), and thus there is a line joining \( x^1 \) and \( x^2 \). If the line \( x^1 x^2 \) is fault free we are done, and thus we assume that this line is faulty, whence there are \( n - 1 \) other faults, in \( Q_n \cap C_4 \). Because there are \( n \) points adjacent to \( x^1 \) in \( Q_n^1 \) and there are \( n - 1 \) other faults, there is one fault free point \( t^0 \in N(x^0) \) in \( Q_n^0 \) with the fault free line \( x^0 t^0 \), and thus also the lines \( x^1 t^1, t^1 t^4, x^2 t^2, t^2 t^3, t^3 t^4 \) and the points \( t^1, t^2, t^3, t^4 \) are fault free. Because the degree of each point in \( Q_n^0 \) is \( n \), there are \( n - 1 \) disjoint points \( v^0 \in N(x^0) \) with \( v^0 \neq t^0 \), and because \( Q_n^0 \) is an \( n \)-cube, there are \( n - 1 \) disjoint points \( w^0 \) such that \( C(x^0, w^0) = \{x^0, v^0, t^0, w^0\} \) (i.e. for each \( v^0 \) there is a point \( w^0 \neq x^0 \) such that \( C(x^0, w^0) = \{x^0, v^0, t^0, w^0\} \)). If \( C(x^0, w^0) \) is fault free (except the point \( x^0 \)) for some \( w^0 \), we are done, since the sets \( C_1(x^1, w^4), C_1(w^4, w^3) \) and \( C_1(w^3, x^2) \) satisfy the demands of the theorem; note that \( C_1(w^3, w^4) = \{w^3, w^4, w^3 w^4\} \). Because there are \( n - 1 \) disjoint
points \( w \) there are also \( n - 1 \) disjoint set systems \( C_1(x^1, w^4), C_1(w^4, w^3), C_1(w^3, x^2), \) and thus \( n - 1 \) faults is enough to damage each of them and then each of the systems contains exactly one faulty element (point or line). If the faulty element is one of the points \( v^2, v^3 \) or one of the lines \( x^2 v^2, v^2 v^3, v^3 w^3 \), then \( C_1(x^1, w^4), C_1(w^4, r^3), C_1(r^3, x^2) \) is the system of fault free sets of the theorem. By the symmetry, the theorem holds also if one of the points \( v^1, v^4 \) or one of the lines \( x^1 v^1, v^1 v^4, v^4 w^4 \) is faulty. If for each of the \( n - 1 \) set systems \( C_1(x^1, w^4), C_1(w^4, w^3), C_1(w^3, x^2) \) one of the elements \( w^4, w^3, w^4 w^3 \) is faulty, we must construct a new \( x^1 - x^2 \) connection. The points \( w^1, w^2 \) corresponding to the point \( w^0 \) (and to \( w^3 \) and \( w^4 \), too) are now fault free as well as the line \( w^1 w^2 \) and the points \( v^1 \) and \( v^2 \), because all \( n \) faults locate elsewhere. Note that \( V(C_1(x^1, w^4)) = \{ x^1, r^1, v^1, w^1 \}, V(C_1(x^2, w^4)) = \{ x^2, r^2, v^2, w^2 \}, \) and thus \( C_1(x^1, w^4), C_1(w^1, w^2), C_1(w^2, x^2) \) is the system of fault free point sets of the theorem. Thus there always exists the asserted \( x^1 - x^2 \) connection of the theorem. The case \( y^1 = y^3 = x^2 \) is similar and hence omitted. This completes the proof.

Let us define the shortest path routing \( \lambda \) between the points \( x \) and \( y \) in the graph \( Q_n \oplus C_4 \) as follows:

**Definition 2** — The \( \lambda \)-paths of the dot product \( Q \oplus C_4 \) are based on the \( \lambda \)-paths of the hypercube \( Q_n \). A \( \lambda \)-path \( x - y \) is always an \( x - y \) geodesic. If there are several \( x - y \) geodesics, then the \( \lambda \)-path \( x - y \) is chosen as follows:

1) If \( x \) and \( y \) belong to a single subhypercube \( Q_n^i \), the routing \( \lambda \) follows that of Dolev et al.\(^2\) for hypercubes: the \( \lambda \)-path from \( x \) to \( y \) moves through the points the coordinates of which differ one at a time from left to right.

2) If \( x^i \neq y^j, x^i \) is in the subhypercube \( Q_n^i \), and \( y^j = y^{i+1} \) in the neighbouring subhypercube \( Q_n^{i+1} \), the \( \lambda \)-route from \( x^i \) to \( y^j \) leaves the subhypercube \( Q_n^i \), as soon as possible (i.e. if the route consists of points \( x^i, z_1, z_2, z_3, ..., y^{i+1} \) then either the point \( z_1 \) or at least the point \( z_2 \) belongs to the subhypercube \( Q_n^{i+1} \)) and thereafter follows the \( \lambda \)-routing of Dolev et al.\(^2\) in \( Q_n^{i+1} \).

3) If \( x^i \neq y^j, x^i \) is in the subhypercube \( Q_n^i \) \( y^j \neq y^{i+2} \) in the subhypercube \( Q_n^{i+2} \) and the \( x^i - y^j \) geodesic goes through the subhypercube \( Q_n^{i+1} \) (or through the subhypercube \( Q_n^{i-1} \)), then the \( \lambda \)-path \( x^i - y^j \) in \( Q_n \oplus C_4 \) leaves the subhypercube \( Q_n^i \) as soon as possible, goes through the subhypercube \( Q_n^{i+1} \) (or through \( Q_n^{i-1} \)) and leaves the subhypercube \( Q_n^{i+1} \) so that the part of the
$x^i - y^j$ path contained in the subhypercube $Q_{n}^{i+2}$ is the shortest possible (i.e. at most two latest points of the $x^i - y^j$ geodesic in the $\lambda$-routing in $Q_n \circ C_4$ belongs to the subhypercube $Q_{n}^{i+2}$). Moreover the part of the $\lambda$-path $x^i - y^j$ belonging to the subhypercube $Q_{n}^{i+1}$ follows Dolev's $\lambda$-routing for hypercubes.

4) As well known, each point $a$ of an $n$-cube $Q_n$ is represented by an $n$-tuple $(a_1, a_2, a_3, ..., a_{n-1}, a_n)$ where each number/coordinate $a_s$ is either 0 or 1 ($s = 1, 2, ..., n$). If $x = x^i \in V(Q_n^i)$ and the point $y = y^j$ also corresponds to the point $x$ but in another subhypercube $Q_n^j, j \neq i$ of the graph $Q_n \circ C_4$, we have two cases: $x^i$ and $x^j$ are adjacent or they are not adjacent. When $x^i$ and $x^j$ are adjacent, then the $\lambda$-path $x^i - x^j$ is the line $x^i, x^j$ and its endpoints. If $x^i$ and $x^j$ are not adjacent, then the $\lambda$-path consists merely of points $x^k$ (corresponding $x$) and $u^l$ the coordinate representation of which is $u = (x_1, x_2, ..., x_{n-1}, x_n)$, i.e. the only difference between points $x$ and $u$ locates on the last coordinate. Thus, for example, $x^i, u^i, u^{i+1}, x^{i+1}$ is the $\lambda$-path $x^i - x^{i+1}$ when $j = i + 1$, and $x^i$ and $x^{i+1}$ are not adjacent.

**Theorem 2** — Let $F$ be the set of faults. If $|F| \leq n$, then $\text{diam} (R(Q_n \circ C_4, \lambda)/F) \leq 2$.

**Proof:** The statement $\text{diam} (R(Q_n \circ C_4, \lambda)/F) \leq 2$ means that for any two points $x^i$ and $y^j$ ($x^i \neq y^j$) there is in the graph $G/F$ the $\lambda$-path $x^i - y^j$ or there are in $G/F$ two $\lambda$-paths $x^i - z$ and $z - y^j$ connecting the points $x^i$ and $y^j$. We shall consider merely the $x^i - y^j$ connection because all other cases are analogous. As in the previous proof, $Q_n^0$ is the projection of the dot product $Q_n \circ C_4$.

Assume first that $x^0 \neq y^0$ and $x^0$ and $y^0$ are fault free. If $y^i = y^1$ and there are at most $n - 1$ faults in $Q_n^1$, the theorem holds by Theorem 2. Let there be $n$ faults in $Q_n^1$ and let $x^1, y^1 \in V_1^i$. Because all faults are in $Q_n^1$, the $\lambda$-paths $x^1 - y^2$ and $y^2 - y^1$ satisfy the demands of the theorem. If $x^1 \in V_1^1$ and $y^1 \in V_2^1$, then the $\lambda$-path $x^1 - x^3$ is fault free (it contains by the definition of $\lambda$-routing merely the point $x^1$ from the subhypercube $Q_{n}^{1}$) as well as the $\lambda$-path $x^3 - y^1$ (note that merely the point $y^1$ from the set $V(Q_n^1)$ belongs to the $\lambda$-path $x^3 - y^1$). Thus these paths satisfy the demands of theorem. The cases $x^1, y^1 \in V_2^1$ and $x^1 \in V_2^1, y^1 \in V_1^1$ are similar those considered above and hence we omit them.

Assume now that $x^0 \neq y^0, x^2$ and $y^0$ are fault free, and $y^1 = y^3$. Two cases arise: (i) $x^1 \in V_1^2$
and \( y^3 \in V_1^3 \), and (ii) \( x^1 \in V_1^1 \) and \( y^3 \in V_2^3 \). Note that the case \( x^1 \in V_2^1, y^3 \in V_1^3 \) is similar to the case (ii) and the case \( x^1 \in V_2^1, y^3 \in V_2^3 \) similar to the case (i).

(i) If the subgraph induced by the pointset \( V(Q_n^1) \cup V(Q_n^2) \cup V(Q_n^3) \) (or the subgraph induced by the pointset \( V(Q_n^1) \cup V(Q_n^4) \cup V(Q_n^3) \)) contains at most \( n - 1 \) faults, then by projecting the induced subgraph onto the hypercube \( Q_n^{00} \) we see that \( x^{00} \) and \( y^{00} \), are not adjacent, because \( x^1 \in V_1^1 \) and \( y^3 \in V_3^1 \), and there is by Theorem 2 a \( \lambda \)-connection \( x^{00} - y^{00} \). If this \( \lambda \)-connection is a single \( \lambda \)-path \( z^{00} - y^{00} \), this path contains an intermediate point \( z^{00} \neq z^{00}, y^{00} \), because \( x^{00} \) and \( y^{00} \), are not adjacent. Thus we have fault free \( \lambda \)-paths \( x^1 - z^2 \) and \( z^2 - y^3 \) (or \( x^1 - z^4 \) and \( z^4 - y^3 \)). If the \( \lambda \)-connection \( x^{00} - y^{00} \) consists of two fault free \( \lambda \)-paths \( x^{00} - z^{00} \) and \( z^{00} - y^{00} \), then the fault free \( \lambda \)-paths \( x^1 - z^2 \) and \( z^2 - y^3 \) (or \( x^1 - z^3 \) and \( z^3 - y^3 \)) are the solution of the problem, and we are done. Thus we assume that both induced subgraphs, one induced by \( V(Q_n^1) \cup V(Q_n^2) \cup V(Q_n^3) \) and another by \( V(Q_n^1) \cup V(Q_n^4) \cup V(Q_n^3) \), contain all \( n \) faults, which is possible only if all faults are contained in the subgraphs \( Q_n^1 \) and \( Q_n^3 \). If now a point \( b^3 \in N(y^3) \) and the line \( b^3 y^3 \) in \( Q_n^3 \) are fault free, then the \( \lambda \)-paths \( x^1 - b^2 \) and \( b^2 - y^3 \) are fault free, and we are done. If each point \( b^3 \in N(y^3) \) (or the corresponding line \( b^3 y^3 \)) is faulty, then all \( n \) faults are in the subgraph \( Q_n^3 \), because \( deg(y^3) = n \) in the subgraph \( Q_n^3 \). Thus the \( \lambda \)-paths \( x^1 - y^4 \) and \( y^4 - y^3 \) are fault free, and we have the paths satisfying the demands of the theorem.

(ii) Note that now \( x^0 \) and \( y^0 \) as well as \( x^{00} \) and \( y^{00} \) may be adjacent. If the sub-graph induced by the pointset \( V(Q_n^1) \cup V(Q_n^2) \cup V(Q_n^3) \) (or the subgraph induced by the pointset \( V(Q_n^1) \cup V(Q_n^4) \cup V(Q_n^3) \)) contains at most \( n - 1 \) faults, then project the induced subgraph onto the hypercube \( Q_n^{00} \). If \( x^{00} \) and \( y^{00} \) are not adjacent, then we see as in the case (i) above that the assertion holds. If both subgraphs induced by the point sets \( V(Q_n^1) \cup V(Q_n^2) \cup V(Q_n^3) \) and \( V(Q_n^1) \cup V(Q_n^4) \cup V(Q_n^3) \) contain \( n \) faults and \( x^{00} \) and \( y^{00} \) are not adjacent, we obtain the solution as in the case (i) above. Thus the new case is that where the points \( x^{00} \) and \( y^{00} \) are adjacent. If the line \( x^2 y^2 \) is fault free, the \( \lambda \)-path with the elements \( x^1, x^1 x^2, x^2, x^2 y^2, y^2, y^2 y^3, y^3 \) satisfy the demands of the theorem independently of the number of faults in the subgraph induced by the point set \( V(Q_n^1) \cup V(Q_n^2) \cup V(Q_n^3) \). If the subgraph induced by the point set \( V(Q_n^1) \cup V(Q_n^4) \cup V(Q_n^3) \) contains at most \( n - 1 \) faults and the line \( x^{00} y^{00} \) in the projection \( Q_n^{00} \) is faculty, then we have in
two $\lambda$-paths $x^0 - z^0$ and $z^0 - y^0$ with $z^0 \neq x^0, y^0$, which paths imply the solution of the problem. Consider now the subgraph induced by the pointset $V(Q^1_n) \cup V(Q^4_n) \cup V(Q^3_n)$. If it contains at most $n - 2$ faults, we can denote the line $x^0 y^0$ in the projection $Q^0_n$ faulty, and thus we have in $Q^0_n$ two $\lambda$-paths $x^0 - z^0$ and $z^0 - y^0$ with $z^0 \neq x^0, y^0$, which paths imply the solution of the problem.

If there are in the subgraph induced by the point set $V(Q^1_n) \cup V(Q^4_n) \cup V(Q^3_n)$ at most $n - 1$ faults such that the line $x^0 y^0$ is faulty, we obtain the solution as above. Thus it remains the case where $x^0$ and $y^0$ are adjacent, the subgraph induced by the point set $V(Q^1_n) \cup V(Q^2_n) \cup V(Q^3_n)$ contains $n$ faults one of which is the line $x^2 y^2$, and the subgraph induced by the pointset $V(Q^1_n) \cup V(Q^2_n) \cup V(Q^3_n)$ contains $n - 1$ fault such that the line $x^0 y^0$ in its projection is fault free. This means that the subgraphs $Q^1_n$ and $Q^2_n$ of $Q_n \bigcirc C_4$ contain $n - 1$ faults and the $n$th fault is the line $x^2 y^2$. Moreover, the lines $x^1 y^1$ and $x^3 y^3$ (corresponding to the line $x^0 y^0$) are fault free. If the difference between points $x$ and $y$ is in the $k$th ($k \geq 2$) coordinate then we use the points $z = (x_1 \ldots \bar{x}_{k-1} x_k x_{k+1} \ldots x_n)$ and $u = (x_1 \ldots \bar{x}_{k-1} \bar{x}_k x_{k+1} \ldots x_n)$ and have the $\lambda$-paths $x^1 - u^2$ (through the points $x^2$ and $z^2$) and $u^2 - y^3$ (through the point $y^2$). If the difference between the points $x$ and $y$ is in the first coordinate, then we use the point $z_r = (\bar{x}_1 x_2 \ldots \bar{x}_{r-1} \bar{x}_r x_{r+1} \ldots x_n)$, $r = 2, \ldots, n$ and have the $\lambda$-paths $x^1, y^1, y^4, z^4_r$ and $z^3_r, z^3_r, y^3$; note that $y = (\bar{x}_1 x_2 \ldots x_n)$. By the assumptions above, the elements $x^1 y^1$ and $y^1$ are fault free as well as the subhypercube $Q^4_n$ and the lines having an endpoint in $Q^4_n$.

Thus if the $\lambda$-connection $x^1 - y^3$ through the points $z^3_r$ and $z^4_r$ is faulty, the only possible faulty elements are $z^3_r$ and $z^4_r y^3$. By the definition of the point $z_r$ there are $n - 1$ disjoint points $z_r$ and hence $n - 1$ disjoint $\lambda$-connections $x^1 - y^3$ through $n - 1$ disjoint points $z_r$. By the assumption, the subhypercube $Q^3_n$ may contain $n - 1$ faults, whence each $\lambda$-connection $x^1 - y^3$ through points $z^3_r, z^4_r$ may be faulty, and thus we assume that each $\lambda$-path $z^4_r y^3$ is faulty (and the $n$th fault is the line $x^2 y^2$). Because the subhypercube $Q^1_n$ is now fault free, we have a fault free $\lambda$-path $x^1 - u^1$ with $u = (\bar{x}_1 \bar{x}_2 x_3 \ldots x_n)$. Now the $\lambda$-path $u^1 - y^3$ through the points $u^1, u^2, y^3$ is fault free, because the only faulty element in $Q^2_n$ is the line $x^2 y^2$. Thus we have a desired $\lambda$-connection in each possible subcase.
Assume now that \( x^0 \neq y^0, x^0 \) and/or \( y^0 \) are faulty, and \( y^f = y^2 \). If there are at most \( n - 1 \) faults in the subgraph induced by the point set \( V(Q_n^1) \cup V(Q_n^2) \), there is the connection of the theorem in that subgraph, and we are done. Thus we assume that all \( n \) faults are in the subgraph induced by \( V(Q_n^1) \cup V(Q_n^2) \). If \( x^1 \in V_2^1 \) and \( y^2 \in V_2^2 \), the \( \lambda \)-paths \( x^1 - y^3 \) and \( y^3 - y^2 \) constitute the \( x^1 - y^2 \) connection of the theorem. If \( x^1 \in V_2^1 \) and \( y^2 \in V_1^2 \), and if there are \( n \) faults in \( Q_n^2 \), then there is an \( \lambda \)-path \( x^1 - y^1 \) in \( Q_n^1 \), and thus this path and the line \( y^1 y^2 \) (since \( y^0 \) is fault free) constitute the connection of the theorem. If there are at most \( n - 1 \) faults in \( Q_n^2 \), there is at least one fault free point \( a^2 \) with the fault free line \( y^2 a^2 \) adjacent to \( y^2 \). Because \( x^0 \) is fault free and all \( n \) faults are in the subgraph induced by \( V(Q_n^1) \cup V(Q_n^2) \), there are \( \lambda \)-paths \( x^1 - a^3 \) and \( a^3 - y^2 \), which satisfy the demands of the theorem. If \( x^1 \in V_1^1 \) and \( y^2 \in V_1^2 \), and if there are at most \( n - 1 \) faults in \( Q_n^2 \), there is a \( \lambda \)-connection \( x^2 - y^2 \) of the theorem by Theorem 2\(^2\), and since the point \( x^0 \) is fault free we can start, by 2 of Definition 2, this connection by the elements \( x^1 \) and \( x^1 x^2 \), and we have the asserted \( \lambda \)-connection \( x^1 - y^2 \). If there are \( n \) faults in \( Q_n^2 \), there is a \( \lambda \)-path \( x^1 - y^1 \) in \( Q_n^1 \). This path and the elements \( y^1 y^2 \) and \( y^2 \) (since \( y^0 \) is fault free) constitute the \( x^1 - y^2 \) connection of the theorem. Let \( x^1 \in V_1^1 \) and \( y^2 \in V_2^2 \). If there are at most \( n - 1 \) faults in \( Q_n^2 \), there is a \( \lambda \)-connection \( x^2 - y^2 \) of the theorem by Theorem 2\(^2\), and since the point \( x^0 \) is fault free we can start, by 2 of Definition 2, this connection by the elements \( x^1 \) and \( x^1 x^2 \), and we have the asserted \( x^1 - y^2 \) connection. If there are \( n \) faults in \( Q_n^2 \) and \( a^1 \) is a point adjacent to \( x^1 \), then \( \lambda \)-paths \( x^1 - a^4 \) and \( a^4 - y^2 \) satisfy the demands of the theorem.

Assume now that \( x^0 \neq y^0, x^0 \) and/or \( y^0 \) are faulty, and \( y^f = y^3 \). As in the cases (i) and (ii) above, one needs to consider the cases (iii) \( x^1 \in V_1^1, y^3 \in V_2^3 \) and (iv) \( x^1 \in V_1^1, y^3 \in V_1^1 \), only.

(iii) If the subgraph induced by the point set \( V(Q_n^1) \cup V(Q_n^4) \cup V(Q_n^3) \) contains \( n \) faults, then the subhypercube \( Q_n^2 \) as well as the lines \( x^1 x^2, y^2 y^3 \) are fault free, and the \( \lambda \)-paths \( x^1 - y^2 \) and \( y^2 - y^3 \) constitute the \( \lambda \)-connection of the theorem. Thus we may assume that the subgraph induced by the point set \( V(Q_n^1) \cup V(Q_n^4) \cup V(Q_n^3) \) contains at most \( n - 1 \) faults. The projection \( Q_n^{00} \) of the subgraph induced by the point set \( V(Q_n^1) \cup V(Q_n^4) \cup V(Q_n^3) \) also contains at most \( n - 1 \) faults, and thus by Theorem 2\(^2\) there is a \( \lambda \)-connection \( x^{00} - y^{00} \). Because \( x^1 \in V_1^1 \) and \( y^3 \in V_3^3 \), \( x^{00} \) and \( y^{00} \) may be adjacent or the \( \lambda \)-connection \( x^{00} - y^{00} \) contains at least two other points between \( x^{00} \) and \( y^{00} \). If the \( \lambda \)-connection \( x^{00} - y^{00} \) consists of a single \( \lambda \)-path containing intermediate points
\(a^{00}\) and \(b^{00}\), then the \(\lambda\)-paths \(x^1 - a^4\) and \(a^4 - y^3\) does not use the (possibly faulty) elements \(y^1, x^1, y^4, x^4, x^3, y^1, y^4, x^4, x^3\), and we are done. If the \(\lambda\)-connection \(x^{00} - y^{00}\) consists of two \(\lambda\)-paths \(x^{00} - c^{00}\) and \(c^{00} - y^{00}\), then at least one of them contains at least one intermediate point. If \(x^{00}\) and \(c^{00}\) are adjacent, then the \(\lambda\)-path \(x^1 - c^4\) does not contain elements \(y^1, y^4\) or \(y^1, y^4\) and the \(\lambda\)-path \(c^4 - y^3\) contains a point \(e^4 \neq y^4\) adjacent to \(c^4\), whence the \(\lambda\)-path \(c^4 - y^3\) does not contain elements \(x^4, x^3, x^4, x^3\). Thus the \(\lambda\)-paths \(x^1 - c^4\) and \(c^4 - y^3\) satisfy the demands of the theorem. If \(c^{00}\) and \(y^{00}\) are adjacent, the considerations are analogous and hence omitted. Thus it remains the case, where \(x^{00}\) and \(y^{00}\) are adjacent, the line \(x^{00} y^{00}\) is contained in the \(\lambda\)-path \(x^{00} - y^{00}\) and the subgraph induced by the point set \(V(Q_n^1) \cup V(Q_n^4) \cup V(Q_n^3)\) contains at most \(n - 1\) faults. Note that when considering \(x^{00} - y^{00}\) connection in the projection \(Q_n^{00}\), the possible faultiness of the points \(x^{00}\) and/or \(y^{00}\) must be ignored, because a connection between faulty points is absurd. Thus if at least one of the elements \(y^1, x^4, y^4, x^3, y^1, y^4, x^3, x^4\) is faulty the nonabsurd projection \(Q_n^{00}\) contains at most \(n - 2\) faults, whence we can denote the line \(x^{00} y^{00}\) faulty and we have case considered above. Thus we may assume now that the elements \(y^1, x^4, y^4, x^3, y^1, y^4, x^3, x^4\) are fault free as well as the elements \(x^1 y^1, y^4 x^4, x^3 y^3\) (if one of the elements \(x^1 y^1, y^4 x^4, x^3 y^3\) is faulty, then the line \(x^{00} y^{00}\) is faulty, and we have a case considered above).

But then, because \(x^0\) and/or \(y^0\) is faulty, one of the elements \(x^1, x^2, x^2, y^2, y^3\) (outside the subgraph induced by the pointset \(V(Q_n^1) \cup V(Q_n^4) \cup V(Q_n^3)\)) must be faulty. Because the projection \(Q_n^{00}\) of the subgraph induced by \(V(Q_n^1) \cup V(Q_n^4) \cup V(Q_n^3)\) is a hypercube, there are \(n - 1\) disjoint points \(m^{00}\) such that \(C(x^{00}, m^{00}) = \{x^{00}, y^{00}, m^{00}, z^{00}\}\); note that for two disjoint points \(m^{00*}\) and \(m^{00**}\) also the corresponding points \(z^{00*}\) and \(z^{00**}\) are disjoint. Because \(x^0\) and \(y^0\) are adjacent, their coordinate representations differ on a single place and we may denote \(y^0 = (y_1 y_2 \ldots y_n)\) and \(x^0 = (y_1 \ldots y_{k-1} \bar{y}_k y_{k+1} \ldots y_n)\). One possible solution of the problem are the \(\lambda\)-paths \(x^1, y^1, y^4, m^4\) and \(m^4, m^3, y^3\) (or \(x^1, z^1, z^4, m^4\) and \(m^4, m^3, y^3\)), and we assume that these paths are faulty, whence there must be \(n - 1\) disjoint faultinesses concerning the elements of these paths in the subgraph induced by the point set \(V(Q_n^1) \cup V(Q_n^4) \cup V(Q_n^3)\). One of the points \(m^{00}\) is \((y_1 y_2 \ldots y_{n-1} \bar{y}_n) = a^{00}\) and one possible \(\lambda\)-connection is \(x^1, y^1, y^4\) and \(y^4, a^4, a^3, y^3\). Because the \(\lambda\)-path \(x^1 - y^4\) contains only fault free elements, the \(\lambda\)-path \(y^4 - y^3\) through the points \(a^4, a^3\) must contain a faulty element. The point \(z^{00}_a\) corresponding to the point \(m^{00} = a^{00}\) in \(C(x^{00}, m^{00}) = \{x^{00}, y^{00}, m^{00}, z^{00}\}\) is \((y_1 y_2 \ldots y_{k-1} \bar{y}_k y_{k+1} \ldots y_{n-1} \bar{y}_n)\), and thus we have the \(\lambda\)-paths \(x^1, z^1_x, z^4_x, x^4, x^3, y^3\), where
the latter one consists of fault free elements, only. This $\lambda$-connection must be fault free because there were only $n - 1$ faulty elements in the subgraph induced by the pointset $V(Q_n^1) \cup V(Q_n^4) \cup V(Q_n^3)$. The only vague case is that where $k = n$, i.e. $x^0 = (y_1, y_2, \ldots, y_{n-1}, y_n)$. But then, by 4 of Definition 2, the $\lambda$-path $x^1 - x^4$ is $x^1, y^1, y^4, x^4$ and we have the fault free $\lambda$-connection $x^1 - x^4$ and $x^4 - y^3$. This completes the proof of the case (iii).

(iv) The problem of the existence of an $x^0 - y^0$ connection is absurd, if $x^0$ or $y^0$ or both are faulty, and therefore in the projection we must ignore the faultiness of $x^0$ and/or $y^0$, whence the projection contains at most $n - 1$ (other) faults. Assume that the elements $y^4$ and $y^3 y^4$ are fault free and consider the projection $Q_n^{00}$ of the subgraph induced by the point set $V(Q_n^1) \cup V(Q_n^4) \cup V(Q_n^3)$. Because $Q_n^{00}$ contains at most $n - 1$ (other) faults, there is a $\lambda$-connection $x^0 - y^0$. If this connection consists of a single path, its correspondence used by 3 of Definition 2 as the last line the line $y^4 y^3$, which is fault free as well as the point $y^4$. If the $x^{00} - y^{00}$ connection consists of two $\lambda$-paths $x^{00} - z^{00}$ and $z^{00} - y^{00}$, then the $\lambda$-paths $x^1 - z^4$ and $z^4 - y^3$ satisfy the assertion. If the elements $x^1 x^2$ and $x^2$ are fault free, the projection of the subgraph induced by $V(Q_n^1) \cup V(Q_n^4) \cup V(Q_n^3)$ contains at most $n - 1$ (other) faults and we obtain a $\lambda$-connection as before. Note that this connection always uses by 2 of Definition 2 the elements $x^1 x^2$ and $x^2$. Assume now that both of the subgraphs induced by the point sets $V(Q_n^1) \cup V(Q_n^4) \cup V(Q_n^3)$ and $V(Q_n^1) \cup V(Q_n^4) \cup V(Q_n^3)$, respectively, contain $n$ faults, which means that all faults locate in subhypercubes $Q_n^1$ and $Q_n^3$. Because $x^0$ and/or $y^0$ are faulty, the location of the faults implies that $x^3$ and/or $y^1$ must now be faulty. If all faults are in $Q_n^1$, then the $\lambda$-paths $x^1 - x^2$ and $x^2 - y^3$ solve the problem, and if all faults are in $Q_n^3$, then the $\lambda$-paths $x^1 - y^4$ and $y^4 - y^3$ give a solution. Thus, let $Q_n^1$ and $Q_n^3$ both contain at least one fault. Then the subgraph induced by the point set $V(Q_n^2) \cup V(Q_n^3)$ as well as its projection $Q_n^{00}$ contain at most $n - 1$ (other) faults and hence there is a $\lambda$-connection $x^{000} - y^{000}$, which implies a $\lambda$-connection $x^2 - y^3$. By 2 of Definition 2 we can continue the first path by means of elements $x^1, x^1 x^2$ and have the system of $\lambda$-paths which proves the assertion of the theorem.

By the considerations above it remains the case where $x^1 x^2$ (or $x^2$) and $y^4 y^3$ (or $y^4$) are faulty, and thus the subgraph induced by $V(Q_n^1) \cup V(Q_n^4) \cup V(Q_n^3)$ as well as its projection $Q_n^{00}$ have at most $n - 2$ (other) faults. The problem is now to find a $\lambda$-connection $x^1 - y^3$ which does
not use the elements $x^1 x^2, x^2, y^4$ and $y^4 y^3$. Because $Q_{n}^{00}$ contains at most $n - 2$ (ther) faults, there is a $\lambda$-connection $x^{00} - y^{00}$. If it is a single path containing at least two adjacent intermediate points $z^{00}$ and $w^{00}$, then the $\lambda$- paths $x^1 - z^4$ and $z^4 - y^3$, where the path $z^4 - y^3$ uses the line $z^2 z^4$ if $z^4 \in V^4_1$ (or the line $w^4 w^3$ if $z^4 \in V^4_2$), avoids the elements $y^4$ and $y^4 y^3$. If the $\lambda$-connection $x^{00} - y^{00}$ consists of two $\lambda$-paths $x^{00} - z^{00}$ and $z^{00} - y^{00}$, where the path $z^{00} - y^{00}$ has at least one intermediate point $w^{00}$, then the $\lambda$-paths $x^1 - z^4$ and $z^4 - y^3$ solve the problem.

If $x^{00}$ and $y^{00}$ are maximally far apart, there are $n$ $\lambda$-connections with $\lambda$-paths $x^0 = (x_1, x_2, \ldots, x_i, \ldots, x_n) - (x_1, x_2, \ldots, x_{i-1}, y_i, \ldots, y_n)$ and $(x_1, x_2, \ldots, x_i, \ldots, y_n) - (y_1, y_2, \ldots, y_n)$ = $y^0$, and because there are at most $n - 2$ (other) faults, at least one of these connections is fault free with $i \geq 2$. This implies the solution of $\lambda$-paths $x^1 = (x_1, x_2, \ldots, x_i, \ldots, y^0) - (x_1, x_2, \ldots, x_{i-1}, y_i, \ldots, y_n)$ (which goes through the points $(x_1, x_2, \ldots, x_i, \ldots, y_i, \ldots, y_n)$, $(x_1, x_2, \ldots, x_{i-1}, y_i, \ldots, y_n)$, $(x_1, x_2, \ldots, x_{i-1}, y_i, \ldots, y_n)$, $(x_1, x_2, \ldots, x_{i-1}, y_i, \ldots, y_n)$) and $(x_1, x_2, \ldots, x_{i-1}, y_i, \ldots, y_n) - (y_1, y_2, \ldots, y_3, \ldots, y_n)$ = $y^3$ (which goes through the points $(x_1, x_2, \ldots, x_i, \ldots, y_i, \ldots, y_n)$, $(x_1, x_2, \ldots, x_{i-1}, y_i, \ldots, y_n)$, $(x_1, x_2, \ldots, x_{i-1}, y_i, \ldots, y_n)$, $(x_1, x_2, \ldots, x_{i-1}, y_i, \ldots, y_n)$, $(y_1, y_2, x_3, \ldots, y_3, \ldots, y_n)$, $(y_1, y_2, x_3, \ldots, y_3, \ldots, y_n)$, $(y_1, y_2, x_3, \ldots, y_3, \ldots, y_n)$)), which solution avoids the elements $y^4$ and $y^4 y^3$. If $y^0$ and $x^0$ are not maximally far apart, then they agree on at least one coordinate $i$ and let the value of this coordinate be $x_i$. If every faulty element has $x_i$ on the $i$th coordinate (a faulty line has $x_i$ on the $i$th coordinate, if either of its endpoints has $x_i$ on the $i$th coordinate), there are $\lambda$-paths $x^0 = (x_1, x_2, \ldots, x_{i-1}, x_i, y_{i+1}, \ldots, y_n)$ and $(x_1, x_2, \ldots, x_{i-1}, x_i, y_{i+1}, \ldots, y_n) - y^0$. These paths imply the solution of $\lambda$-paths $(x_1, x_2, \ldots, x_{i-1}, x_i, y_{i+1}, \ldots, y_n) - (x_1, x_2, \ldots, x_{i-1}, y_i, \ldots, y_n)$ = $x^4 y_i, \ldots, y_n$ and $(x_1, x_2, \ldots, x_{i-1}, y_i, \ldots, y_n) - (y_1, y_2, \ldots, y_3, \ldots, y_n)$. This solution does not use the elements $y^4$ and $y^4 y^3$, if $i \geq 2$. If $i = 1$, we use the $\lambda$-path $(x_1, x_2, \ldots, x_1) - (x_1, y_2, \ldots, y_{j-1}, y_j, y_{j+1}, \ldots, y_n)$ and $(x_1, y_2, \ldots, y_{j-1}, y_j, y_{j+1}, \ldots, y_n) - (y_1, y_2, \ldots, y_{j-1}, y_j, y_{j+1}, \ldots, y_n)$. Because there are at most $n - 2$ other faults there is at least one fault free line inside $Q_n \cap C_4$, and thus there is at least one coordinate $y_1 \neq y_1$ such that the line $(y_1, y_2, \ldots, y_{j-1}, y_j, y_{j+1}, \ldots, y_n), (y_1, y_2, \ldots, y_{j-1}, y_j, y_{j+1}, \ldots, y_n)$ is fault free. If $y^0$ and $x^0$ are not maximally far apart, then they agree on at least one coordinate $i$ and let the value of this coordinate be $x_i$. If every faulty element has not the value $x_i$ in the $i$th coordinate, then by removing from $Q_n \cap C_4$ each point (and all incident lines) having the value $x_i$ in the $i$th coordinate and
thereafter ignoring the coordinate $i$ with the value $x_i$ we obtain the graph $Q_{n-1} \odot C_4$ with at most $n-1$ faults (at least one fault is also removed). We can now look the solution in this smaller graph $Q_{n-1} \odot C_4$ which is an induced subgraph of $Q_n \odot C_4$. This reduction process can be continued up to the graph $Q_2 \odot C_4 = C_8$ where at least one desired $\lambda$-connection can be found by inspection. This proves (iv).

Let $x^0 \neq y^0, x^0$ and/or $y^0$ be faulty, and $y^2 = y^2$. We have four subcases: (v) $x^1 \in V_2^1$ and $y^2 \in V_2^2$, (vi) $x^1 \in V_2^1$ and $y^2 \in V_1^2$, (vii) $x^1 \in V_1^1$ and $y^2 \in V_2^2$, and (viii) $x^1 \in V_1^1$ and $y^2 \in V_1^2$.

(v) Because $x^0$ and/or $y^0$ are faulty, there are at most $n-1$ (other) faults in the projection $Q_n^{00}$ of the graph induced by the point set $V(Q_1^n) \cup V(Q_2^n)$. Thus there is at least one $\lambda$-connection $x^{00} - y^{00}$ in $Q_n^{00}$ because $x^1 \in V_2^1$ and $y^2 \in V_2^2$, a $\lambda$-connection $x^1 - y^2$ in the graph induced by the point set $V(Q_1^n) \cup V(Q_2^n)$ does not use the possibly faulty elements $x^1 x^4, x^4, x^3, x^3 x^2, y^1 y^4, y^4, y^3, y^3 y^2$, and thus in the case (v) the $\lambda$-connection $x^{00} - y^{00}$ in $Q_n^{00}$ generates the $\lambda$-connection $x^1 - y^2$ of the theorem.

(vi) Because there are in $Q_n^{00}$ at most $n-1$ faults there is at least one fault free $x^{00} - y^{00}$-connection in $Q_n^{00}$. If this connection consists of a pair of $\lambda$-paths $x^{00} - z^{00}, z^{00} - y^{00}$, the $\lambda$-paths $x^1 - z^2, z^2 - y^2$ solve the problem. If the $\lambda$-connection $x^{00} - y^{00}$ consists of a single $\lambda$-path with $d(x^{00}, y^{00}) \geq 3$ and it contains the points $x^{00}, z_1^{00}, z_2^{00}, z_3^{00}, ..., y^{00}$, then this $\lambda$-path generates the solution with the points $x^1, z_1^1, z_2^1, z_3^1, z_4^1, ..., y^2$. Note that $d(x^{00}, y^{00}) \neq 2$, because $x^1 \in V_2^1$ and $y^2 \in V_2^1$. Thus, if $d(x^{00}, y^{00}) = 1$, then the $\lambda$-path $x^{00} - y^{00}$ (the line $x^{00} y^{00}$) generates the $\lambda$-path $x^1, y^1, y^2$ where the point $y^1$ or the line $y^1 y^2$ may be faulty. If the elements $y^1$ and $y^1 y^2$, are fault free we are done, and thus we assume that at least one of them is faulty. If there are at most $n - 2$ other faults in $Q_n^{00}$, then we can mark the line $x^{00} y^{00}$ faulty, have at most $n - 1$ faults and find a solution as above.

Thus the graph induced by the pointset $V(Q_1^n) \cup V(Q_2^n)$ contains $n-1$ other faults and the faulty element $y^1$ or $y^1 y^2$, in total $n$ faults. But then the subhypercube $Q_n^2$ can contain at most $n-1$ faults and hence there is a fault free $\lambda$-connection $x^2 - y^2$. If this connection is the line $x^2 y^2$ we have the solution $x^1 - x^3, x^3 - y^2$ of $\lambda$-paths because all faults are in the subgraph induced by the point set $V(Q_1^n) \cup V(Q_2^n)$. If the connection consists of two $\lambda$-paths $x^2 - w^2, w^2 - y^2$, we have
the solution $x^1 - w^3$ and $w^3 - y^2$ of $\lambda$-paths. This completes the proof of the case (vi).

(vii) If the subhypercube $Q_n^2$ contains all $n$ faults, then the $\lambda$-paths $x^1 - x^4$ and $x^4 - y^2$ are the solution of the problem. Note that by 3 of Definition 2 the $\lambda$-path $x^4 - y^2$ uses the fault free line $y^3$. Assume that the elements $x^1$, $x^2$ and $x^3$ are fault free. If $Q_n^2$ contains at most $n - 1$ faults, there is fault free $\lambda$-connection $x^2 - y^2$ to the beginning of which we can add the elements $x^1$ and $x^1$, $x^2$ by 2 of Definition 2. Thus we assume that at least one of the elements $x^1$, $x^2$, $x^3$ is faulty and thus there are at most $n - 1$ other faults. If all $n - 1$ other faults are in the subgraph induced by the point set $V(Q_n^1) \cup V(Q_n^2)$, then its projection $Q_n^0$, where we have ignored the faultiness of $x^0$, contains at least one fault free $\lambda$-connection $x^0 - y^0$. If this connection consists of a pair of $\lambda$-paths $x^0 - w^0$, $w^0 - y^0$, then the $\lambda$-paths $x^1 - w^1$, $w^1 - y^2$ are the solution of the problem. If the connection consists of a single $\lambda$-path $x^0 - y^0$ with $d(x^0, y^0) \geq 3$, then the path contains a point $w^0 \neq x^0$, $y^0$ with $w^1 \in V^1_1$ and the $\lambda$-paths $x^1 - w^1$, $w^1 - y^2$ satisfy the demands of the theorem.

Because $x^1 \in V^1_1$ and $y^2 \in V^2_2$, the case $d(x^0, y^0) = 2$ is absurd, and hence the case $d(x^0, y^0) = 1$ remains.

If there are at most $n - 2$ other faults in the subgraph induced by the point set $V(Q_n^1) \cup V(Q_n^2)$, then we can denote the line $x^0 - y^0$ faulty and obtain the case of at most $n - 1$ faults with $d(x^0, y^0) \neq 1$ considered above. Thus we have the case: there are exactly $n - 1$ other faults in the subgraph induced by the point set $V(Q_n^1) \cup V(Q_n^2)$ with $d(x^0, y^0) = 1$. Because $x^0$ and $y^0$ are adjacent, their coordinate representations differ on one coordinate $i$. Let $x^1 = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ and $y^1 = (x_1, \ldots, x_{i-1}, \bar{x}_{i+1}, \ldots, x_n)$. If $i \neq n$, then we can choose a point $b^1 = (x_1, \ldots, x_{i-1}, w_{i+1}, x_{i+2}, \ldots, x_n)$ which is adjacent to $y^1$, $b^1 \in V^1_1$ and the $\lambda$-path $x^1 - b^1$ goes through the point $y^1$. Let $x^1 = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. We have now $n - 1$ disjoint $\lambda$-paths with the points $x^1 = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$, $(x_1, x_2, \ldots, x_{i-1}, \bar{x}_{i+1}, x_{i+2}, \ldots, x_n)$, $(x_1, x_2, \ldots, x_{i-1}, \bar{x}_{i+1}, \ldots, x_n)$, $(x_1, x_2, \ldots, x_{i-1}, \bar{x}_{i+1}, \ldots, x_n)$, $(x_1, x_2, \ldots, x_{i-1}, \bar{x}_{i+1}, \ldots, x_n)$, $(x_1, x_2, \ldots, x_{i-1}, \bar{x}_{i+1}, \ldots, x_n)$, $(x_1, x_2, \ldots, x_{i-1}, \bar{x}_{i+1}, \ldots, x_n)$, $(x_1, x_2, \ldots, x_{i-1}, \bar{x}_{i+1}, \ldots, x_n)$, $(x_1, x_2, \ldots, x_{i-1}, \bar{x}_{i+1}, \ldots, x_n)$, $(x_1, x_2, \ldots, x_{i-1}, \bar{x}_{i+1}, \ldots, x_n)$, $(x_1, x_2, \ldots, x_{i-1}, \bar{x}_{i+1}, \ldots, x_n)$, $(x_1, x_2, \ldots, x_{i-1}, \bar{x}_{i+1}, \ldots, x_n)$. Because there are at most $n - 1$ other faults in the subgraph induced by the point set $V(Q_n^1) \cup V(Q_n^2)$, one of these $n$ disjoint $\lambda$-paths is fault free, and thus one of the $\lambda$-connections $x^1 - b_i^4$, $b_i^4 - y^2$ ($i = 1, \ldots, n - 1$) and $x^1 - x^4$, $x^4 - y^2$ satisfies the demands of the
theorem, and the proof of the case (vii) is complete.

(viii) If all $n$ faults are in the subhypercube $Q_n^2$, then the $\lambda$-paths $x^1 - y^1$ and $y^1 - y^2$ solve the problem. If $Q_n^2$ contains at most $n - 1$ faults, there is a $\lambda$-connection $x^2 - y^2$ to which we can add the elements $x^1, x^1 x^2$ by 2 of Definition 2. This is a solution provided that the elements $x^1$ and $x^1 x^2$ are fault free. Thus we assume that at least one of the elements $x^1, x^1 x^2$ is faulty, and we have at most $n - 1$ other faults in the projection $Q_n^{00}$ of the subgraph induced by the point set $V(Q_n^1) \cup V(Q_n^2)$. If $\lambda$-connection $x^{00} - y^{00}$ is the single $\lambda$-path, the desired $\lambda$-connection consists of $\lambda$-paths $x^1 - y^1$ and $y^1 - y^2$ provided that the elements $y^1, y^1 y^2$ are fault free. If the $\lambda$-connection $x^{00} - y^{00}$ consists of two $\lambda$-paths $x^{00} - w^{00}$ and $w^{00} - y^{00}$, the solution of our problem is the pair of $\lambda$-paths $x^1 - w^1, w^1 - y^2$, and if $w^1 \notin N(y^1)$, this solution does not use the elements $y^1$ and $y^1 y^2$. Thus we assume that at least one of the elements $y^1, y^1 y^2$ is faulty. This faultiness in common with the faultiness of elements $x^1 x^2, x^2$ implies at most $n - 2$ other faults. Because $x^1 \in V_1^1$ and $y^2 \in V_2^2$, we have $d(x^{00}, y^{00}) = 2, 4, 6, \ldots$. If $d(x^{00}, y^{00}) \geq 4$, the $\lambda$-path $x^{00} - y^{00}$ contains an intermediate point $w^1 \neq x^1 y^1$ with $w^1 \in V_1^1$. If the $\lambda$-path $x^{00} - y^{00}$ with $d(x^{00}, y^{00}) \geq 4$ is fault free, we have the solution of $\lambda$-paths $x^1 - w^1, w^1 - y^2$. We have the same solution if the fault free $\lambda$-connection $x^{00} - y^{00}$ consists of two $\lambda$-paths $x^{00} - w^{00}$ and $w^{00} - y^{00}$ where $w^{00} \notin N(y^{00})$ and $d(x^{00}, y^{00}) \geq 4$. Thus we have still two cases to consider: (A) The fault free $\lambda$-connection $x^{00} - y^{00}$ consists of two $\lambda$-paths $x^{00} - w^{00}$ and $w^{00} - y^{00}$ with $d(x^{00}, y^{00}) \geq 4$ and $w^{00} \in N(y^{00})$, and (B) the fault free $\lambda$-connection $x^{00} - y^{00}$ consists of the two $\lambda$-paths $x^{00} - w^{00}$ and $w^{00} - y^{00}$ with $d(x^{00}, y^{00}) = 2$ and $w^{00} \in N(y^{00})$, or the $\lambda$-connection $x^{00} - y^{00}$ consists of the single $\lambda$-path $x^{00} - y^{00}$ with $d(x^{00}, y^{00}) = 2$ and the only intermediate point $w^{00}$ is adjacent to $y^{00}$.

(viii A) By Lemma 2, there are $n - 1$ disjoint $\lambda$-connections $x^{00} - y^{00}$ one of which is $\lambda$-path $x^{00} - y^{00}$ and the other $n - 2$ consists of two $\lambda$-paths $x^{00} - w^{00}$ and $w^{00} - y^{00}$ with $w^{00} \notin N(y^{00})$. Since there are at most $n - 2$ other faults, one of these connections is fault free and generates the asserted connection $x^1 - y^2$.

(viii B) By Lemma 2, there are $n - 1$ disjoint $\lambda$-connections $x^{00} - y^{00}$ one of which is the $\lambda$-path $x^{00}, z^{00}, y^{00}$. If one of the $n - 2$ $\lambda$-connections consisting of two $\lambda$-paths $x^{00} - w^{00}$ and $w^{00} - y^{00}$ with $w^{00} \notin N(y^{00})$ is fault free, we are done, and thus we assume that each of them is faulty. Note that then all $n$ faults are in the subgraph induced by the point set $V(Q_n^1) \cup V(Q_n^2)$ and the single $\lambda$-path $x^{00}, z^{00}, y^{00}$ is fault free. Let the coordinate representation of $x$ be $(x_1 x_2 \ldots x_n)$. 
The proof of Lemma 2 that shows the pair of $\lambda$-paths do not use the point $(x_1 x_2 \ldots x_{n-1} \bar{x}_n) = u$ in first path (starting from $x$) of the pair. Thus the point $u^1$ in the subhypercube $Q_n^1$ must be fault free and the $\lambda$-paths $x^1, u^1, u^4, x^4$ and $x^1, x^3, z^3, z^2, x^2$ imply now the solution of the problem. This completes the proof of the case (viii).

Let $x^0 = y^0$ and $x^0$ be fault free. If $y^1 = x^2$ and $x^1 \in V_1^1, x^2 \in V_1^2$, the line $x^1 x^2$ is fault free since $x^0$ is fault free, and the line $x^1 x^2$ is the solution of the problem. If $x^1 \in V_2^1, x^2 \in V_2^2$ and the subgraph induced by the point set $V(Q_n^1) \cup V(Q_n^2)$ contains at most $n - 1$ faults, there is at least one point $b^1 \in N(x^1), b^1 \neq x^4$ such that $x^1 b^1, b^1 b^2, b^2 x^2$ are fault free, and thus the $\lambda$-paths $x^1 - b^1$ and $b^1 - x^2$ are the paths of the theorem. If the subgraph induced by the pointset $V(Q_n^1) \cup V(Q_n^2)$ contains all $n$ faults, the lines $x^1 x^4, x^3 x^2$ are fault free and there is a fault free point $b^4 \in N(x^4), b^4 \neq x^1$, and thus the $\lambda$-paths $x^1 - b^4$ and $b^4 - x^2$ are the solution of the problem. If $y^1 = y^3 = x^1 \in V_1^1, x^3 \in V_1^3$ and the subgraph induced by the point set $V(Q_n^2) \cup V(Q_n^3)$ contains at most $n - 1$ faults, there is at least one point $b^2 \in N(x^2), b^2 \neq x^1$ such that $x^2 b^2, b^2 b^3, b^3 x^3$ are fault free, and thus the $\lambda$-paths $x^1 - b^2$ and $b^2 - x^3$ are the paths of the theorem. If the subgraph induced by the point set $V(Q_n^2) \cup V(Q_n^3)$ contains all $n$ faults, the line $x^4 x^3$ is fault free and there is a fault free point $b^1 \in N(x^1), b^1 \neq x^2$, and thus the $\lambda$-paths $x^1 - b^4$ and $b^4 - x^3$ are the solution of the problem. The case $y^1 = y^3 = x^1 \in V_2^1, x^3 \in V_2^3$ is similar to the case $x^1 \in V_1^1, x^3 \in V_1^3$ and hence omitted.

Let $x^0 = y^0$ and $x^0$ be faulty. If $y^1 = y^2 = x^2, x^1 \in V_1^1, x^2 \in V_1^2$ and $x^1 x^2$ is fault free, there is nothing to prove. Thus we assume that the line $x^1 x^2$ is faulty and thus there are at most $n - 1$ other faults. Let $x = (x_1 x_2 \ldots x_n)$, $u_i = (x_1 x_2 \ldots x_{i-1} \bar{x}_i \bar{x}_{i+1} \ldots x_n)$, $w_i = (x_1 x_2 \ldots x_{i-1} x_i \bar{x}_{i+1} \ldots x_n)$ and $u_i = (x_1 x_2 \ldots x_{i-1} x_i \bar{x}_{i+1} \bar{x}_{i+2} \ldots x_n); (i = 1, \ldots, n - 1)$. Now we have $n - 1$ disjoint pairs of $\lambda$-paths $x^1, v_1 u_1$ and $w_i^1 w_{i+1}^2, u_i, x^2$. If at least one of them is fault free, we are done, and thus we assume that each of them is faulty. We can still construct one disjoint pair of $\lambda$-paths more. Because the line $x^1 x^2$ and the $n - 1$ disjoint pairs of $\lambda$-paths are faulty, all $n$ faults are in the subgraph induced by the point set $V(Q_n^1) \cup V(Q_n^2)$. Any path of the $n - 1$ pairs constructed above does not use the point $r = (x_1 x_2 \ldots x_i \ldots x_{n-1} \bar{x}_n)$ in $Q_n^1$ and the point $t = (\bar{x}_1 x_2 \ldots x_i \ldots x_n)$. Thus, by 3 of Definition 2, we have the following fault free pair of $\lambda$-paths: $x^1 - x^4$ through the points $r^1$ and $r^4$ and $x^4 - x^2$ through $x^3, r^3$ and $r^2$. This proves the case $y^1 = y^2 = x^2, x^1 \in V_1^1, x^2 \in V_1^2$. If
$x^1 \in V^1_2, x^2 \in V^2_2$ and the subgraph induced by the point set $V(Q^1_n) \cup V(Q^2_n)$ contains at most $n - 1$ faults, there is at least one point $b^1 \in N(x^1), b^1 \neq x^4$ such that $x^1, b^1, b_2^1, b_2^2$ and $b_2^2 x^2$ are fault free, and the $\lambda$-paths $x^1 - b^1$ and $b^1 - x^2$ satisfy the demands of the theorem. If the subgraph contains all $n$ faults, then the lines $x^1 x^4$ and $x^3 x^2$ are fault free, and the $\lambda$-paths $x^1 - b^4$ (with $b^4 \in N(x^4), b^4 \neq x^1$) and $b^4 - x^2$ imply the assertion of the theorem.

Let now $y^1 = y^3 = x^3, x^1 \in V^1_2, x^3 \in V^3_2$. If the line $x^2 x^3$ and the point $x^2$ are fault free, and there is at least one fault free point $b^1 \in N(x^1), b^1 \neq x^4$, then the $\lambda$-paths $x^1 - b^2$ and $b^2 - x^3$ are the paths of the theorem. If such a point $b^1$ does not exist, all the points $b^1 \neq x^4$ adjacent to $x^1$ (or the corresponding line $x^1 b^1$) are faulty, which implies $n$ faults, and thus the line $x^1 x^4$ and $x^4$ must be fault free, and the $\lambda$-paths $x^1 b^4$ and $b^4 x^3$ with $b^4 \in N(x^4), b^4 \neq x^1$ constitute a solution of the problem. We similarly find a solution of the problem, if the elements $x^1 x^4$ and $x^4$ are fault free. Thus we assume that at least two of the elements $x^1 x^4, x^4, x^2 x^3, x^2$ are faulty and hence we have at most $n - 2$ other faults. Consider the projection $Q^0_n$: Since $Q^0_n$ is a hypercube, there is for any two disjoint points $a^0, b^0 \in N(x^0)$ a third point $c^0$ such that $d(x^0, c^0) = 2$ and $C(x^0, c^0) = \{x^0, c^0, a^0, b^0\}$. On the other hand, since there are $n$ points adjacent to $x^0$ and at most $n - 2$ (other) faults in $Q^0_n$, there must be at least one point $c^0$ such that the convex $C(x^0, c^0) = \{x^0, c^0, a^0, b^0\}$ is fault free (except the point $x^0$). Since the $\lambda$-paths are geodesics and the convex contains all corresponding geodesics, the $\lambda$-paths $x^1 - c^2$ and $c^2 - x^3$ satisfy the demands of the theorem. Note that according to 2 of Definition 2, the path $c^2 - x^3$ avoids the line $x^2 x^3$.

The case $x^1 \in V^1_1, x^3 \in V^3_1$ is analogous to the case $x^1 \in V^1_2, x^3 \in V^3_2$ and hence omitted. This completes the proof.

**Lemma 2** — Let $x$ and $y$ be two points of a hypercube $Q_n$ and let $d(x, y) \geq 2$. Then there are $n - 2$ pairs of $\lambda$-paths $x - z$ and $z - y$ with $d(x, y) \geq 2$ such that the $\lambda$-path $x - y$ and all $n - 2$ pairs of $\lambda$-paths are disjoint (except points $x$ and $y$).

**PROOF**: Let the $s + 1$th coordinate be the first one where the points $x$ and $y$ differ ($0 \leq s \leq n - 1$), and let $r$th coordinate be the last one where the points $x$ and $y$ differ ($2 \leq r \leq n$). At first we consider the coordinates $s + 1, s + 2, \ldots, t$. Let $s + 1 \leq j < t$. If $x_j = y_j$, we have two $\lambda$-paths $x, (x_1 \ldots x_{j-1} x_j y_j + x_{j+1} x_{j+2} \ldots x_n), \ldots; (x_1 \ldots x_{j-1} x_j y_j + 1 y_{j+1} \ldots y_n) = y$: note that $x_1 = y_1, \ldots, x_s = y_s$ by the choice of the points $x$ and $y$. The distance $d(z, y)$ is at least two because of the disjoint coordinates $s + 1$ and $j$. If $x_j \neq y_j$ (i.e. $x_j = y_j$), we have two $\lambda$-paths $x, x_1 \ldots x_{j-1} x_j y_j + 1 x_{j+1} x_{j+2} \ldots x_n, \ldots; (x_1 \ldots x_{j-1} x_j y_j + 1 \ldots y_{j+n}) = z$ and $z, (x_1 \ldots x_s y_{s+1} x_{s+2} \ldots x_{j-1} x_j y_j + 1 \ldots y_n), \ldots; (x_1 \ldots x_s y_{s+1} y_{s+2} \ldots y_{j-1} y_j y_j + 1 \ldots y_n) = y$. As above $d(z, y) \geq 2$ because of the disjoint coordinates $s + 1$ and
According to the construction, the pairs of the $\lambda$-paths for $j = s + 2, \ldots, t - 1$ are disjoint. In the case $j = t$ we obtain the single $\lambda$-path $x - y$, which is disjoint from the pairs of $\lambda$-paths constructed above because of the coordinate $x_j$. The number of pairs of $\lambda$-paths constructed here is $(t - 1) - (s + 2) + 1 = t - s - 2$.

When considering coordinates $1, \ldots, s$ we obtain the following pairs of $\lambda$-paths:

\begin{align*}
&x(x_1 \ldots x_{k-1} \overline{x}_k x_{k+1} \ldots x_s x_{s+1} \ldots x_n), (x_1 \ldots x_{k-1} \overline{x}_k x_{k+1} \ldots x_s x_{s+1} \ldots x_n), (x_1 \ldots x_{k-1} \overline{x}_k x_{k+1} \ldots x_s x_{s+1} \ldots x_n), \ldots, (x_1 \ldots x_{k-1} \overline{x}_k x_{k+1} \ldots x_s x_{s+1} \ldots x_n) = z \quad \text{and} \quad z, (x_1 \ldots x_{k-1} \overline{x}_k x_{k+1} x_{k+2} \ldots x_s x_{s+1} x_{s+2} \ldots y_n), (x_1 \ldots x_{k-1} x_k x_{k+1} x_{k+2} \ldots x_s y_{s+1} y_{s+2} \ldots y_n) = y,
&\text{where } x_i = y_i \text{ for } i = 1, \ldots, s. \text{ When } k \text{ has the values } 1, \ldots, s \text{ we have } s \text{ new pairs of disjoint } \lambda\text{-paths with } d(z, y) = 2. \text{ When considering coordinates } t, \ldots, n \text{ we obtain the following pairs of } \lambda\text{-paths:}
&x(x_1 \ldots x_s x_{s+1} \ldots x_t x_{t+1} \ldots x_{k-1} \overline{x}_k x_{k+1} \ldots x_n), (x_1 \ldots x_s x_{s+1} \ldots x_t x_{t+1} \ldots x_{k-1} \overline{x}_k x_{k+1} x_{k+2} \ldots x_n), \ldots, (x_1 \ldots x_s y_{s+1} \ldots x_t x_{t+1} \ldots x_{k-1} \overline{x}_k x_{k+1} x_{k+2} \ldots x_n), (x_1 \ldots x_s y_{s+1} \ldots y_t x_{t+1} \ldots x_{k-1} \overline{x}_k x_{k+1} x_{k+2} \ldots x_n), (x_1 \ldots x_s \overline{x}_k x_{k+1} \overline{x}_k x_{k+1} x_{k+2} \ldots x_n), (x_1 \ldots x_s \overline{x}_k x_{k+1} x_{k+2} \ldots x_n), \ldots, (x_1 \ldots x_s \overline{x}_k x_{k+1} \overline{x}_k x_{k+1} x_{k+2} \ldots x_n),
&\text{with } x_i = y_i \text{ for } i = 1, \ldots, s \text{ and } i = t + 1, \ldots, n. \text{ When } k \text{ has the values } t, \ldots, n - 1, \text{ we have } (n - 1) - t + 1 = n - t \text{ new disjoint pairs of } \lambda\text{-paths } x - y. \text{ Moreover } d(z, y) = 2 \text{ by the coordinates } k \text{ and } k + 1. \text{ Thus we have } t - s - 2 + s + n - t = n - 2 \text{ disjoint pairs of } \lambda\text{-paths as asserted.}

Note that the grouping of points in $Q_n$ into two maximum independent sets $V_1$ and $V_2$ is not the only possible grouping. The other groupings make up the topic of further research. Similarly, the dot product where $Q_n$ is replaced by $Q_n \odot C_4$ is a topic for further research as well as the properties of the more general graphs like \((\ldots (((Q_n \odot C_4)) \odot C_4) \odot C_4) \ldots\).

REFERENCES