SYMmetric DUAL MultIOBJECTIVE PROGRamming PROBLEMS WITH PSEUdo-INVEXITY

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(Received 14 March 2001; accepted 5 November 2001)

A pair of symmetric dual multiobjective programming problems is formulated and duality theorems are established for pseudo-invex functions.

Key Words: Symmetric Dual Multiobjective Programming; Pseudo-Invex; Proper Efficiency

1. INTRODUCTION

Symmetric duality in nonlinear programming was introduced by Dorn\textsuperscript{4}. According to him a program and its dual are symmetric if the dual of the dual is the primal. In the single objective cases. Dantzing et al.\textsuperscript{2} first formulated a pair of symmetric dual nonlinear programs in which the dual of the dual equals the primal, and established the weak and strong duality theorems for these problems concerning convex and concave functions. In 1981, Mond and Weir\textsuperscript{12} formulated another pair of symmetric dual nonlinear programs under weaker assumptions on the objective and constraints.

Duality in multiobjective programming has been studied extensively in the literature. For example, Egudo\textsuperscript{5} has dealt with proper efficiency. Hanan\textsuperscript{8} and Kornbluth\textsuperscript{10} used duality theory for sensitivity analysis in multiobjective linear programming and isermann\textsuperscript{9} developed some relations between a dual pair. Mazzoleni\textsuperscript{11} discussed duality and reciprocity for vector programming and Rodder\textsuperscript{13} discussed generalized saddle point theory for linear vector optimum problems. But it should be noted that there is no enough literature dealing with the multiobjective symmetric dual problem.

The purpose of this paper is to formulate a pair of symmetric dual multiobjective programming problems under suitable pseudo-invexity assumptions and this will fill up a gap in the existing literature. Subsequently the weak and the strong duality theorems are established for pseudo-invex functions.
2. NOTATION AND DEFINITIONS

Let $C_1, C_2$ be closed convex cones in $R^n$ and $R^m$ respectively. For $i = 1, 2$, $C_i^*$ is called the polar cone of $C_i$ and is defined as follows.

$$C_i^* = \{ z : x^T z \leq 0 \, \forall \, x \in C_i \}$$

where $x^T$ denotes the transpose of $x$.

**Definition 1** — $k : C_1 \times C_2 \mapsto R$ is said to be pseudo-invex in the first variable at $u$ if \( \exists \) $\eta_1 : C_1 \times C_1 \mapsto R^n$ such that

$$\eta(x, u)^T \nabla_1 k(u, y) \geq 0$$

$$\Rightarrow k(x, y) \geq k(u, y) \, \forall \, x \in C_1, y \in C_2$$

**Definition 2** — For a vector function $k : C_1 \times C_2 \mapsto R^l$, $\nabla_1 k(x, y)$ and $\nabla_2 k(x, y)$ respectively denote the first order partial differentials with respect to first and second variables. Similarly $\nabla_{11} k(x, y)$ and $\nabla_{22} k(x, y)$ denote the second order partial differentials with respect to first and second variables respectively. The notation $\nabla_{21} k(x, y)$ stands for the partial differential of $\nabla_1 k(x, y)$ with respect to the second variable.

**Definition 3** — $k : C_1 \times C_2 \mapsto R$ is said to be pseudo-invex in the second variable at $u$, if there exists $\eta_2 : C_2 \times C_2 \mapsto R^m$ such that

$$\eta_2(y, u)^T \nabla_1 k(x, u) \geq 0$$

$$\Rightarrow k(x, y) \geq k(x, u) \, \forall \, x \in C_1, y \in C_2$$

**Definition 4** — If $f : C_1 \times C_2 \mapsto R^l$, is a vector valued function. then $f$ is pseudo-invex in the first variables (second) variable means all its components are pseudo-invex in the first variable (second variable)

3. THE PRIMAL AND DUAL PROBLEMS

We formulate the following pair of vector valued symmetric dual programs.

$$(SPV) \text{ Min } f(x, y) = (k_1(x, y) - y^T \nabla_2 k_1(x, y), \ldots, k_l(x, y) - y^T \nabla_2 k_l(x, y))$$

subject to $(x, y) \in C_1 \times C_2$,

$$\nabla_2 k_i(x, y) \in C_2^*$$

for $i = 1, 2, \ldots, l$.

where each $k_i(x, y), i = 1, 2, \ldots, l$ is a real-valued function defined on $C_1 \times C_2$ and $C_1, C_2$ are
closed convex cones in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. The dual of (SPV) is given:

$$(SDV) \quad \text{Max } g(u, v) = (k_1(u, v) - u^T \nabla_1 k_1(u, v), \ldots, k_l(u, v) - u^T \nabla_1 k_l(u, v))$$

subject to

$$(u, v) \in C_1 \times C_2,$$

$$- \nabla_1 k_i(u, v) \in C_{i1}^*, \text{ for } i = 1, 2, \ldots, l.$$

The corresponding parametric problem is as follows

$$(SPV) \quad \text{Min } r^T f(x, y)$$

subject to

$$(x, y) \in C_1 \times C_2,$$

$$\nabla_2 k_i(x, y) \in C_{i2}^*, \text{ for } i = 1, 2, \ldots, l$$

$$(r_1, r_2, \ldots, r_l) \in \mathbb{R}^l, \text{ with } \sum_{i=1}^l r_i = 1, r_i \geq 0.$$

$$(SDV) \quad \text{Max } r^T g(u, v)$$

subject to

$$(u, v) \in C_1 \times C_2,$$

$$- \nabla_1 k_i(u, v) \in C_{i1}^*, \text{ for } i = 1, 2, \ldots, l$$

$$\sum_{i=1}^l r_i = 1, r_i \geq 0.$$

**Definition** $^6$ — A point $(\overline{x}, \overline{y})$ is said to be properly efficient for (SPV) if it is efficient for (SPV) and there exists a section $M > 0$ such that

$$k_i(\overline{x}, \overline{y}) - \overline{y}^T \nabla_2 k_i(\overline{x}, \overline{y}) - k_i(x, y) + y^T \nabla_2 k_i(x, y) \leq M (k_j(x, y) - y^T \nabla_2 k_j(x, y - k_j(\overline{x}, \overline{y}) + y^T \nabla_2 k_j(x, y))$$

for some $j$ such that

$$k_j(x, y) - y^T \nabla_2 k_j(x, y) > k_j(\overline{x}, \overline{y}) - \overline{y}^T \nabla_2 k_j(\overline{x}, \overline{y})$$

and

$$k_i(x, y) - y^T \nabla_2 k_i(x, y) < k_i(x, y) - \overline{y}^T \nabla_2 k_i(x, y)$$

for

$$(x, y) \in C_1 \times C_2.$$

We now state the following two lemmas whose proofs are the extensions of Geoffrion's$^6$
Theorems 1 and 2 respectively.

**Lemma 1** — If for fixed $0 < r \in R^l$, $(\bar{x}, \bar{y})$ is a solution of $(SPV)$, then $(\bar{x}, \bar{y})$ is a properly-efficient solution of $(SPV)$.

**Lemma 2** — If $(\bar{x}, \bar{y})$ is properly efficient solution of $(SPV)$ and a constraint qualification is satisfied at $(\bar{x}, \bar{y})$, then there exist $\bar{r} \in R^l$, $\bar{y} \in C_2$ such that $(\bar{x}, \bar{r}, \bar{y})$ is feasible for $(SDV)$.

Geoffrion proved the second lemma by assuming that a Kuhn-Tucker constraint qualification is satisfied there. But Hartley in his Theorem 3 showed that the Kuhn-Tucker constraint qualification can be replaced by any other constraint qualification for differentiable functions.

**Theorem 1** — If for fixed $0 < r \in R^l$, $(\bar{u}, \bar{v})$ solves $(SDV_\lambda)$, then $(\bar{u}, \bar{r}, \bar{v})$ is a properly-efficient solution of $(SDV)$.

**PROOF** : See Das and Nanda.

**Theorem 2 (Weak Duality)** — For fixed $y$, let $k_i(\bar{x}, \cdot)$ and $k_i(\cdot, \bar{y})$ be pseudo-invex in second and first variables respectively (with respect to $\eta_2$ and $\eta_1$). Let $(\bar{x}, \bar{y})$ and $(\bar{u}, \bar{v})$ be the feasible solutions of $(SPV)$ and $(SDV)$ respectively with $\eta_2(\bar{v}, \bar{y}) \in C_2$ and $\eta_1(\bar{x}, \bar{u}) \in C_1$. Then $f(\bar{x}, \bar{y}) \geq g(\bar{u}, \bar{v})$ for all $\bar{x}, \bar{u} \in C_1$ and $\bar{y}, \bar{v} \in C_2$.

**PROOF** : As $-k_i(x, y)$ are pseudo-invex with respect to $\eta_2$ in second variable on $C_2 \forall i = 1, 2, \ldots, l$. $\bar{u}, \bar{v} \in C_2$ with $\eta_2(\bar{u}, \bar{y}) \in C_2$ and since $\nabla_x k_i(\bar{x}, \bar{y}) \in C_2^*$, we have

$$\eta_2^T(\bar{u}, \bar{y}) \nabla_x k_i(\bar{x}, \bar{y}) \leq 0$$

i.e.

$$\eta_2^T(\bar{u}, \bar{y}) \nabla_x (-k_i(\bar{x}, \bar{y})) \geq 0$$

$$\Rightarrow k_i(\bar{x}, \bar{y}) \geq k_i(\bar{u}, \bar{v}) \quad \forall \; i = 1, 2, \ldots, l \quad \text{... (1)}$$

Similarly as $k_i(x, y)$ are pseudo-invex with respect to $\eta_1$ in first variable on $C_1 \forall i = 1, 2, \ldots, l$.

$x, u \in C_1$ with $q_1(x, u) \in C_1$ and since $-\nabla_1 k_i(u, v) \in C_1^*$, we have

$$-\eta_1(\bar{x}, u) \nabla_1 (k_i(u, v)) \leq 0$$

$$\Rightarrow k_i(\bar{x}, \bar{v}) \geq k_i(u, v) \quad \forall \; i = 1, 2, \ldots, l \quad \text{... (2)}$$

From (1) and (2) we have

$$k_i(\bar{x}, \bar{y}) \geq k_i(\bar{u}, \bar{v}) \quad \text{... (3)}$$

Also from the primal and dual constraints we have. $\nabla_x k_i(\bar{x}, \bar{y}) \in C_2^*$ and $-\nabla_1 k_i(\bar{u}, \bar{v}) \in C_1^*.$ Thus for $\bar{y} \in C_2$ and $\bar{u} \in C_1$, we have

$$\bar{y}^T \nabla_x k_i(\bar{x}, \bar{y}) \leq 0 \quad \text{... (4)}$$
\[ -\mathbf{u}^T \mathbf{V}_1 k_i (\mathbf{u}, \mathbf{v}) \leq 0 \]

i.e.

\[ -\mathbf{y}^T \mathbf{V}_2 k_i (\mathbf{x}, \mathbf{y}) \geq 0 \quad \text{and} \quad \mathbf{u}^T \mathbf{V}_1 k_i (\mathbf{u}, \mathbf{v}) \geq 0. \]

From (3)

\[ k_i (\mathbf{x}, \mathbf{y}) - \mathbf{y}^T \mathbf{V}_2 k_i (\mathbf{x}, \mathbf{y}) + \mathbf{u}^T \mathbf{V}_1 k_i (\mathbf{u}, \mathbf{v}) \geq \mathbf{y}^T \mathbf{V}_2 k_i (\mathbf{x}, \mathbf{y}) - \mathbf{u}^T \mathbf{V}_1 k_i (\mathbf{u}, \mathbf{v}) \]

i.e. \[ k_i (\mathbf{x}, \mathbf{y}) - \mathbf{y}^T \mathbf{V}_2 k_i (\mathbf{x}, \mathbf{y}) \geq k_i (\mathbf{u}, \mathbf{v}) - \mathbf{u}^T \mathbf{V}_1 k_i (\mathbf{u}, \mathbf{v}) \]

\[ \Rightarrow \sum_{i=1}^{l} r_i [k_i (\mathbf{x}, \mathbf{y}) - \mathbf{y}^T \mathbf{V}_2 k_i (\mathbf{x}, \mathbf{y})] \geq \sum_{i=1}^{l} r_i [k_i (\mathbf{u}, \mathbf{v}) - \mathbf{u}^T \mathbf{V}_1 k_i (\mathbf{u}, \mathbf{v})] \]

\[ \Rightarrow r^T f (\mathbf{x}, \mathbf{y}) \geq r^T g (\mathbf{x}, \mathbf{y}). \]

Hence the Theorem follows:

**Theorem 3 (Strong Duality)** — Let \((\mathbf{x}, \mathbf{y})\) be a properly efficient solution of \((SPV)\) at which a constraint qualification is satisfied. Then there exists \(r > 0\) such that \((\mathbf{x}, \mathbf{r}, \mathbf{y})\) and \(\mathbf{y}^T \mathbf{V}_2 k_i (\mathbf{x}, \mathbf{y}) = 0\). Assume the conditions of the Theorem 2 are satisfied, then \((\mathbf{x}, \mathbf{r}, \mathbf{y})\) is properly efficient solution of \((SDV)\).

**Proof:** By Lemma 2. \(\exists r\) such that \((x, r, y)\) is feasible in \((SDV)\) and \(\mathbf{y}^T \mathbf{V}_2 k_i (\mathbf{x}, \mathbf{y}) = 0\). By the weak duality Theorem 2 for all feasible \((\mathbf{u}, \mathbf{r}, \mathbf{v})\) in \((SDV)\) we have

\[ \Rightarrow r^T f (x, y) \geq r^T g (u, v). \]

Thus for \((x, y)\) solves \((SDV)\). Since \(r > 0\), \((x, r, y)\) is properly efficient solution of \((SDV)\) by Theorem 1.

**REFERENCES**