SUFFICIENT CONDITIONS FOR CARATHÉODORY FUNCTIONS

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For Carathéodory functions \( p(z) \) which are analytic in the open unit disk \( U \) with \( p (0) = 1 \), Miller (Bull. Amer. Math. Soc. 81 (1975), 79-81) has shown some sufficient conditions applying the differential inequalities. The object of the present paper is to derive some improvements of results by Miller.

Key Words: Analytic Function; Carathéodory Function; Subordination

1. INTRODUCTION

Let \( A \) be the class of functions \( p(z) \) of the form

\[
p (z) = 1 + p_1 z + p_2 z^2 + \ldots
\]

which are analytic in the open unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \). If \( p (z) \) in \( A \) satisfies \( \text{Re} \ p(z) > 0 \) for \( z \in U \), then we say that \( p (z) \) is the Carathéodory function. For Carathéodory functions, Miller\(^1\) has given
Theorem A — Let \( p(z) \) be in the class \( A \).

(i) If \( \text{Re} \left\{ p(z)^2 + z p'(z) \right\} > 0 \) \((z \in U)\), then \( \text{Re} \ p(z) > 0 \) \((z \in U)\).

(ii) If \( \text{Re} \left\{ p(z) + \alpha z p'(z) \right\} > 0 \) \((z \in U)\) for some \( \alpha (\alpha \geq 0) \), then \( \text{Re} \ p(z) > 0 \) \((z \in U)\).

(iii) If \( p(z) \neq 0 \) \((z \in U)\) and \( \text{Re} \left\{ p(z) - \frac{z p'(z)}{p(z)^2} \right\} > 0 \) \((z \in U)\), then \( \text{Re} \ p(z) > 0 \) \((z \in U)\).

Let \( f(z) \) and \( g(z) \) be analytic in \( U \). If there exists an analytic function \( w(z) \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) \((z \in U)\) such that \( f(z) = g(w(z)) \), then \( f(z) \) is said to be subordinate to \( g(z) \) in \( U \). We denote this subordination by \( f(z) < g(z) \). We note that the subordination \( f(z) < g(z) \) implies that \( f(U) \subset g(U) \). Applying the subordination principles, we improve Theorem A by Miller\(^1\). To prove our results for Carathéodory functions, we have to recall here the following lemma due to Nunokawa\(^3\) (also due to Miller and Mocanu\(^2\)).

Lemma — Let \( p(z) \in A \) and suppose that there exists a point \( z_0 \in U \) such that \( \text{Re} \ p(z) > 0 \) for \( |z| < |z_0| \) and \( \text{Re} \ p(z_0) = 0 \) with \( p(z_0) \neq 0 \). Then we have

\[
 z_0 p'(z_0) \leq -\frac{1}{2} (1 + a^2),
\]

where \( p(z_0) = ia \ (a \neq 0) \).

2. Subordination Theorems for Carathéodory Functions

Our first result for Carathéodory functions is contained in

Theorem 1 — Let \( p(z) \in A \) and \( w(z) \) be analytic in \( U \) with \( w(0) = \alpha \) and \( w(z) \neq k \) \((k \in \mathbb{R}, z \in U)\). If

\[
\alpha p(z)^2 + \beta z p'(z) < w(z),
\]

then \( \text{Re} \ p(z) > 0 \) \((z \in U)\), where \( \beta > 0, \alpha \geq -\frac{\beta}{2} \), and \( k \leq -\frac{\beta}{2} \).

PROOF: Let us suppose that there exists a point \( z_0 \in U \) such that

\[
\text{Re} \ p(z) > 0 \ \text{for} \ |z| < |z_0|,
\]

and

\[
\text{Re} \ p(z_0) = 0 \ (p(z_0) \neq 0).
\]

Then Lemma gives that \( p(z_0) = ia \ (a \neq 0) \) and \( z_0 p'(z_0) \leq -\frac{1}{2} (1 + a^2) \). It follows that

\[
\alpha p(z_0)^2 + \beta z_0 p'(z_0) = -\alpha a^2 + \beta z_0 p'(z_0)
\]
\[ \leq \frac{1}{2} \left( \beta + (2 \alpha + \beta) a^2 \right) \]

\[ \leq -\frac{\beta}{2}. \]  \hspace{1cm} \text{(2.2)}

Since \( w(0) = \alpha \) and \( w(e^{i\theta}) \leq -\frac{\beta}{2} \), the inequality (2.2) contradicts our condition (2.1).

Therefore \( p(z) > 0 \) for all \( z \in U \). \hspace{1cm} \square

Remark 1: Theorem 1 is the improvement of (i) of Theorem A by Miller\(^1\).

Corollary 1 — If \( p(z) \in A \) satisfies

\[ \alpha p(z)^2 + \beta z p'(z) < \frac{2 \alpha + \beta}{2} \left( \frac{1+z}{1-z} \right)^2 - \frac{\beta}{2}, \]  \hspace{1cm} \text{(2.3)}

where \( \beta > 0 \) and \( \alpha \geq -\frac{\beta}{2} \), then \( \Re p(z) > 0 \) (\( z \in U \)).

PROOF: Taking

\[ w(z) = \frac{2 \alpha + \beta}{2} \left( \frac{1+z}{1-z} \right)^2 - \frac{\beta}{2}, \]  \hspace{1cm} \text{(2.4)}

in Theorem 1, we see that \( w(z) \) is analytic in \( U \), \( w(0) = \alpha \) and

\[ w(e^{i\theta}) = \frac{2 \alpha + \beta}{2} \left( \frac{1+e^{i\theta}}{1-e^{i\theta}} \right)^2 - \frac{\beta}{2} \leq -\frac{\beta}{2}. \]  \hspace{1cm} \text{(2.5)}

Thus \( w(z) \) satisfies the conditions in Theorem 1. \hspace{1cm} \square

Theorem 2 — Let \( p(z) \in A \) and \( w(z) \) be analytic in \( U \) with \( w(0) = \alpha \) and \( w(z) \neq ik \) \( (k \in \mathbb{R}, z \in U) \). If

\[ \alpha p(z) + \beta \frac{z p'(z)}{p(z)} < w(z), \]  \hspace{1cm} \text{(2.6)}

then \( \Re p(z) > 0 \) (\( z \in U \)), where \( \alpha > 0, \beta > 0 \) and \( k^2 \geq \beta (2 \alpha + \beta) \).

PROOF: From the subordination (2.6), we have \( p(z) \neq 0 \) in \( U \), because if \( p(z) \) has a zero of order \( l \) at \( z = z_0 \in U \), then we have \( p(z) = (z - z_0)^l q(z) \), where \( q(z) \) is analytic in \( U \), \( q(z_0) \neq 0 \), and \( l \) is a positive integer.

Letting \( z \to z_0 \) such that

\[ \arg (z - z_0) = \arg (z_0) - \frac{\pi}{2}, \]

we have
\[
\lim_{z \to z_0} \text{Im} \left( \alpha p(z) + \beta \frac{zp'(z)}{p(z)} \right) = \lim_{z \to z_0} \text{Im} \left( \alpha p(z) + \frac{\beta z (lq(z) + (z-z_0)q'(z))}{(z-z_0)q(z)} \right) = +\infty.
\]

This contradicts (2.6) and so we conclude that \( p(z) \neq 0 \) for all \( z \in U \). We assume that there exists a point \( z_0 \in U \) such that

\[ \text{Re } p(z) > 0 \text{ for } |z| < |z_0| \]

and

\[ \text{Re } p(z_0) = 0. \]

Then using Lemma, we have

\[
\alpha p(z_0) + \beta \frac{z_0 p'(z_0)}{p(z_0)} = i \alpha a + \frac{\beta}{ia} z_0 p'(z_0)
\]

\[ = i \left( \alpha a - \frac{\beta}{a} z_0 p'(z_0) \right) = i \nu, \quad \ldots \ (2.7)
\]

where \( \nu \) is real, because \( z_0 p'(z_0) \leq -\frac{1}{2} (1 + a^2) \). Furthermore, we have, if \( a > 0 \), then

\[ \nu \geq \alpha a + \frac{\beta}{2a} (1 + a^2) \geq \sqrt{\beta (2 \alpha + \beta)}, \quad \ldots \ (2.8) \]

and if \( a < 0 \), then

\[ \nu \leq -\alpha b - \frac{\beta}{2b} (1 + a^2) (b = -a > 0) \leq -\sqrt{\beta (2 \alpha + \beta)}. \quad \ldots \ (2.9) \]

This contradicts our condition that \( w(e^{i\theta}) = ik (|k| \geq \sqrt{\beta (2 \alpha + \beta)}) \). Thus we conclude that

\[ \text{Re } p(z) > 0 \text{ for all } z \in U. \]

Using Theorem 2, we have the following corollary.

**Corollary 2** — If \( p(z) \in A \) satisfies

\[ p(z) + \frac{zp'(z)}{p(z)} < \frac{1 + 4z + z^2}{1 - z^2}, \quad \ldots \ (2.10) \]

then \( \text{Re } p(z) > 0 \) (\( z \in U \)).
PROOF : Let us consider the case of $\alpha = \beta = 1$ in Theorem 2. Defining the function $w(z)$ by

$$w(z) = \frac{1 + 4z + z^2}{1 - z^2}, \quad \ldots \quad (2.11)$$

we know that $w(z)$ is analytic in $U$, $w(0) = 1$, and

$$w(e^{i\theta}) = \frac{2 + \cos \theta}{\sin \theta} \cdot i. \quad \ldots \quad (2.12)$$

Letting

$$g(\theta) = \left( \frac{2 + \cos \theta}{\sin \theta} \right)^2 (0 \leq \theta \leq 2\pi), \quad \ldots \quad (2.13)$$

we have $g'(\theta) = 0$ when $\cos \theta = -\frac{1}{2}$.

It follows from the above that $g(\theta) \geq 3$, that is, that $w(z) \neq ik (|k| \geq \sqrt{3})$.

Next, we derive

**Theorem 3** — If $p(z) \in A$ satisfies

$$\text{Re} \left\{ \alpha p(z) - \beta \frac{z p'(z)}{p(z)^2} \right\} > -\frac{\beta}{2} \quad (z \in U) \quad \ldots \quad (2.14)$$

for some $\alpha \geq 0$ and $\beta > 0$, then $\text{Re} \ p(z) > 0 \ (z \in U)$.

PROOF : Applying the same method as the proof of Theorem 2, the condition (2.14), gives us that $p(z) \neq 0$ in $U$, because if $p(z)$ has a zero of order $l$ at a point $z = z_0 \in U$, then we have $p(z) = (z - z_0)^l q(z)$, where $q(z)$ is analytic in $U$, $q(z_0) \neq 0$ and $l$ is a positive integer. Letting $z \to z_0$ such that

$$\arg (z - z_0) = \frac{\arg (z_0) - \arg (q(z_0))}{l + 1},$$

we see that

$$\lim_{z \to z_0} \left( \alpha p(z) - \beta \frac{z p'(z)}{p(z)^2} \right) = \lim_{z \to z_0} \left( \alpha p(z) - \beta \frac{izq(z) + (z - z_0) q'(z)}{(z - z_0)^l + q(z)^2} \right)$$

$$= -\infty.$$ 

This contradicts our condition (2.14) and so we have $p(z) \neq 0$ in $U$.

By means of Lemma, if there exists a point $z_0 \in U$ such that
Re $p(z) > 0$ for $|z| < |z_0|$

and $\Re p(z_0) = 0,$

then $p(z_0) = ia$ ($a \neq 0$) and $z_0 p'(z_0) \leq -\frac{1}{2} (1 + a^2).$

This implies that

$$\Re \left\{ \alpha p(z_0) - \beta z_0 p'(z_0) \right\} \leq -\frac{\beta}{2a^2} (1 + a^2) \leq -\frac{\beta}{2}$$

... (2.15)

which contradicts our condition (2.14). Thus $\Re p(z) > 0$ for all $z \in U.$

Remark 2: Theorem 3 is the improvement of (iii) of Theorem A by Miller$^1$.

Finally we have

Corollary 3 — If $p(z) \in A$ satisfies

$$\alpha p(z) - \beta zp'(z) < \frac{2 \alpha + \beta}{2} \left( \frac{1 + z}{1 - z} \right)^2 - \frac{\beta}{2}$$

... (2.16)

for some $\alpha \geq 0$ and $\beta > 0,$ then $\Re p(z) > 0$ ($z \in U$).

PROOF: Since the function

$$w(z) = \frac{2 \alpha + \beta}{2} \left( \frac{1 + z}{1 - z} \right)^2 - \frac{\beta}{2}$$

... (2.17)

maps the open unit disk $U$ onto the complex domain which has the slit

$$\delta = \left\{ w : \Re(w) < -\frac{\beta}{2} \right\},$$

the proof of Corollary 3 follows from the above.

REFERENCES