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STUDY OF AN ELLIPTIC EQUATION WITH A SINGULAR POTENTIAL

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This paper deals with solutions of the following semi-linear elliptic equation

$$\Delta u + u - |u|^{-2\theta} u = 0 \text{ in } I\!R^N$$

where $0 < \theta < \frac{1}{2}$. First, we prove the existence of a gound state and infinitely may radial solutions which are compactly supported. Second, we study the behaviour of the radial solutions and we give a classification of these solutions. These are of two kinds: solutions that tend to zero at infinity which are compactly supported and solutions that oscillats indifinitely and go to either 1 or −1 at infinity.

Key Words : Elliptic Equation; Singular Potential Folder Inequality

1. INTRODUCTION

In this paper is devoted to the study of the following semi linear elliptic equation

$$-\Delta u = g(u) \text{ in } I\!R^N, \qquad \dots (1.1)$$

where $g : I\!R \to I\!R$ is defined by $g(s) = s - |s|^{-2\theta} s, 0 < \theta < \frac{1}{2}$ and $u : I\!R^N \to I\!R, N \geq 3$.

Eq. (1.1), or more generally

$$-\Delta u = g(u) \text{ in } I\!R^N, \qquad \dots (1.2)$$

where $f : I\!R \to I\!R$ is a given continuous function, arises in various domains. For example, it appears in Astrophysics as the nonlinear scalar field equation, in chemical reaction, in statistical mechanical and others.

Eq. (1.2) has been intensively studied by many authors, such as Lions and Beresticki[1&2], Wi M. Ni and Nussbaum[9], Yanaguida[10], Kwong[11], Franchi, Lanconelli and Serrin[12&13], and the references therein.

However, most of these works have been done only for positive solutions and in the case where f is locally Lipschitz continuous in $(0, \infty)$. But in many physical interesting situations, f is not Lipschitz continuous. In this case various difficulties arise.

In 1996, C. Cortazar, M. Elgueta and P. Felmer[5], have considered equation (1.2) where $f(s) = s^p - s^q$ with $0 < q < 1 < p < \frac{N+2}{N-2}$. In this work, the authors have proved that positive solution of (1.2) is unique, radial, and has a compact support. In 1998, the same authors have proved the uniqueness of positive solution of (1.2) for a more general function f.

The aim of this work is the following: First, we study the existence of non trivial solutions of (1.1). Second, we show that results in [5] are still valid in our case. In particular the results in [5] regarding the compactness support of positive solutions can be generalized to solutions of (1.1) which tend to zero at infinity. Finally, we study the behaviour of all radial solutions of (1.1) and we give a classification of these solutions. The paper is organized as follows :

Section I is concerned with the existence of non trivial solutions of (1.1). It is divided into two parts. In the first part, as in [1], we consider a constrained minimizing problem and we get

Theorem 1.1 — *Eq. (1.1) has a non trivial solution u which is nonnegative, radial, non-increasing with $r = |x|$ (in the sense that if $|x| \leq |y|$ then $u(x) \geq u(y)$), and tends to zero at infinity. (This solution is called ground State)*

The natural question that follows theorem 1.1, is the existence of infinitely many solutions of (1.1), which is the subject of the second part. By applying the critical points theory developed by P. L. Liond and H. Berestycki in [2], we get

Theorem 1.2 — *Eq. (1.1) has infinitely many radial solutions which tend to zero at infinity.*

In section 2, we prove that each solution of (1.1) that tends to zero at infinity has compact support.

In section 3, which is divided into parts, we study the behaviour of the radial solutions of (1.1). In the first part, we prove that a radial nonnegative solution of (1.1) with a compact support is decreasing on its support. In the second part, we show that a radial solution with a compact suppport, has only a finite number of zeros on its support.

Finally, in section 4 we show that if u is a radial solution of (1.1) satisfying $u(0) = \lambda \in (-\lambda_\theta, \lambda_\theta)$ and $u'(0) = 0$, where λ_θ is the positive zero of G such that $G(s) = \int_0^s g(x)\, dx$, then $u(r) \to 1$ or -1 and $u'(r) \to 0$ as $r \to \infty$. At the end of this section, we classify all radial solutions of (1.1) as follows : either solutions that tend to zero at infinity and which are compactly supported, or solutions that tend to 1 or -1 at infinity.

2. Existence of Nontrivial Solutions of (1.1)

In this first section, we prove theorem 1.1 and theorem 1.2. Let $\mathcal{H} = H^1(\mathbb{R}^N) \cap L^{2(1-\theta)}(\mathbb{R}^N)$ endowed with the norm $\| \|_{\mathcal{H}} = \| \|_{H^1} \mid \| \|_{L^{2(1-\theta)}}$ by

$$S(u)=\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla(u)|^2-u^2+\frac{1}{1-\theta}|u|^{2(1-\theta)})\,dx.$$

it is well known that the critical points of functional S are the qeak solutions of eq. (1.1) that belong to $\mathcal{H}$. For that, we need a minimal regularity of S.

Lemma 2.1 — $S\in C^1(\mathcal{H},\mathbb{R})$

PROOF : Let $u\in E$, $S(u)=\frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla(u)|^2-u^2+\frac{1}{1-\theta}|u|^{2(1-\theta)}\right)dx$, the first term $\int_{\mathbb{R}^N}(|\nabla(u)|^2-u^2)\,dx$ is C^∞. For the second term, to show that is $C^1(\mathcal{H},\mathbb{R})$, and by using a classical arguments[3], it is enough to prove that

for u and $v\in\mathcal{H}$:

$$a)\quad\left|\frac{1}{t}\int_{\mathbb{R}^N}\frac{1}{2(1-\theta)}|u+tv|^{2(1-\theta)}-\frac{1}{2(1-\theta)}|u|^{2(1-\theta)}-t|u|^{-2\theta}uv\,dx\right|\xrightarrow[t\to 0]{t>0}0$$

and b) if $u_n\to u$ in $\mathcal{H}$ (strongly), then

$$\sup_{\|v\|_{\mathcal{H}}\le 1}\left|\int_{\mathbb{R}^N}(|u_n|^{-2\theta}u_n-||u|^{-2\theta}u)v\,dx\right|\xrightarrow[n\to 0]{}0$$

PROOF OF (*a*) : If $t\in(0,1)$, almost every where in $\mathbb{R}^N$ one has

$$\left|\frac{1}{t}\left\{\frac{|u+tv|^{2(1-\theta)}}{2(1-\theta)}-|u\frac{|^{2(1-\theta)}}{2(1-\theta)}-t|u|^{-2\theta}uv\right\}\right|\underset{t\to 0}{\to}0 \text{ a.e. } x\in\mathbb{R}^N.$$

Applying Lebesgue's dominated convergence theorem, the conclusion follows.

PROOF OF (*b*) : If $u_n\to u$ in $\mathcal{H}$ (strongly) then by standard arguments, $u_n\to u$ a.e. in $\mathbb{R}^N$ and there exists $u_0\in L^{2(1-\theta)}$ such that (taking subsequence if necessary) $|u_n|,|u|\le u_0$ a,e, in $\mathbb{R}^N$, therefore we have

$$|u_n|^{-2\theta}u_n-|u|^{-2\theta}u\,\Big|^{\frac{2(1-\theta)}{1-2\theta}}\le 2^{\frac{2(1-\theta)}{1-2\theta}}u_0^{2(1-\theta)}$$

this shows that $|u_n|^{-2\theta}u_n\to|u|^{-2\theta}u$ in $L^{\frac{2(1-\theta)}{1-2\theta}}$, and by Hölder inequality

$$\sup_{\|v\|_{\mathcal{H}}\le 1}\left|\int_{\mathbb{R}^N}(|u_n|^{-2\theta}u_n-|u|^{2\theta}u)\,v\,dx\right|\le$$

$$\left\{ \sup_{\| v \|_{\mathcal{H}} \le 1} \| v \|_{L^2(1-\theta)} \right\} \| | u_n |^{-2\theta} u_n - | u |^{-2\theta} u \|_{L^{\frac{2(1-\theta)}{1-2\theta}}}^{1-2\theta} \underset{n \to \infty}{\longrightarrow} 0.$$

The proof is thereby completed.

Let us now look for critical points of S. For that we follow closely the method used by P. L. Lions and H. Berestycki in ([1] and [2])

2.1. Existence of a Ground State of Equation (1.1)

We consider the constrained minimization problem

$$J = \inf_{w \in \mathcal{H}} \{ T(w); V(w) = 1 \}, \qquad \text{... (2.1)}$$

where $T(w) = \frac{1}{2} \int_{\mathbb{R}^N} | \nabla (w) |^2 dx$ and $V(w) = \frac{1}{2} \int_{\mathbb{R}^3} | w |^2 - \frac{1}{1-\theta} | w |^{2\,1-\theta)} dx.$

The problem (2.1) leads to a solution of (1.1). Indeed, if u is a solution of (2.1), since T and V are C^1, there exists a lagrangian multiplies β such that $T'(u) = \beta V'(u)$, that is at least in the distribution sense

$$- \Delta u = \beta (u - | u |^{-2\theta} u) \text{ in } \mathbb{R}^N \qquad \text{... (2.2)}$$

We observe that $\beta > 0$. Indeed, multiplying (2.2) by u and integrating by parts, we get

$$\int_{\mathbb{R}^N} | \nabla (u) |^2 dx = \beta \int_{\mathbb{R}^N} (| u |^2 - u |^{2(1-\theta)}) dx$$

$\int_{\mathbb{R}^N} (| u |^2 - | u |^{2(1-\theta)}) dx > 2V(u) = 2$, thus $\beta > 0$. Let $u_\alpha (x) = u \left(\frac{x}{\alpha} \right) \alpha \ge 0$, in $\mathbb{R}^N$ then u_α satisfies

$$- \Delta u_\alpha = \frac{\beta}{\alpha^2} (u_\alpha - | u_\alpha |^{-2\theta} u_\alpha) \qquad \text{... (2.3)}$$

choosing $\alpha = \sqrt{\beta}$, we obtain a solution of (1.1).

***Theorem* 2.1** — *The minimization problem (2.1) has a solution $v \in \mathcal{H}$ which is nonnegative, radial and non increasing with $r = |x|$. Furthermore, there exists of Lagrangian multiplies $\beta > 0$ such that v satisfies (2.2).*

PROOF : We prove theorem 2.1 in 3 steps

Step 1 — Selection of the Minimization Sequence

The set $\{ w \in \mathcal{H}$ such that $V(w) = 1 \}$ is non empty (see [1]). There exists $(u_n) \subset \mathcal{H}$ a minimizing sequence of problem (2.1), such that $V(u_n)=1$ and $\lim_{n\to\infty} T(u_n)=J\geq 0$. We associate to (u_n) a sequence denoted by (v_n) which is radial, nonnegative and non-increasing, with $r=\|x\|\cdot(v_n)$. is also a minimizing sequence of problem 2.1), with $V(v_n)=1$ and $\lim_{n\to\infty} T(v_n)=J\geq 0$ (Schwartz spherical rearrangement of $|u_n|$ see [3]).

Step 2 — (v_n) Is Bounded in $\mathcal{H}$

Since $T(v_n)\to J$ as $n\to\infty$, $\|\nabla(v)_n\|^2_{L^2}$ is bounded, which implies by Sobolev imbedding theorems that there exists a positive constant M such that $\|v_n\|_{L^{2^*}}\leq M$, where $2^*=\frac{2N}{N-2}$. On the other and, for each $\varepsilon>0$, there exists $C_\varepsilon>0$ such that for all $s\in I\!R$

$$\frac{s^2}{2}\leq C_\varepsilon |s|^{2^*}+\varepsilon\frac{|s|^{2(1-\theta)}}{2(1-\theta)}. \qquad \text{... (2.4)}$$

By putting $s=v_n$ and $\varepsilon=\theta$ in (2.4), we obtain

$$\frac{1}{2}\int_{I\!R^N} v_n^2\,dx\leq\frac{\theta}{2(1-\theta)}\int_{I\!R^N}|v_n|^{2(1-\theta)}\,dx+C_\theta M^{2^*}$$

Taking the above inequality in $V(v_n)=1$, hence,

$$1+\frac{1}{2(1-\theta)}\int_{I\!R^N}|v_n|^{2(1-\theta)}\,dx\leq\frac{\theta}{2(1-\theta)}\int_{I\!R^N}|v_n|^{2(1-\theta)}\,dx+C_\theta M^{2^*}.$$

which gives $\displaystyle\int_{I\!R^N}|v_n|^{2(1-\theta)}\,dx\leq 2C_\theta M^{2^*}$

and $\displaystyle\int_{I\!R^N} v_n^2\,dx\leq\frac{1+\theta}{1-\theta}C_\theta M^{2^*}.$

Thus (v_n) is bounded in $\mathcal{H}$.

Step 3 — Convergence of the Sequence (v_n)

(v_n) is bounded in $\mathcal{H}$, therefore we can extract a subsequence of (v_n), also denoted (v_n), such that (v_n) converges weakly in $\mathcal{H}$ almost everywhere in $I\!R^N$ to a function $v\in\mathcal{H}$ and

$T(v) \leq \lim_{n\to\infty} t(v_n) = J$. (v_n) is radial non-negative and non-increasing with $r = \|x\|$. Then, v is radial, nonnegative, and non-increasing with $r = \|x\|$. Now let $f(s) = |s|^{2(1-\theta)} + |s|^{2^*}$, so we have the following properties :-

1. $\dfrac{s^2}{2(|s|^{2(1-\theta)} + |s|^{2^*})} \to 0$ as $s \to 0$ or $s \to \infty$.

2. $\sup_{n>1} \int_{\mathbb{R}^N} f(v_n)\,dx < \infty$

3. $|v_n|^{-2\theta} v_n + |v_n|^{2^*} \to |v|^{-2\theta} v + |v|^{2^*}$, a.e. in $\mathbb{R}^N$

4. $|v_n(x)| \leq c_N \|x\|^{-\frac{N}{2}}$, $v_n \to 0$ as $|x| \to \infty$ uniformly in n, (see [3]).

Therefore, the compactness lemma of Strauss[3], implies

$$\int_{\mathbb{R}^N} v_n^2\,dx \to \int_{\mathbb{R}^N} v^2\,dx \text{ as } n \to \infty.$$

Fatou's lemma gives,

$$\frac{1}{2}\int_{\mathbb{R}^N} v^2\,dx \geq 1 + \frac{1}{2(1-\theta)} \int_{\mathbb{R}^N} |v|^{-2\theta} v\,dx,$$

so, we have $V(v) \geq 1$. On the other hand, we also know that $T(v) \leq \overline{\lim_{n\to\infty}}\, T(v_n) = J$. Suppose that $V(v) > 1$. Then, by a scale change $v_\alpha(x) = v\left(\dfrac{x}{\alpha}\right)$ we obtain $V(v_\alpha) = \alpha^N V(v) = 1$ for $0 < \alpha = \left(\dfrac{1}{V(v)}\right)^{\frac{1}{N}} < 1$. Also $T(v_\alpha) = \alpha^{N-2} T(v) \leq \alpha^{N-2} J$ but by the definition of J, $T(v_\alpha) \geq J$. This implies that $J = 0$, whence $T(v) = 0$ and then $v = 0$, a contradiction with $V(v) > 1$. So, $V(v) = 1$, $T(v) = J$ and v is solution of (2.1). Hence, there exists a lagrangian multiples $\beta > 0$ such that v is a weak solution of (2.2). The proof follows.

PROOF OF THEOREM 1.1

Let $v \in \mathcal{H}$ be a weak solution of (2.2). Using standard bootstrap arguments, since g satisfies $|g(s)| \leq C(1 + |s|)$ for every $s \in \mathbb{R}$, v is a classical solution of (2.2); (see [4]). Furthermore,

$|v(x)| \to 0$ uniformly as $|x| \to +\infty$ (see [5]). Let $v_\alpha(x) = v\left(\frac{x}{\alpha}\right)$. v_α is a classical solution of (2.3). Choosing $\alpha = \sqrt{\beta}$, $v_{\sqrt{\beta}}$ is a classical solution of (1.1)

Remark 2.1 : The solution v of (1.1) obtained by theorem 1.2 is the minimal one in the sense

$$0 < S(v) \le S(u) \text{ for any solution } u \text{ of (1.1) (see [1]).}$$

2.2 Existence of Infinitely Many Radial Solutions of (1.1)

In the second part of this section we prove theorem 1.2. We seek solutions of (1.1) which are radially symmetric but not necessarily positive. For this, we apply the critical points theory developed by P.L. Lions and H. Berestycki in [2]. We begin by recalling the two general theorems of that theory :

Let H be an infinite dimensional real Hilbert space and $F \subset H$, an infinite dimensional real Banach space such $\|x\|_H \le \|x\|_F$ for all $x \in F$. We consider the manifold $M = \{x \in F, \|x\|_H = 1\}$ and a C^1 functional $J : F \to I\!R$. We denote by $J_{|M}$ the restriction of J to M. Let $\Sigma(M)$ be the set of compact and symmetric (with respect to the origin) subsets of M. We recall that the genus $\gamma(A)$ of a set $A \in \Sigma(M)$, is defined as the least integer $n \ge 1$ such that there exists an odd continuous mapping $\varphi : A \to S^{n-1}$. We denote by $\Gamma_k = \{A \in \Sigma(M); \gamma(A) \ge k\}$. Lastly, we recall that $V_{|M}$ is said to satisfy the weak Palais-Smale $(P.S)$^+^ condition if the following holds :

For any a, $b > 0$ and for any sequence $(u_n)_{n \ge 1} \subset M$ such that $a \le V(u_n) \le b$ and $\|V'_{|M}(u_n)\| \to 0$, there exists a subsequence (u_n) which converges to M.

Theorem 1 — *We suppose that* J *is bounded from above by* M *and* $J_{|M}$ *satisfies Palais-Smale condition (PS). Let*

$$b_k = \sup_{A \in \Gamma_k} \inf_{x \in A} J(x)$$

then, for any $k \ge 1$, b_k is a critical value of J and $b_1 \ge b_2 \ge \ldots \ge b_k \ge \ldots$. If J satisfies only the weak Palais-Smale condition $(PS)^+$, then b_k is a critical value provided that $b_k > 0$.

Theorem 2 — *We suppose that F is separable, reflexive, and a dense subspace of H. Let*

$$S = \left\{x \in F; J(x) \ge 0, \|x\|_H \le 1\right\}$$

In addition of theorem 1, we assume $J(0) = 0$ and that

$$(*.*) \text{ if } (x_n) \subset S, x_n \to x \text{ in } H \text{ and } x \in F, \text{ then } J(x) = \overline{\lim_{n\to\infty}}\, J(x_n),$$

Lastly, we suppose that $b_k > 0$ for all $k \geq 1$. We then have $\lim_{k \to \infty} \downarrow b_k = 0$.

Remark 2.2 : The critical value b_k of $J, k \geq 1$ are distinct (see Theorem 8 [2])

To apply the above two theorems in our case, we have to precise the functional framework which will be used. Let

$$H = \left\{ u \in L^{2^*} (\mathbb{R}^N); \frac{\partial u}{\partial x_i} \in L^2 (\mathbb{R}^N) \text{ for } 1 \leq i \leq N \right\},$$

H is a Hilbert space endowed with the scalar product

$$(u, v) = \int_{\mathbb{R}^N} \nabla (u) \nabla (v) \, dx$$

let $\mathcal{H}_r = \{u \in \mathcal{H}, u \text{ is radial}\}$, $\mathcal{H}_r$ is a Banach space endowed with the $\mathcal{H}$ norm. Lastly, we consider the following unbounded manifold of $\mathcal{H}_r M = \left\{ u \in \mathcal{H}_r; T(u) = 1 \right\}$, and we denote by $V_{|M}$ the trace of V on M. The proof of theorem 1.2 will be derived from the followinf result :

Theorem 2.2 — *There exists infinitely many distinct critical values β_k $(k \geq 1)$ of $V_{|M}$ given by*

$$\beta_k = \sup_{A \in \Gamma_k} \inf_{x \in A} V_{IM}(x).$$

Moreover $\beta_k > 0$, $\lim_{k \to +\infty} \beta_k = 0$ and for each $k \in \mathbb{N}$, there exists a critical point $v_k \in M$ corresponding to β_k, and $\alpha_k > 0$ such that

$$-\Delta v_k = \alpha_k g(v_k). \qquad \text{... (2.5)}$$

PROOF : We have to verify the hypotheses of theorem 1 and theore 2 :-

a. $V_{|M}$ is bounded from above

b. $V_{|M}$ satisfies the weak Palais-Smale condition

c. The condition (*.*) of theorem 2 is satisfied

d. $\beta_k > 0$ for $k \geq 1$.

and the conclusion follows

Step 1 — $V_{|M}$ is Bounded From Above

Put $g_1(s) = s$, $g_2(s) = |s|^{1-2\theta} s$, $G_1(s) = \frac{s^2}{2}$ and $G_2(s) = |s|^{2(1-\theta)}$. We have $g = g_1 - g_2$ and

$G = G_1 - G_2$. Let $u \in M \, \| u \|_H = 1$, the Sobolev imbedding theorem gives $\| u \|_{L^{2^*}} \leq C$. Take $\varepsilon = \frac{1}{2}$ in (2.4), we obtain

$$V(u) \leq C - \frac{1}{2} \int_{\mathbb{R}^N} G_2(u)\, dx, \qquad \text{... (2.6)}$$

therefore, $V_{|M} \leq C$.

Step 2 — Condition $(P.S)^+$

First, we remark that Palais-Smale condition is not satisfying in our case (we can take the following sequence $u_n = \varphi_0 \, \omega_n^{-1} \sin(\omega_n x_1)$ where $\varphi_0 \in \mathcal{D}(\mathbb{R}^N)$ and $\omega_n \to 0$ as $n \to \infty$).

Let $(u_n) \subset M$ such that $V_{|M}(u_n) \geq 0$, the inequality (2.6) gives

$$\int_{\mathbb{R}^N} G_2(u_n)\, dx \leq C.$$

Since $V_{|M} \leq C$, we deduce that $\| u_n \|_L^2 \leq C$ and we conclude that the sequence (u_n) is bounded in $\mathcal{H}_r$. So there exists a subsequence $u_n \rightharpoonup u$ in $\mathcal{H}_r$, the arguments used in step 3 of theorem 2.1 shows

$$\int_{\mathbb{R}^N} G_1(u_n)\, dx \to \int_{\mathbb{R}^N} G_1(u)\, dx \text{ (this means } u_n \to u \text{ in } L^2)$$

and
$$\int_{\mathbb{R}^N} G_2(u)\, dx \leq \lim_{n \to \infty} \int_{\mathbb{R}^N} G_2(u_n)\, dx$$

we obtain
$$V(u) \geq \overline{\lim_{n \to \infty}} \, V(u_n) \geq a$$

thus $u \not\equiv 0$. Using now the hypotheses $\| V'_{|M}(u_n) \| \to 0$, applying lemma 3 in [2], this is equivalent to $V'(u_n) + < V'(u_n), u_n > \Delta u_n \to 0$ in H^{-1} which means

$$\langle V'(u_n), u_n \rangle \Delta u_n + g(u_n) \to 0 \text{ in } \mathcal{H}'_r. \qquad \text{... (2.7)}$$

Put $\lambda_n = \langle V'(u_n), u_n \rangle = \int_{\mathbb{R}^N} g(u_n)\, u_n\, dx$ and let us show that λ_n is bounded. We have $| g(s)\, s | \leq | s |^2 + | s |^{2(1-\theta)}$, thus

$$\lambda_n = \langle V'(u_n), u_n \rangle = \int_{\mathbb{R}^N} g(u_n)\, u_n\, dx \leq \| u_n \|_L^2{}^2 + \| u_n \|_{L^2(1-\theta)}^{2(1-\theta)} \leq C$$

So $\lambda_n \to \lambda$, since $u_n \to u$ in $\mathcal{H}_r$, it follows from (2.7)

$$g(u_n) \to -\lambda \Delta u \text{ in } \mathcal{H}_r' \qquad \text{... (2.8)}$$

On the other hand we have, $\frac{g(s)}{s^l} \to 0$ as $|s| \to \infty$, then theorem of Strauss^3^ gives $\int_B |g(u_n) - g(u)|\, dx \underset{n\to\infty}{\to} 0$ for any bounded Borel set B of $\mathbb{R}^N$, Hence, we deduce that $g(u_n) \to g(u)$ in distributions sense and from (2.8)

$$-\lambda \Delta u = g(u). \qquad \text{... (2.9)}$$

The Pohozaev's identity

$$\frac{N-2}{2} \lambda \int_{\mathbb{R}^N} |\nabla u|^2\, dx = NV(u) > 0$$

shows that $\lambda > 0$, multiplying (2.9) by u

$$\lambda \int_{\mathbb{R}^N} |\nabla u|^2\, dx = \int_{\mathbb{R}^N} g(u)\, dx. \qquad \text{... (2.10)}$$

The arguments earlier employed shows that

$$\int_{\mathbb{R}^N} g_1(u_n)\, u_n\, dx \to \int_{\mathbb{R}^N} g_1(u)\, u\, dx$$

and

$$\int_{\mathbb{R}^N} g_2(u)\, u\, dx \to \lim_{n\to\infty} \int_{\mathbb{R}^N} g_2(u_n)\, u_n\, dx.$$

This implies

$$0 < \lambda = \lim_{n\to\infty} \int_{\mathbb{R}^N} g(u_n)\, u_n\, dx \leq \int_{\mathbb{R}^N} g(u)\, u\, dx, \qquad \text{... (2.11)}$$

then from (2.10) we derive $\int_{\mathbb{R}^N} |\nabla u|^2\, dx \geq 1$.

Since $u_n \to u$ we have $\int_{\mathbb{R}^N} |\nabla u|^2\, dx \leq 1$. Therefore, $\int_{\mathbb{R}^N} |\nabla u|^2\, dx = 1$ and

$\int_{\mathbb{R}^N} g(u)\,u\,dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} g(u_n)\,u_n\,dx$, the first $\int_{\mathbb{R}^N} |\nabla u|^2\,dx = 1$ shows that $u_n \to u$ in H ($u \in M$ and since $\int_{\mathbb{R}^N} g_1(u_n)\,u_n\,dx \to \int_{\mathbb{R}^N} g_1(u)\,u\,dx$, we obtain $\int_{\mathbb{R}^N} g_2(u_n)\,u_n\,dx \to \int_{\mathbb{R}^N} g_2(u)\,u\,dx$, which means $\| u_n - u \|_{\mathcal{H}_r} \to 0$. Thus we conclude $u_n \to u$ strongly in $\mathcal{H}_r$.

Step 3 — Condition (*.*) of Theorem 2.

Let $S = \{u \in \mathcal{H}_r$ such that $V(u) \geq 0$ and $\| u \|_H \leq 1\}$ and consider $(u_n) \subset S$ such that $u_n \to u$. We want to show that

$$V(u) \geq \overline{\lim_{n\to\infty}}\, V(u_n).$$

since $\| u_n \| H \leq 1$, then $\| \nabla u_n \|_{L^2} \leq C$ and $\| u_n \|_L^2 \leq C$.

The arguments employed in the previous step show that (u_n) is bounded in $\mathcal{H}_r$ and it follows that $u_n \to u$ in $\mathcal{H}_r$ also the technique of the same step shows that

$$G_1(u_n) \underset{L^1}{\overset{n\to\infty}{\to}} G_1(u)$$

and
$$\int_{\mathbb{R}^N} g_2(u)\,dx \leq \lim_{n\to\infty} \int_{\mathbb{R}^N} G_2(u_n)\,dx$$

Hence, $V(u) \geq \overline{\lim_{n\to\infty}}\, V(u_n)$ and V is weakly sequentially upper semi continuous on S

Step 4 — $\beta_k > 0$ for all $k \geq 1$, for This Step see [2]

PROOF OF THEOREM 1.2

The hypotheses of theorem 1 and theorem 2 are satisfied, then $(\beta_k)_{k\geq 1}$ are critical values of $V_{|M}$ and $\lim_{k\to\infty} \downarrow \beta_k = 0$. There facts show that $V_{|M}$ has infinitely many critical points on M. Let $v_k \in M$ be a critical point of $V_{|M}$ associated with the value β_k, this means $V(v_k = \beta_k$, $T(v_k) = 1$ and

$$-V'(v_k) = \langle V'(v_k), v_k \rangle\, \Delta v_k.$$

Let $\alpha_k = \langle V'(v_k), v_k \rangle$, thus we have $-\alpha_k \Delta u = g(v_k)$.

Using the Pohozaev identity, we derive $\alpha_k = \frac{2N}{N-2} \beta_k$ is positive, hence

$$-\Delta v_k = \frac{1}{\alpha_k} g(v_k)$$

Let $u_k = v_k(\widehat{\alpha_k})$, hen u_k is a solution of (1.1) and this complete the proof

3. Compactness Supports of Solutions of (1.1)

In this section we study the compactness supports of solutions of (1.1). Note that g is decreasing on $(-a_\theta, a_\theta)$, increasing on $(-\infty, a_\theta) \cup (a_\theta, \infty)$ where $a_\theta = (1-2\theta)^{\frac{1}{2\theta}}$, we have the following theorem :

Theorem 3.1 — *Let* u *be a solution of (1.1) which tends to zero at infinity, then it has a compact support.*

Proof : We follow closely the proof of theorem 1.1 in [5], since $|u(x)| \to 0$ as $|x| \to +\infty$, there exists $R_0 > 0$ such that $|u(x)| < a_\theta$ if $|x| \geq R_0$. By the argument used in [5] (proof of theorem 1.1)), there exists a function v defined on $D = I\!R^N \; B(0, R_0)$ solution of

$$\Delta v + g(v) \leq 0 \text{ in } D|$$

which verifies $v(x) = a_\theta$ in $\partial D, 0 \leq v(x) \leq a_\theta$ for all x in D
and $v(x) = 0$ for any x such that $|x| \geq R_0 + A$,
where A is a positive constant. On the other hand, we have

$$\Delta u + g(u) = 0 \text{ in } D, \; -a_\theta \leq u(x) \leq a_\theta \text{ for all } x \in \partial D$$

and $\lim\limits_{|x| \to \infty} u(x) = 0$. Since $0 \leq v(x) \leq a_\theta, -a_\theta \leq u(x) \leq a_\theta$ for $x \in \partial D$ and g is decreasing in $(-a_\theta, a_\theta)$, we apply the comparison principle to obtain

$$-a_\theta \leq u(x) \leq v(x) \text{ in } D.$$

So $u(x) \leq 0$ for any x such that $|x| \geq R_0 + A$. But $(-u)$ is also a solution of (1.1) and $0 \leq -u(x) \leq a_\theta$ if $|x| \geq R_1 = R_0 + A$, then by repeating the same techniques with (- u), we obtain - $u(x)$ = 0 for any x such that $|x| \geq R_1 + A_1$, where A_1 is a positive constant, thus the proof.

4. Radial Solutions of (1.1)

Inthis section, we study the behaviour of the radial solutions of the eq. (1.1). Let v be a radial solution of (1.1), then $u(r) = v(x)$ where $r = \|x\|$ satisfies

$$u'' + \frac{N-1}{r} u' = -(u - |u|^{-2\theta} u). \qquad \text{... (4.1)}$$

We consider the boundary conditions

$$u'(0) = 0, \; u(0) = \lambda. \qquad \text{... (4.2)}$$

For each $\lambda \in I\!R$, there exists a unique C^2-solution of (4.1)-(4.2) see [9]). We define the energy function associated to a solution of (4.1)

$$E(r) = \frac{1}{2} u'(r)^2 + G(g(r),$$

where $G(s) = \int_0^r g(s)\, ds$. G is coercive, even, and $G(\lambda_\theta) = G(-\lambda_\theta) = 0$, where $\lambda_\theta - \left(\frac{1}{1-\theta} \right)^{\frac{1}{2\theta}}$. By a simple computation, it follows that

$$E'(r) = -\frac{N-1}{r} u'(r)^2.$$

So E is a non-increasing function.

We begin by setting the following useful lemmas :-

Lemma 4.1 — Let u be a solution of (4.1)-(4.2), then u is C^2-bounded.

PROOF : E is non-increasing, $E(r) \le E(0)$ for all $r \ge 0$, then $G(u(r)) \le G(u(0))$. Since G is even and coercive, there exists a unique real β such that $G(\beta) = G(u(0))$ and either $\beta \le u(r) \le u(0)$ or $u(0) \le u(r) \le \beta$ for all $r \in (0, \infty)$, it follows that u is bounded. Thereof, it yields $u')^2(r) \le 2E(0) - G(u(r)$, which gives u' is bounded and from (4.1) we get u'' is bounded, thus the result.

Lemma 4.2 — Let $0 < \lambda < 1$ (resp $\lambda > 1$) and u be a solution of (4.1)-(4.2) with $u(0) = \lambda$. Then, there exists $r_0 > 0$ such that $u(r_0) > 1$ and u is increasing on $(0, r_0)$ (resp $u(r_0) > 1$ and u is decreasing on $(0, r_0)$.

PROOF : For $\lambda > 1$ see [5]. Let $0 < \lambda < 1$, we follow closely the method used in [5]. We argue by contradiction, assuming $u(r) \le 1$ for all $r > 0$. We claim $u(r) < 1$ for all $r \ge 0$; indeed suppose there exists $r_1 > 0$ such that $u(r_1) = 1$. Since $u(r) \le 1$ for all $r \ge 0$, we must have $u'(r_1) = 0$, but this implies $u \equiv 1$, a contradiction, From (4.1)-(4.2) we get $r^{N-1} u'(t) = -\int_0^r s^{N-1} g(u(s)\, ds > 0$, then u is increasing and there exists $\alpha \ge 0$ such that

$a_\theta < u(r) < 1$ for all $r \geq \alpha$. Set $w(r) = -\frac{(r^{n-1} u'(r))'}{r^{N-1} u'(r)}$, w is a solution of the following equation

$$w'(r) = w^2(r) + g'(u(r)) - \frac{N-1}{r} w(r). \qquad \dots (4.3)$$

Since $a_\theta < u(r) < 1$ for all $r \geq \alpha$, there exists $k > 0$ and $\alpha_1 > \alpha$ such that $g'(u(r)) \geq k$, for $r > \alpha_1$. It follows that w is bounded and there exists r_2 large enough so that $w'(r) \geq \frac{k}{2}$. Therefore, sicne $w(r) < 0$, there exists $r_1 > 0$ such that $w(r_1) = 0$, this implies $u(r_1) = 1$, a contradiction with $u(r) < 1$ for all $r \geq 0$. Then, there exists $r_0 > 0$ such that $u(r_0) > 1$. Furthermore, since $0 < u(r) \leq 1$ for $r \in (0, r_{00})$, where r_{00} the first real such that $u(r_{00}) = 1$, we have

$$r^{N-1} u'(r) = \int_0^r s^{N-1} g(u(s))\, ds > 0 \text{ for } s \in (0, r_{00}).$$

Then by continuity $u' > 0$ on $(r_{00}, r_{00} + \delta)$, This implies $u > 1$ on $(r_{00}, r_{00} + \delta)$ and u is increasing on $(0, r_{00} + \delta)$, we denote $r_0 = r_{00} + \delta$, thus the desired result follows.

Remark 4.1 : If u is a solution of (4.1)-(4.2), then $(-u)$ is also a solution of (4.1)-(4.2). Then Lemma (4.2) still valid for $\lambda < -1$ or $-1 < \lambda < 0$: there exists $r_0 > 0$ such that $u(r_0) > -1$ and u is decreasing on $(0, r_0)$ (resp. $u(r_0) < -1$ and u is increasing on $(0, r_0)$.

Lemma 4.3 — Let u be a non-trivial compactly supported solution of (4.1)-(4.2); suppose there exists $r_0 \geq 0$ such that $u(r_0) = 0$ and $u'(r_0) = 0$, then $u \equiv 0$ on (r_0, ∞).

PROOF : Suppose there exists $r_1 > r_0$ such that $u(r_1) \neq 0$. Since E is non-increasing, there exists $r_2 \geq r_1$ such that $E(r) \leq E(r_2) < E(r_1) \leq 0$ and then $G(u(r)) < 0$ for all $r \geq r_2$. But G is coercive and even, so there exist either $0 < \alpha < 1$ and $0 < \beta < 1$ or $(-1 < \alpha < 0$ amd $-1 < \beta < 0)$, such that $\beta \leq u(r) \leq \alpha$ for all $r \geq r_2$, a contradiction. Hence the proof .

4.1 Behaviour of Radial Non-Negative Solutions of (1.1) With Compact Support

In this subsection, we study the behaviour of the nonnegative radial solution of (4.1)-(4.2).

Theorem 4.1 — *Let u be a nonnegative radial solution of (4.1)-(4.2) that has compact support, then u is a decreasing function on its support.*

PROOF : Without loss of generality we can suppose $u(0) = \lambda > \lambda_\theta$ (lemma 5.1). We present the proof in two steps.

Step 1 — We rewrite eq. (4.1) as

$$(r^{N-1} u')' = -r^{N-1} (u - |u|^{-2\theta} u)\, ds.$$

Consider $r_1 = \inf \{s > 0$ such that $u(r) > 1$, for any $r < s\}$.

Since u is continuous then $u > 1$ on $(0, r_1)$, the function $g(s) = s - |s|^{-2\theta} s \le 0$ on $(0, 1)$ and $g \ge 0$ on $(1, \infty)$, this implies

$$r^{N-1} u' < 0, \text{ therefore } u \text{ is decreasing on } (0, r_1),$$

furthermore $u(r_1) = 1$, and there exists $\gamma > 0$ such that $u(r) < 1$ *and* $u'(r) < 0$ *for* $r \in (r_1, r_1 + \gamma)$ *(lemma 4.2).*

Step 2 — For $r > r_1$, suppose that there exists $r^* > r_1$ such that $0 < u(r^*) < 1$ and $u'(r^*) = 0$. Since $u(r) \to 0$ as $r \to \infty$, there exists $r_* > r^*$ such that $u(r_*) < u(r^*)$, By the non-increase of the energy function E, we obtain $G(u(r_*)) \le G(u(r^*))$, but the function G is decreasing on $(0, 1)$, which implies $G u(r_*)) > G(u(r^*))$, a contradiction. Then u has no critical points on $A = \{r > r_1$ such that $0 < u(r) < 1\}$ and u remains decreasing on Al. u *has a compact support, so there exists* $r_0 > r_1$ *such that* $u(r_0) = 0$ *(*r_0 *is the first zero of* u*), then* $A = (r_1, r_0)$*. Since* $u(r) \ge 0$ *for every* $r \ge 0$ *and* $u'(r_0) = 0$*, lemma 4.3 implies* $u \equiv 0$ *on* (r_0, ∞)*, thereby the desired result.*

Let us now consider radial solutions that change sign.

4.2 Behaviour of Radial Solution of (4.1)-(4.2) with Compact Support and Changing Sign

Theorem 4.2 — *Let u be a solution of (4.1)-(4.2) with compact support and changing sign, and* $(r_i)_{1 \le i \le n}$ *the zeros of u in its support, then —*

1. for every i, there exists a unique $t_i \in (r_i, r_{i+1})$ such that $u'(t_i) = 0$ and $u(t_i) > \lambda_\theta$ or $u(t_i) < -\lambda_\theta$.

2. If $u(t_i) \ge \lambda_\theta$ (resp $u(t_i) \le -\lambda_\theta$), then u is increasing (resp decreasing), on (r_i, t_i), decreasing (resp increasing) on (t_i, r_{i+1}) and $|u(t_i)| > |u(t_{i+2})|$ for every $0 \le i \le n-2$.

PROOF : Without loss of generality, we can suppose $u(0) = \lambda > \lambda_\theta$. We present the proof of five steps

Step 1 — The steps 1 and 2 in the proof of theorem 4.1 show that there exists $r_1 > 0$ such that $u(r_1) = 0, u'(r_1) < 0$, and u is decreasing on $(0, r_1)$ (r_1 is the first zero of u). Since u changes sign, there exists $\gamma > 0$ such that $u < 0$ and $u' < 0$ on $(r_1, r_1 + \gamma)$. Let

$$l_1 = \sup \{r > r_1 \text{ such that } -1 < u(s) < 0, \text{ for any } r_1 \le s \le r\},$$

then $l_1 < \infty$ and $-1 < u(r) < 0$ for every $r \in (r_1, l_1)$. We claim that u is decreasing on (r_1, l_1). Indeed, suppose that there exists $r^* \varepsilon (r_1, l_1)$ such that $u'(r^*) = 0$, since u has a compact support there exists $r_* > r^*$ such that $u(r_*) = 0$. The energy function E is decreasing $E(r_*) < E(r^*) < E(r_1)$, then $\frac{1}{2}(u'(r_*))^2 < G(u(r^*)) < 0$, a contradiction. Thus, we have no critical points on (r_1, l_1). Since $g > 0$ on $(-1, 0)$, eq. (4.1) implies $r^{N-1} u' < 0$, then u is decreasing on (r_1, l_1), so $u(l_1) = -1, u'(l_1) < 0$ and there exists $\gamma > 0$ such that $u(r) < -1$ for $r \in (l_1, l_1 + \gamma)$ (see Remark 4.1).

Step 2 — $(r > l_1)$ Since u has a compact support, there exists $t_1 > l_1$ such that $u'(t_1) = 0$. For $r \in (l_1, t_1)$, we have

$$r^{N-1} u'(r) = \int_r^{t_1} s^{N-1} g(s)\, ds < 0,$$

so u is decreasing on (l_1, t_1). For $r > t_1$, we have

$$r^{N-1} u'(r) = -\int_{t_1}^{r} s^{N-1} g(s)\, ds.$$

Let $l_2 = \sup \{r > t_1 \text{ such that } u(s) < -1, \text{ for any } t_1 \le s \le r\}$,

$l_2 < \infty$ and $u(r) < -1$. For $r \in (t_1, l_2), g < 0$ on $(-\infty, -1)$ then u is increasing on (t_1, l_2), $u(l_2) = -1, u'(l_2) > 0$ and there exists $\gamma > 0$ such that $u'(r) > 0, u(r) > -1$ for $r \in (l_2, l_2 + \gamma)$ see lemma (4.2)

Step 3 — $(r > l_2)$ — Let

$$r_2 = \sup \{r > l_2 \text{ such that } -1 < u(s) < 0, \text{ for } l_1 \le s \le r\}$$

u is contnuous, $-1 < u(r) < 0$ on (l_2, r_2) and $u(r_2) = 0$. We must have $u' > 0$ on (l_2, r_2). Indeed, suppose there exists $r^* \in (l_2, r_2)$ such that $u'(r^*) = 0$, hence we have $\frac{1}{2}(u'(r_2))^2 < G(u(r^*)) < 0$, a contradiction. Then u is increasing on (t_1, r_2). On the other hand, $\frac{1}{2}(u'(r_{2)})^2 < G(u(t_1) < \frac{1}{2}(u'(r_1))^2$, then $u'(r_1) < 0$ and $G(u(t_1) \geq 0$ which imply $u(t_1) \leq -\lambda_\theta$.

Step 4 — $(r > r_2)$ We have two possibilities: either $u'(r_2) = 0$ and by lemma 4.3 we get $u \equiv 0$ on (r_2, ∞) $(n = 2)$ and the conclusion of the theorem follows; or $u'(r_2) > 0$. In this case by lemma 4.3 and the continuity of u, there exists $\gamma > 0$ such that $u(r) > 0$ in $r \in (r_2, r_2 + \gamma)$. Let

$$l_3 = \sup \{r > r_2 \text{ such that } 0 < u(s) < 1, \text{ for any } r_2 \leq s \leq r\},$$

$0 < u < 1$ on (r_2, l_3). We must have $u'(r) > 0$ on (r_2, l_3). Indeed, suppose there exists $r_* \in (r_2, l_3), u'(r_*) = 0$. Since u is compactly supported, there exists $r^* > r_*$ such that $\frac{1}{2}(u'(r))^2 < G(u(r_*)) \leq 0$, a contradiction.

Therefore, u is increasing on $(r_2, l_3), u(l_3) = 1$ and $u'(l_3) > 0$, there exists $\gamma > 0$ such that $u(r) > 1$ on $(l_3, l_3 + \gamma)$. Again since u has a compact support, there exists $t_2 > l_3$ such that $u'(t_2) = 0$. From (4.1), we get $r^{N-1} u'(r) = \int_r^{t_2} r^{N-1} (u - |u|^{-2\theta} u)\, dr > 0$ for $r \in (l_3, t_2)$, hence u is increasing on (l_3, t_2).

Step 5 — $(r > t_2)$ The same arguments employed in steps 1-2 of the proof of theorem 4.1, imply the existence of $r_3 > t_2$, such that $u(r_3) = 0, u(r) > 0$ on (r_2, r_3), and u is increasing on (t_1, t_2). t_1, t_2 are the unique successive critical points of u on $(r_1, r_2), (r_2, r_3)$ and $u'(r_1) < 0, u'(r_2) > 0$. Furthermore, as in the previous step, we have $G(u(t_2)) \geq 0$ which implies $u(t_2) \geq \lambda_\theta$, and $G(u(t_2)) < G(u, 0))$ then $u(0)) > u(t_2)$. Now, we repeat by a recurrent way the same arguments, to obtain the desired result.

Remark 4.2 : Let u be a radial solution of (4.1)-(4.2) with a compact support. Theorem 4.2 assures the following property: Let r_1, r_2 be successive zeros of u and $r_0 \in (r_1, r_2)$ such that $u'(r_0) = 0$. Then $u(r_0) \geq \lambda_\theta$ or $u(r_0) \leq -\lambda_\theta$.

We conclude this subsection by the following theorem

Theorem 4.3 — *Let u be a solution of (4.1)-(4.2) that has a compact support. Then u has a finite number of zeros in its support.*

PROOF : Suppose there exists an infinite number of zeros r_n in the support of u. then $r_n \to r_0$ as $n \to \infty$, by continuity, $u(r_0) = 0$. For every n there exists $s_n \in (r_n, r_{n+1})$ such that $u'(s_n) = 0, u(s_n) \geq \lambda_\theta$ or $u(s_n) \leq -\lambda_\theta$ and $s_n \to r_0$ as $n \to \infty$. Also by continuity $u(s_n) \to u(r_0) \geq \lambda_\theta$, a contradiction. Thus the proof.

Let us now study the radial solutons of (1.1) which do not tend to zero at infinity

5. A Class of Radial Solutions of (4.1)-(4.2) which do not Tend to Zero at Infinity

The two following lemmas give us a class of radial solutions for eqns. (4.1)-(4.2) which doesn't tend to zero at infinity —

Lemma 5.1 — Let $\lambda \in (, \lambda_\theta)$ (resp $\lambda \in (-\lambda_\theta, 0)$) and u be a solution of (4.1)-(4.2) with $u(0) = \lambda$. Then, there exists a unique β such that either $0 < \lambda \leq u(r) \leq \beta$ or $0 < \beta \leq u(r) \leq \lambda$ (resp $\lambda \leq u(r) \leq \beta < 0$ or $\beta \leq u(r) \leq \lambda < 0$) for all $r \geq 0$.

PROOF : If u is a solution of (4.1)-(4.2) then $(-u)$ is also a solution of (4.1)-(4.2), so we only study the case of $\lambda \in (0, \lambda_\theta)$. As we already know, the energy function $E(r)$ is non-increasing, $G(u(r)) \leq E(r) \leq E(0) = G(\lambda) < 0$ for every $r > 0$. Since G is evenand coercive, there exists a unique positive $\beta \neq \lambda$ which verifies $G(\lambda) = G(\beta)$ and $\beta \leq u(r) \leq \lambda$ or $\lambda \leq u(r) \leq \beta$ (u doesn't tend to zero at infinity), thus the result.

Lemma 5.2 — Let $\lambda \in (0, \lambda_\theta)$ (resp. $\lambda \in (-\lambda_\theta, 0)$) and u is a solution of (4.1)-(4.2) with $u(0) = \lambda$, Then u oscillates indefinitely on $(0, \infty)$ around 1 (resp around - 1).

PROOF : Let $\lambda \in (0, \lambda_\theta)$ and suppose $\lambda < 1$ (the calse $\lambda > 1$ is the same). According to lemma 4.2 there exists r_0 such that $u(r_0) > 1$ and u is increasing on $(0, r_0)$. For $r > r_0$, there exists $r_1 > r_0$ such that $u(r_1) > 1$, and $u'(r_1) = 0$. Indeed suppose that $u'(r) > 0$ for all $r > r_0$, then since u

is bounded, $u(r) \to a > 1+\delta, \delta > 0, u'(r) \to 0$ and $u''(r) \to 0$ as $r \to \infty$. But from (4.1), we obtain $g(u(r)) \to 0$ as $r \to \infty$, this implies $u(r) \to 1$ or -1 or 0 as $r \to \infty$, a contradiction. Thus, there exists $r_1 > r_0$ such that $u(r_1) > 1$ and $u'(r_1) = 0$. We apply again lemma 4.2 and the same techniques as above, there exists $r_2 > r_1$ such that $u(r_2) < 1, u'(r_2) = 0$ and u is decreasing on (r_1, r_2). We repeat, by à recurrent way, the same arguments, to obtain the desired result. For $\lambda \in (\lambda_\theta, 0)$, it is enough to change u on $(-u)$.

Lemma 5.3 — Let u be a solution of (4.1) then there exists a periodic solution u of

$$u'' + f(u) = 0 \qquad \dots (4.3)$$

such that $$\lim_{r \to \infty} |u(r) - \bar{u}(s)| + |u'(r) - \bar{u}'(s)| = 0.$$

PROOF : (see [7])

Now we state an interesting result of this section

Theorem 5.1 — *Let $\lambda \in (0, \lambda_\theta)$, (resp. $\lambda \in (-\lambda_\theta, 0)$) and u is the solution of (4.1)-(4.2) such that $u(0) = \lambda$. Then $u(r) \to 1$ and $u'(r) \to 0$ if $r \to \infty$ (resp $u(r) \to -1$ and $u'(r) \to 0$ if $r \to \infty$).*

PROOF : Let $\lambda \in (0, \lambda_\theta)$, suppose that $\lambda > 1$ ($\lambda < 1$ will be the same). According to lemma 5.2, u oscillates indefinitely around 1. Let's denote by α_n the local maximums of u, and by β_n the locals minimums of u. There exists an increasing sequence of positive reels (r_n) with $r_0 = 0$ such that $u'(r_{2n}) = 0$, $u(r_{2n}) = \alpha_n > 1$ and $u'(r_{2n+1}) = 0, 0 < u(r_{2n+1}) = \beta_n < 1$. (r_n) satisfies $r_{2n+1} - r_{2n} > \dfrac{2((\alpha_n - \beta_n))}{f(\lambda_\theta))}$. Thereof, if $\alpha_n - \beta_n > c > 0$ then r_n are isolated and $r_n \to \infty$ as $n \to \infty$. First, we remark that α_n is a decreasing sequence and β_n is an increasing one. In fact, $E(r_{2n}) > E(r_{2n+2}), E(r_{2n+1}) > E(r_{2n+3})$. This is equivalent to $G(\alpha_n) > G(\alpha_{n+1})$ and $G(\beta_n) > G(\beta_{n+1})$. but G is increasing on (l, ∞), decreasing on $(0, 1)$. Then $\alpha_n > \alpha_{n+1}$ and $\beta_n < \beta_{n+1}$. Let's set $\lambda_n = \alpha_n - \beta_n$. The sequence λ_n is positive and decreasing, it converges to $\lambda_0 \geq 0$. If $\lambda_0 = 0$, then $u \to 1$ and so $u'(r) \to 0$ as $r \to 0$, we obtain the desired result. Let us prove $\lambda_0 = 0$. Suppose $\lambda_0 > 0$, then $\alpha_n - \beta_n > c > 0$ for all $r > 0$. According to lemma 5.3, $u'(t)$ does not tend to zero as $t \to \infty$. This means that there exists ε_0 such that for all $j > 1$ there exists $t_j > j$ such that $|u'(t_j)| > \varepsilon_0$. Furthermore, for every $j \geq 1$ there exists r_{k_j} such that $r_{k_j} \leq t_j < r_{k_{j+1}}$, ... (4.4)

by continuity of u there exists $s_{k_j}, s_{k_{j+1}}$ such that $r_{k_j} \leq s_{k_j} \leq t_j \leq s_{k_{j+1}} \leq r_{n_{j+1}}$ and

$$|u'(s)| > \frac{\varepsilon_0}{2} \text{ for any } s \in (s_{k_j}, s_{k_{j+1}}). \qquad \text{... (4.5)}$$

On the other hand, let n be large enough and $(i_1, i_2, \ldots, i_m)$ the integers $\in \{1, \ldots, n\}$ for which, there exists $t_i \in (r_{i_k}, r_{i_k+1})$ satisfies (4.4), (4.5) and $s_{k_{i_m}} \to \infty$ as $r_n \to \infty$. We then have

$$E(r_n) - E(r_1) = -(N-1) \int_{r_1}^{r_n} \frac{(u'(s))^2}{s} ds = -\sum_{i=1}^{n} (N-1) \int_{r_i}^{r_i+1} \frac{(u'(s))^2}{s} ds \leq$$

$$-\sum_{l=i_1}^{i_m} (N-1) \int_{s_{k_l}}^{s_{k_l}+1} \frac{(u'(s))^2}{s} ds \leq -\frac{\varepsilon_0^2}{2} \log\left(\frac{s_{k_{i_m}}}{s_{k_{i_1}}}\right) \to -\infty \text{ as } m \to \infty.$$

But E is bounded, thus the contradiction. For $\lambda \in (-\lambda_\theta, 0)$, we change u on $(-u)$. Thus the proof.

We finish this paper by a classification result of the radial solutions.

Theorem 5.2 — *For every* λ, *the energy function* E *associates to (4.1)-(4.2) satisfies either:*

a. $E(r) \to 0$, as $r \to \infty$ and this corresponds to compactly supported solutions

or b. $E(r) \to G(1)$ or $G(-1)$, as $r \to \infty$ and this corresponds to solutions that tends to 1 or -1 at infinity.

PROOF : Let u be a solution of (4.1)-(4.2). Since u is bounded, it has an infinite or a finite number of oscillations at infinity. Firstly, suppose u oscillates indefinitely, then $u'(r) \to 0$ as $r \to \infty$. If not, the argument employed in the proof of the precedent theorem gives a contradiction. Lemma 5.3 gives $u(r) \to \alpha$, and (4.1) implies $g(\alpha) = 0$. So $\alpha = 0$, 1 or -1. If we suppose $\alpha = 0$ then u is compactly supported (Theorem 3.1) and has a finite number of oscillations, a contradiction. So, $u(r) \to 1$ or -1 and $E(r) \to G(1)$ or $G(-1)$. Let us now suppose u oscillates a finite number of times, since u is bounded, $u(r) \to \alpha$ and $u'(r) \to 0$ as $r \to \infty$. Eq. (4.1) gives $g(\alpha) = 0$ so $\alpha =$ 0 or 1 or -1. Suppose $\alpha = 1$ or -1, lemmas (4.2) and (5.2) give u oscillates idefinitely, a contradiction. $u(r) \to 0$, u is compactly supported and $E(r) \to 0$, hence the result.

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REFERENCES

1. H. Beretstycki and P. L. Lions, *Arch. Rach. Mech. Anal.* **82** (1983) 313-45.
2. H. Berestycki and P. L. Lions, *Arch. Rach. mech. Anal.* **82** (1983) 347-75.
3. W. A. Stauss, *Comm. Math. Phys.* **55** (1977) 148-62.
4. M. Struwe, *Variational methods*, Springer-Verlag, 1990.
5. C. Cortazar, M. Elgueta and P. Felmer, *Adv. Diff. Eq.* **1** N2 (1996) 199-218.
6. C. Cortazar, M. Elgueta and P. Felmer, *Arch. Rach. Mech. Anal.* **142** (1998) 2, 127-41.
7. R. D. Benguria, J. DOlbeaut and M. Esteban, *Cahiers du coromade* 9919.
8. D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order.* Springer-Verlag, 1983.
9. W. M. Ni and R. Nussbaum, *Comm. pure and appl. Math.*, **38** (1983) 67-108.
10. E. Yanaguida, *Arch. Rach. Mech. Anal.* **115** (1991) 257-74.
11. M. K. Kwong, *Arch. Rach. Mech. Anal.* **105** (1989) 243-66.
12. B. Franchi, E. Lanconelli and J. Serrin, *Atti. Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Natur.* **8** 79 (1985) 121-26.
13. B. Franchi, E. Lanconelliand J. Serrin, *In*: Nonlinear Diffusion Equations and their Equilibrium States II. *Springer-Verlag, new York-Berlin 1988*