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PERIODIC BOUNDARY VALUE PROBLEM FOR FIRST ORDER IMPULSIVE DIFFERENTIAL EQUATIONS WITH SUPREMUM

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In this paper, by means of the upper and lower solutions and the monotone iterative technique, the existence of maximal and minimal solutions of the periodic boundary value problem for first order impulsive differential equations with supremum is considered.

Key Words : Impulsive Differential Equation; Upper and Lower Solution; Monotone Iterative Technique; Periodic Boundary Value Problem

1. INTRODUCTION

The theory of impulsive differential equations is emerging as an important area of investigating since it is much richer than the corresponding theory of differential equations [see, 1, 2, 4-8, 10]. In this paper, we consider the periodic boundary value problem for first order impulsive differential equations with supremum (PBVP)

$$\left.\begin{array}{ll} x'(t)=f\left(t, x(t), \sup\limits_{s\in[t-h,t]} x(s)\right) & t\neq t_k,\ t\in[0,T] \\ \Delta x(t_k)=I_k(x(t_k)), & k=1,2,\dots,p, \\ x(t)=x(0), & t\in[-h,0], \\ x(0)=x(T), & \end{array}\right\} \quad \dots (1)$$

where $f\in C([0,T]\times R\times R, R)$, $I_k\in C(R,R)$, $\Delta x(t_k)=x(t_k^+)-x(t_k)$ $(k = 1, 2, \dots, p)$, h and T are positive constants, $0<t_1<t_2<\dots<t_k<\dots<t_p<T$ are fixed points.

The method of upper and lower solutions coupled with the monotone iterative technique has been widely used in treatment of nonlinear differential equations in recent years [3-10]. The basic idea of this method is that using the upper and lower solutions as an initial iteration one can construct monotone sequences from a corresponding linear equation, and these sequences converge monotonically to the maximal and minimal solutions of the nonlinear equation. When the method is applied to impulsive differential equations, it usually needs a suitable impulse differential inequality as a comparison principle. In sectiion 2, we establish an impulsive differential inequality with supremum as a comparison principle, i.e., Lemma 2. Then we discuss the existence and uniqueness of the solutions for a linear PBVP for impulsive differential equation with supremum, i.e., Lemma 3, Lemma 4. Finally, by use of the monotone iterative technique and the method of upper and lower solutions we obtain the existence theorem of external solutions for the PBVP (1).

2. Preliminaries

Let $J\subset R$ be an interval, we define $PC(J, R) = \{x: J\to R; x(t)$ is continuous everywhere except some t_k at which $x(t_k^-)$ and $x(t_k^+)$ exist and $x(t_k^-)=x(t_k)\}$; $PC'(J,R)=\{x\in PC(J,R): x(t)$ is continuously differentiable everywhere except some t_k at which $x'(t_k^-)$ and $x'(t_k^+)$ exist$\}$. Let $\Omega=PC([-h,T],R)\cap PC'([0,T],R)$. A function $x\in\Omega$ is called a solution of PBVP (1) if it satisfies (1).

Definition 1 — A function $\alpha\in\Omega$ is called a lower solution of PBVP (1) if

$$\left\{\begin{array}{ll} \alpha'(t)\leq f\left(t, \alpha(t)\ \sup\limits_{s\in[t-h,t]} \alpha(s)\right), & t\neq t_k,\ t\in[0,T] \\ \Delta\alpha(t_k)\leq I_k(\alpha(t_k)), & k=1,2,\dots,p, \\ \alpha(t)=\alpha(0), & t\in[-h,0], \\ \alpha(0)\leq\alpha(T). & \end{array}\right. \quad \dots (2)$$

Definition 2 — A function $\beta\in\Omega$ is called an upper solution of PBVP (1) if

$$\begin{cases} \beta'(t) \geq f\left(t, \beta(t) \sup_{s \in [t-h,t]} \beta(s)\right), & t \neq t_k, \; t \in [0, T] \\ \Delta \beta(t_k) \geq I_k(\beta(t_k)), & k = 1, 2, \ldots, p, \\ \beta(t) = \beta(0), & t \in [-h, 0], \\ \beta(0) \geq \beta(T). \end{cases} \quad \ldots (3)$$

Lemma 1[1] — Assume that

(a1) the sequence $\{t_k\}$ satisfies $0 \leq t_0 < t_1 < t_2 < \ldots < t_k < \ldots$ with $\lim_{k \to \infty} t_k = \infty$;

(a2) $m \in PC'(R_+, R)$ is left continuous at t_k for $k = 1, 2, \ldots$;

(a3) for $k = 1, 2, \ldots, t \geq t_0$,

$$m'(t) \leq p(t)\, m(t) + q(t), \quad t \neq t_k$$

$$m(t_k^+) \leq d_k\, m(t_k) + b_k$$

where $p, q \in C(R_+, R)$, $d_k \geq 0$ and b_k are real constants.

Then,

$$m(t) \leq m(t_0) \prod_{t_0 < t_k < t} d_k \exp\left(\int_{t_0}^{t} p(s)\, ds\right) + \int_{t_0}^{t} \prod_{s < t_k < t} d_k \exp\left(\int_{s}^{t} p(\sigma)\, d\sigma\right) q(s) ds$$

$$+ \sum_{t_0 < t_k < t} \prod_{t_k < t_j < 1} d_j \exp\left(\int_{t_k}^{t} p(s)\, ds\right) b_k.$$

We shall prove a comparison lemma which plays an important role in the proof our results.

Lemma 2 — Let the function $m \in \Omega$ satisfy the inequalities

$$\begin{cases} m'(t) \leq -M\, m(t) + N \sup_{s \in [t-h,t]} m(s), & t \neq t_k, \; t \in [0, T] \\ \Delta m(t_k) \leq -L_k\, m(t_k), & k = 1, 2, \ldots, p, \\ m(t) = m(0), & t \in [-h, 0], \\ m(0) \leq m(T), \end{cases} \quad \ldots (4)$$

where $M > 0, N > 0, 0 \leq L_k < 1$ $(k = 1, 2, \ldots, p)$. Assume that the following inequality

$$0<\frac{N}{M}\left\{\chi_{[0,t_1]}(t)+\frac{\sum_{k=1}^{p}\chi_{(t_k,t_{k+1}]}(t)\sum_{i=0}^{k-1}\prod_{t_i<t_j<t}(1-L_j)\exp[-M(t-t_{i+1})]}{\left[1-\sum_{k=1}^{p}\chi_{(t_k,t_{k+1}]}(t)\prod_{0<t_j<t}(1-L_j)\exp(-Mt)\right]}\right\}<1 \quad \ldots (5)$$

holds for $t\in[0,T]$, where $\chi_{[0,t_1]}$, $\chi_{(t_k,t_{k+1}]}$ are characteristic functions of the sets $[0,t_1]$, $(t_k,t_{k+1}]$ respectively and $k=1,2,\ldots,p$, $t_0=0$, $t_{p+1}=T$.

Then $m(t)\le 0$ for $t\in[-h,T]$.

PROOF: From (4), by Lemma 1 we get

$$m(t)\le m(0)\prod_{0<t_k<t}(1-L_k)\exp(-Mt)$$

$$+N\int_0^t\prod_{s<t_k<t}(1-L_k)\exp[-M(t-s)]\sup_{\xi\in[t-h,t]}m(\xi)\,ds. \quad \ldots (6)$$

Suppose that the conclusion of the lemma is not true, i.e., that there exist an $\varepsilon>0$ and some point $\bar{t}\in[0,T]$ such that $m(\bar{t})=\sup_{t\in[0,T]}m(t)=\varepsilon>0$.

Then $m(t)=\varepsilon$ for $t\in[-h,T]$.

Consider the following two cases :

Case 1 — $\bar{t}\in(t_1,T]$. In this case, there exists some positive integer k such that $\bar{t}\in(t_k,t_{k+1}]$, where $k=1,2,\ldots,p$, $t_{p+1}=T$. From (6) we have

$$m(\bar{t})\le m(0)\prod_{0\le t_k<\bar{t}}(1-L_k)\exp(-M\bar{t})$$

$$+N\int_0^{\bar{t}}\prod_{s<t_k<\bar{t}}(1-L_k)\exp[-M(\bar{t}-s)]\sup_{\xi\in[\bar{t}-h,\bar{t}]}m(\xi)\,ds. \quad \ldots (7)$$

Since $m(0)\le m(\bar{t})$, therefore we obtain the following inequality

$$m(\bar{t})\left[1-\prod_{0<t_k<\bar{t}}(1-L_k)\exp(-M\bar{t})\right]$$

$$\leq N \int_0^{\bar{t}} \prod_{s < t_k < \bar{t}} (1 - L_k) \exp[-M(\bar{t} - s)] \sup_{\xi \in [\bar{t} - h, \bar{t}]} m(\xi)\, ds$$

$$\leq Nm(\bar{t}) \left[\int_0^{t_1} \prod_{s < t_k < \bar{t}} (1 - L_k) \exp[-M(\bar{t} - s)] ds \right.$$

$$+ \int_{t_1^+}^{t_2} \prod_{s < t_k < \bar{t}} (1 - L_k) \exp[-M(\bar{t} - s)]\, ds + \dots$$

$$\left. + \int_{t_k^+}^{\bar{t}} \prod_{s < t_k < \bar{t}} (1 - L_k) \exp[-M(\bar{t} - s)]\, ds \right]$$

$$\leq \frac{Nm(\bar{t})}{M} \left[\sum_{0 < t_k < \bar{t}} (1 - L_k) \exp[-M(\bar{t} - t_1)] \right.$$

$$+ \prod_{t_1 < t_k < \bar{t}} (1 - L_k) \exp[-M(\bar{t} - t_2)] + \dots$$

$$\left. + \prod_{t_{k-1} < t_k < \bar{t}} (1 - L_k) \exp[-M(\bar{t} - t_k)] \right].$$

i.e.
$$m(\bar{t}) \leq \frac{Nm(\bar{t}) \left[\sum_{i=0}^{k-1} \prod_{t_i < t_j < \bar{t}} (1 - L_j) \exp[-M(\bar{t} - t_{i+1}] \right]}{M \left[1 - \prod_{0 < t_j < \bar{t}} (1 - L_j) \exp(-M\bar{t}) \right]} < m(\bar{t}),$$

leading to a contradiction.

Case 2 — $\bar{t} \in [0, t_1]$. If $\bar{t} = 0$, then $\varepsilon = m(0) \leq m(T)$. From (4) we have

$$0 \leq m'(T) \leq -Mm(T) + N \sup_{s \in [T - h, T]} m(s) \leq -M\varepsilon + N\varepsilon < 0,$$

which is a contradiction. If $\bar{t} \in (0, t_1]$, then $m'(\bar{t}) \geq 0$. From (4) we get

$$0 \leq m'(\bar{t}) \leq -Mm(\bar{t}) + N \sup_{s \in [\bar{t} - h, \bar{t}]} m(s) \leq -M\varepsilon + N\varepsilon < 0,$$

which is also a contradiction. Thus lemma 2 is proved.

Let us consider the following periodic boundary value problem of a linear impulsive differential equation with supremum (PBVP)

$$\begin{cases} u'(t) + Mu(t) = N \sup\limits_{s \in [t-h,\, t]} u(s) + \sigma(t), & t \neq t_k, \quad t \in [0, T], \\ \Delta u(t_k) = -L_k u(t_k) + I_k(\eta(t_k)) + L_k \eta(t_k), & k = 1, 2, \ldots, p, \\ u(t) = u(0), & t \in [-h, 0], \\ u(0) = u(T), \end{cases} \quad \ldots (8)$$

where M, N, L_k $(k = 1, 2, \ldots, p)$ are constants, $I_k \in C([0, T], R)$, $\sigma \in PC([0, T], R)$ and $\eta \in \Omega$.

Lemma 3 — $u \in \Omega$ is a solution of PBVP (8) if and only if $u \in PC([-h, T], R)$ is a solution of the following impulsive integral equation

$$u(t) = \begin{cases} \int_0^T G(t, s) \left(N \sup\limits_{s \in [T-h,\, T]} u(s) + \sigma(s) \right) ds \\ \quad + \sum\limits_{0 < t_k < T} G(t, t_k)(-L_k u(t_k) + I_k(\eta(t_k)) + L_k \eta(t_k)), \quad t \in [0, T] \\ u(0), \quad t \in [-h, 0), \end{cases} \quad \ldots (9)$$

where $$G(t, s) = \frac{1}{e^{MT} - 1} \begin{cases} e^{-M(t-s-T)}, & 0 \leq s \leq t \leq T, \\ e^{-M(t-s)}, & 0 \leq t \leq s \leq T. \end{cases}$$

PROOF : Assume that $u \in \Omega$ is a solution of PBVP (8). By the variation of parameters formula, we get

$$u(t) = u(0) e^{-Mt} + \int_0^t e^{-M(t-s)} \left(\sup\limits_{s \in [t-h,\, t]} u(s) + \sigma(s) \right) ds$$

$$+ \sum_{0 < t_k < t} e^{-M(t-t_k)} [-L_k u(t_k) + I_k(\eta(t_k)) + L_k \eta(t_k)]. \quad \ldots (10)$$

Setting $t = T$ in (10), we have

$$u(T) = u(0) e^{-MT} + \int_0^T e^{-M(T-s)} \left(N \sup\limits_{s \in [T-h,\, T]} u(s) + \sigma(s) \right) ds$$

$$+ \sum_{0 < t_k < T} e^{-M(t-t_k)} [-L_k u(t_k) + I_k(\eta(t_k)) + L_k \eta(t_k)]. \quad \ldots (11)$$

From the boundary condition $u(0) = u(T)$, we obtain

$$u(0) = \frac{1}{e^{MT} - 1}\left[\int_0^T e^{Ms}\left(N \sup_{s \in [T-h, T]} u(s) + \sigma(s)\right)ds + \sum_{0<t_k<T} e^{Mt_k}[-L_k u(t_k) + I_k(\eta(t_k)) + L_k \eta(t_k)]\right] \quad \text{... (12)}$$

Substituting (12) into (10), we see that $u \in PC[([-h, T], R)$ saisfies (9)

$$\begin{cases} u'(t) + Mu(t) = N \sup_{s \in [t-h, t]} u(s) + \sigma(t), & t \neq t_k, \ t \in [0, T], \\ \Delta u(t_k) = -L_k u(t_k) + I_k(\eta(t_k)) + L_k \eta(t_k), & k = 1, 2, ..., p, \\ u(t) = u(0), & t - [-h, 0]. \end{cases}$$

Setting $t = 0$, T in (9) respectively, we have

$$u(T) = \frac{1}{e^{MT} - 1}\left[\int_0^T e^{Ms}\left(N \sup_{s \in [T-h, T]} u(s) + \sigma(s)\right)ds + \left[\sum_{0<t_k<T} e^{Mt_k}[-L_k u(t_k) + I_k(\eta(t_k)) + L_k \eta(t_k)\right]\right.$$

$$= u(0).$$

Therefore, $u \in \Omega$ is a solution of PBVP (8). Thus Lemma 3 is proved.

Lemma 4 — Assume that $M > 0$, $N > 0$, $0 \leq L_k < 1$ $(k = 1, 2, ..., p)$, $I_k \in C([0, T], R)$, $\sigma \in PC([0, T], R)$, $\eta \in \Omega$, and the following inequality holds

$$\frac{N}{M} + \frac{1}{1 - e^{-MT}} \sum_{k=1}^{p} L_k < 1. \quad \text{... (13)}$$

Then PBVP (8) possesses a unique solution in $PC([-h, T], R)$.

PROOF : Let $\Omega_0 = \{u \in PC([-h, T], R) : u(t) \equiv u(0) \text{ for } t \in [-h, 0]\}$ with norm $\|u\| = \sup \{|u(t)| : t \in [-h, T]\}$, then Ω_0 is a Banach space.

For any $u \in \Omega_0$, consider the operator F defined by the formula

$$(Fu)(t)=\begin{cases}\int_0^T G(t,s)\left(N\sup_{s\in[T-h,T]}u(s)+\sigma(s)\right)ds \\ \quad+\sum_{0<t_k<T} G(t,t_k)(-L_k u(t_k)+I_k(\eta(t_k))+L_k\eta(t_k)), & t\in[0,T], \\ (Fu)(0), & t\in[-h,0).\end{cases}$$

Then $(Fu)\in\Omega_0$ i.e., $F\Omega_0\subset\Omega_0$.

For every $u, \upsilon\in\Omega_0, t\in[0,T]$, we have

$$|(Fu)(t)-(F\upsilon)(t)|\le N\int_0^T G(t,s)\left|\sup_{s\in[T-h,T]}u(s)-\sup_{s\in[T-h,T]}\upsilon(s)\right|ds$$

$$+\sum_{0<t_k<T} G(t,t_k)\,L_k\,|u(t_k)-\upsilon(t_k)|$$

$$\le\left\{\frac{N}{M}+\frac{1}{1-e^{-MT}}\sum_{k=1}^{p}L_k\right\}\|u-\upsilon\|.$$

Hence,
$$\|Fu-F\upsilon\|=\sup_{t\in[-h\,T]}|(Fu)(t)-(F\upsilon)(t)|$$

$$=\sup_{t\in[0,T]}|(Fu)(t)-(F\upsilon)(t)|$$

$$\le\alpha\|u-\upsilon\|,$$

where
$$\alpha=\frac{N}{M}+\frac{1}{1-e^{-MT}}\sum_{k=1}^{p}L_k<1.$$

Thus the operator F is a contraction on Ω_0. That is, there is a unique element $u\in\Omega_0$ such that $u = Fu$. This u is the unique solution of PBVP (8). The proof of the Lemma is complete.

3. The Main Result

Theorem 1 — *Let the following conditions hold :*

(i) The functions $\alpha, \beta\in\Omega\cap\Omega_0$ *are lower and upper solutions of PBVP (1) such that*

$$\alpha(t)\le\beta(t)\quad\text{for}\quad t\in[-h,T].$$

(ii) The function $f\in C([0,T]\times R\times R, R)$ *satisfies*

$$f\left(t, x(t), \sup_{s\in[t-h,t]} x(s)\right) - f\left(t, y(t), \sup_{s\in[t-h,t]} y(s)\right)$$

$$\geq -M(x(t)-y(t)) + N\left(\sup_{s\in[t-h,t]} x(s) - \sup_{s\in[t-h,t]} y(s)\right),$$

whenever $\alpha(t) \leq y(t) \leq x(t) \leq \beta(t)$, *where* M, N *are positive constants.*

(iii) The functions $I_k \in C(R, R)$ *satisfy*

$$I_k(x) - I_k(y) \geq -L_k(x-y),$$

whenever $\alpha(t_k) \leq y \leq x \leq \beta(t_k)$ *(k = 1, 2, ..., p), where* $0 \leq L_k < 1$ *(k = 1, 2, ..., p).*

(iv) The inequalities (5) and (13) hold.

Then, there exist monotone sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ *with* $\alpha_0(t) = \alpha(t)$, $\beta_0(t) = \beta(t)$, *such that* $\lim_{n\to\infty} \alpha_n(t) = \rho(t)$, $\lim_{n\to\infty} \beta_n(t) = r(t)$ *uniformly on [–h, T], and* ρ, r *are the minimal and the maximal solutions of PBVP (1) respectively, such that*

$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n \leq \rho \leq x \leq r \leq \beta_n \leq \ldots \leq \beta_2 \leq \beta_1 \leq \beta_0 \text{ on } [-h, T],$$

where x *is any solution of the PBVP (1) such that* $\alpha(t) \leq x(t) \leq \beta(t)$ *on [–h, T].*

PROOF : Let $[\alpha, \beta] = \{x \in \Omega \cap \Omega_0 : \alpha(t) \leq x(t) \leq \beta(t), t \in [-h, T]\}$. For any $\eta \in [\alpha, \beta]$, consider PBVP (8) with

$$\sigma(t) = f\left(t, \eta(t), \sup_{s\in[t-h,t]} \eta(s)\right) + M\eta(t) - N \sup_{s\in[t-h,t]} \eta(s).$$

By Lemma 3 and Lemma 4, PBVP (8) possesses a unique solution $u \in \Omega \cap \Omega_0$. We define an operator A by $u = A\eta$, then the operator A has the following properties :

(a) $\alpha \leq A\alpha$, $\beta \geq A\beta$;

(b) A is monotone nondecreasing on $[\alpha, \beta]$ i.e., for any $\eta_1, \eta_2 \in [\alpha, \beta]$, $\eta_1 \leq \eta_2$ implies $A\eta_1 \leq A\eta_2$.

To prove (a), set $m = \alpha_0 - \alpha_1$, where $\alpha_1 = A\alpha_0$. Then we have

$$m'(t) = \alpha_0'(t) - \alpha_1'(t)$$

$$\leq f\left(t, \alpha_0(t), \sup_{s\in[t-h,t]} \alpha_0(s)\right) - \left[-M\alpha_1(t) + N \sup_{s\in[t-h,t]} \alpha_1(s)\right.$$

$$\left. + f\left(t, \alpha_0(t), \sup_{s\in[t-h,t]} \alpha_0(s)\right) + M\alpha_0(t) - N \sup_{s\in[t-h,t]} \alpha_0(s)\right]$$

$$\leq -Mm(t) + N \sup_{s \in [t-h, t]} m(s), t \neq t_k, t \in [0, T],$$

$$\Delta m(t_k) = \Delta \alpha_0(t_k) - \Delta \alpha_1(t_k)$$

$$\leq I_k(\alpha_0(t_k)) - [-L_k \alpha_1(t_k) + I_k(\alpha_0(t_k)) + L_k \alpha_0(t_k)]$$

$$= -L_k m(t_k), \ k = 1, 2, \ldots, p,$$

$$m(t) = m(0) \leq m(T), t \in [-h, 0].$$

By Lemma 2, we get $m(t) \leq 0$ on $[-h, T]$, i.e., $\alpha \leq A\alpha$. Similar arguments show that $\beta \leq A\beta$.

To prove (b) let $u_1 - A\eta_1$, $u_2 = A\eta_2$, where $\eta_1 \leq \eta_2$ on $[-h, T]$ and $\eta_1, \eta_2 \in [\alpha, \beta]$.

Set $m = u_1 - u_2$. Usig (ii) (iii) ad (8), we get

$$m'(t) = u_1'(t) - u_2'(t)$$

$$= \left[-Mu_1(t) + N \sup_{s \in [t-h, t]} u_1(s) + f\left(t, \eta_1(t), \sup_{s \in [t-h, t]} \eta_1(s)\right)\right.$$

$$\left. + M\eta_1(t) - N \sup_{s \in [t-h, t]} \eta_1(s)\right] - \left[-Mu_2(t) + N \sup_{s \in [t-h, t]} u_2(s)\right.$$

$$\left. + f\left(t, \eta_2(t), \sup_{s \in [t-h, t]} \eta_2(s)\right) + M\eta_2(t) - N \sup_{s \in [t-h, t]} \eta_2(s)\right]$$

$$\leq -M(u_1(t) - u_2(t)) + N\left(\sup_{s \in [t-h, t]} u_1(s) - \sup_{s \in [t-h, t]} u_2(s)\right)$$

$$\leq -Mm(t) + N \sup_{s \in [t-h, t]} m(s), \ t \neq t_k, \ t \in [0, T],$$

$$\Delta m(t_k) = \Delta u_1(t_k) - \Delta u_2(t_k)$$

$$= [-L_k u_1(t_k) + I_k(\eta_1(t_k)) + L_k \eta_1(t_k)]$$

$$- L_k u_2(t_k) + I_k(\eta_2(t_k)) + L_k \eta_2(t_k)]$$

$$\leq -L_k m(t_k), \ k = 1, 2, \ldots, p,$$

$$m(t) = m(0) = m(T), t \in [-h, 0].$$

In view of Lemma 2, we have $m(t) \leq 0$ on $[-h, T]$, i.e., $u_1 \leq u_2$.

It is now easy to define the sequences of functions $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$ such that $\alpha_{n+1} = A\,\alpha_n$, $\beta_{n+1} = A\,\beta_n$. From (a), (b), we obtain

$$\alpha_0 \le \alpha_1 \le \alpha_2 \le \ldots \le \alpha_n \le \ldots \beta_n \le \ldots \le \beta_2 \le \beta_1 \le \beta_0, \text{ on } [-h, T]$$

and each $\alpha_n, \beta_n \in \Omega \cap \Omega_0$ $(n = 1, 2, \ldots)$ satisfies

$$\alpha_n(t) = \begin{cases} \int_0^T G(t,s) \left(N \sup_{s \in [t\ h,\, t]} \alpha_n(s) + \sigma_{n-1}(s) \right) ds \\ + \sum_{0 < t_k < T} G(t, t_k)(-L_k\,\alpha_n(t_k) + I_k(\alpha_{n-1}(t_k)) + L_k\,\alpha_{n-1}(t_k)), \quad t \in [0, T], \\ \alpha_n(0), t \in [\ h, 0], \end{cases}$$

$$\beta_n(t) = \begin{cases} \int_0^T G(t,s) \left(N \sup_{s \in [t\ h,\, t]} \beta_n(s) + \bar{\sigma}_{n-1}(s) \right) ds \\ + \sum_{0 < t_k < T} G(t, t_k)(-L_k\,\beta_n(t_k) + I_k(\beta_{n-1}(t_k)) + L_k\,\beta_{n-1}(t_k)), \quad t \in [0, T], \\ \beta_n(0), t \in [\ h, 0], \end{cases}$$

where

$$\sigma_{n-1}(t) = f(t, \alpha_{n-1}(t), \sup_{s \in [t-h,\, t]} + M\,\alpha_{n-1}(t) - N \sup_{s \in [t-h,\, t]} \alpha_{n-1}(s)$$

$$\bar{\sigma}_{n-1}(t) = f(t, \beta_{n-1}(t), \sup_{s \in [t-h,\, t]} + M\,\beta_{n-1}(t) - N \sup_{s \in [t-h,\, t]} \beta_{n-1}(s)$$

Therefore, there exist ρ, r such that $\lim_{n \to \infty} \alpha_n(t) = \rho(t)$, $\lim_{n \to \infty} \beta_n(t) = r(t)$ uniformly on $[-h, T]$. Clearly ρ, r satisfy PBVP (1). To prove that ρ, r are the minimal and maximal solutions of PBVP (1), let $x(t)$ be any solution of PBVP (1) such that $x \in [\alpha, \beta]$. Suppose that there exists a positive integer n such that $\alpha_n(t) \le x(t) \le \beta_n(t)$ on $[-h, T]$. Then, setting $m = \alpha_{n+1} - x$, we have

$$m'(t) = \alpha'_{n+1}(t) - x'(t)$$

$$= \left[-M\,\alpha_{n+1}(t) + N \sup_{s \in [t-h,\, t]} \alpha_{n+1}(s) + f\left(t, \alpha_n(t), \sup_{s \in [t,\, h,\, t]} \alpha_n(s) \right) \right.$$

$$\left. + M\,\alpha_n(t) - N \sup_{s \in [t-h,\, t]} \alpha_n(s) \right] - f\left(t, x(t), \sup_{s \in [t-h,\, t]} x(s) \right)$$

$$\leq -Mm(t) + N \sup_{s \in [t-h,\, t]} m(s),\; t \neq t_k,\; t \in [0, T]$$

$$\Delta m(t_k) = \Delta \alpha_{n+1}(t_k) - \Delta x(t_k)$$

$$= [-L_k \alpha_{n+1}(t_k) + I_k(\alpha_n(t_k)) + L_k \alpha_n(t_k)] - I_k(x(t_k))$$

$$\leq -L_k m(t_k),\; k = 1, 2, \ldots, p,$$

$$m(t) = m(0) = m(T),\; t \in [-h, 0].$$

By Lemma 2, $m(t) \leq 0$ on $[-h, T]$, i.e., $\alpha_{n+1}(t) \leq x(t)$ on $[-h, T]$. Similarly, we obtain $x(t) \leq \beta_{n+1}(t)$ on $[-h, T]$. Since $\alpha_0(t) \leq x(t) \leq \beta_0(t)$ on $[-h, \mathrm{T}]$, by induction we get $\alpha_n(t) \leq x(t) \leq \beta_n(t)$ on $[-h, T]$ for every n. Therefore, $\rho(t) \leq x(t) \leq r(t)$ on $[-h, T]$ by taking limit as $n \to \infty$. The proof of the theorem is complete.

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