Indian J. pure appl. Math., 34(1) : 157-162, January 2003

GENERAL PRINCIPLES FOR ISHIKAWA ITERATIVE PROCESS FOR MULTI-VALUED MAPPINGS

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In this paper, we show two generic theorems for the Ishikawa iterative scheme of a pair of multi-valued mappings. Our results extend and improve the corresponding results due to Guay-Singh, Hu-Huang-Rhoades and Rashwan-Saddeek

Key Words : Ishikawa Iterative Scheme; Multi-Valued Mappings; Common Fixed Point

1. INTRODUCTION AND PRELIMINARIES

Guay-Singh[2] and Rashwan-Saddeek[5] established convergence of the Ishikawa iterative schemes for a pair of single-valued contractive type mappings. Beg-Azam[1] proved a similar result for multi-valued mappings. Hu-Huang-Rhoades[3] obtained a generic theorem for the Ishikawa iterative scheme of a pair of multi-valued mappings. The purpose of this paper is to establish two generic theorems for the Ishikawa iterative scheme of a pair of multi-valued mappings. Our results extend and improve the corresponding results due to Guay-Singh[2], Hu-Huang-Rhoades[3] and Rashwan-Saddeek[5].

Let (X, d) be a metric space, $CB(X)$ the collection of nonempty, closed and bounded subsets of X, and $H(A, B)$ the Hausdorff metric on X. Define $d(x, A) = \inf \{d(x, a) : a \in A\}$ for any $A \subseteq X$.

Lemma 1.1 (Nadler[4]) — Suppose that A, B are in $CB(X)$ and a is in A. Then for any $\varepsilon > 0$ there exists some $b \in B$ such that $d(a, b) \leq H(A, B) + \varepsilon$.

Definition 1.1 — Let K be a nonempty convex subset of a normed linear space $(X, \|\cdot\|)$ and let $S, T: K \to CB(K)$ be multi-valued mappings. For any $x_0 \in K$, the sequence $\{x_n\}_{n\geq 0}$ defined by

$$y_n = (1 - b_n) x_n + b_n t_n, \quad t_n \in Tx_n, n \geq 0,$$

$$x_{n+1} = (1 - a_n) x_n + a_n s_n, \quad s_n \in Sy_n, n \geq 0, \qquad \text{... (1.1)}$$

is called the *Ishikawa iterative scheme*, where $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ are sequences in [0, 1].

It follows from Lemma 1.1 that there exist $t_n \in Tx_n, s_n \in Sy_n$ such that

$$\| t_n - s_n \| \leq H(Tx_n, Sy_n) + \varepsilon_n, \quad n \geq 0, \qquad \text{... (1.2)}$$

where $$\lim_{n\to\infty} \varepsilon_n = 0 \text{ and } \varepsilon_n > 0 \text{ for all } n \geq 0. \qquad \text{... (1.3)}$$

Let R^+ denote the set of nonnegative real numbers and

$$\Phi = \{\phi : \phi : (R^+)^2 \to R^+ \text{ satisfies (1.4), (1.5) and (1.6)}\},$$

$$\Psi = \{\psi : \psi : R^+ \to R^+ \text{ satisfies that } \psi(t) < t \text{ for any } t > 0\},$$

where $\phi(x, y)$ is nondecreasing in each coordinate variable, ... (1.4)

$$\phi(0, t) < t \text{ for each } t > 0 \qquad \text{... (1.5)}$$

and $$\limsup_{n\to\infty} \phi(x_n, y_n) \leq \phi\left(\limsup_{n\to\infty} x_n, \limsup_{n\to\infty} y_n\right) \qquad \text{... (1.6)}$$

for any sequences $\{x_n\}_n \geq 0$ and $\{y_n\}_{n\geq 0}$ in R^+.

2. Main Results

Our main results are as follows :

Theorem 2.1 — *Let K be a nonempty closed convex subset of a normed linear space $(X, \|\cdot\|)$ and let $S, T: K \to CB(K)$ be multi-valued mappings. Suppose that the Ishikawa iterative scheme (1.1) with $\{a_n\}_{n\geq 0}, \{b_n\}_{n\geq 0} \subset [0, 1]$ satisfying*

$$\liminf_{n\to\infty} a_n = a > 0, \qquad \text{... (2.1)}$$

and $\{s_n\}_{n\geq 0}$, $\{t_n\}_{n\geq 0}$ *and* $\{\varepsilon_n\}_{n\geq 0}$ *satisfying (1.2) and (1.3) converges strongly to a point p. If there exists an* $\phi \in \Phi$ *and a nonnegative sequence* $\{r_n\}_{n\geq 0}$ *with*

$$\lim_{n\to\infty} r_n = 0 \qquad \text{... (2.2)}$$

such that, for all n sufficiently large, S and T satisfy

$$H(Tx_n, Sy_n) \leq \phi(\| x_n - s_n \|, \| x_n - t_n \|) + r_n \qquad \text{... (2.3)}$$

and
$$H(Sp, Tx_n) \leq \phi\ (\max\ \{\| x_n - p \|, d(p, Tx_n), d(x_n, Tx_n)\},$$

$$\max\ \{d(p, Sp), d(x_n, Sp)\}, \qquad \text{... (2.4)}$$

is a fixed point of S. Furthermore, if there exists an $\psi \in \Psi$ such that

$$H(Tp, Sp) \leq \psi(d(p, Tp), + d(p, Sp)), \qquad \text{... (2.5)}$$

then p is a common fixed point of S and T.

PROOF : It follows from (1.1) that for all $n \geq 0$,

$$\| x_{n+1} - p \| = \| (1 - a_n)(x_n - p) + a_n (s_n - p) \|$$

$$\geq a_n \| s_n - p \| - (1 - a_n) \| x_n - p \|,$$

which implies that

$$a_n \| s_n - p \| \leq \| x_{n+1} - p \| + (1 - a_n) \| x_n - p \| \qquad \text{... (2.6)}$$

for any $n \geq 0$. Since $\lim_{n\to\infty} x_n = p$, by (2.1) and (2.6), we obtain that

$$\lim_{n\to\infty} \| s_n - p \| = 0. \qquad \text{... (2.7)}$$

In view of (1.2), (1.4) and (2.3), we have

$$\| t_n - p \| \leq \| t_n - s_n \| + \| s_n - p \|$$

$$\leq H(Tx_n, Sy_n) + \varepsilon_n + \| s_n - p \|$$

$$\leq \phi(\| x_n - s_n \|, \| x_n - t_n \|) + r_n + \varepsilon_n + \| s_n - p \|$$

$$\leq \phi(\| x_n - p \| + \| s_n - p \|, \| x_n - p \| + \| t_n - p \|)$$

$$+ r_n + \varepsilon_n + \| s_n - p \| \qquad \text{... (2.8)}$$

for all n sufficiently large. By virtue of $\lim_{n \to \infty} x_n = p$, (1.3), (1.4), (1.6), (2.2), (2.7) and (2.8), we easily conclude that

$$\limsup_{n \to \infty} \| t_n - p \|$$
$$\leq \limsup_{n \to \infty} \phi (\| x_n - p \| + \| s_n - p \|, \| x_n - p \| + \| t_n - p \|)$$
$$+ \limsup_{n \to \infty} (r_n + \varepsilon_n + \| s_n - p \|) \qquad \dots (2.9)$$
$$\leq \phi \left(\limsup_{n \to \infty} \phi (\| x_n - p \| + \| s_n - p \|), \limsup_{n \to \infty} (\| x_n - p \| + \| t_n - p \|) \right)$$
$$\leq \phi \left(0, \limsup_{n \to \infty} \| t_n - p \| \right).$$

Thus (1.5) and (2.9) ensures that $\limsup_{n \to \infty} \| t_n - p \| = 0$, which means that

$$\lim_{n \to \infty} t_n = p. \qquad \dots (2.10)$$

Using (2.4) and $t_n \in Tx_n$, $s_n \in Sy_n$, we get that

$$d(p, Sp) \leq \| t_n - p \| + d(t_n Sp)$$
$$\leq \| t_n - p \| + H(Tx_n, Sp)$$
$$\leq \| t_n - p \| + \phi \max \{\| x_n - p \|, d(p, Tx_n), d(x_n, Tx_n)\}, \max \{d(p, Sp), d(x_n, Sp)\})$$
$$\leq \| t_n - p \| + \phi \max \{\| x_n - p \|, \| t_n - p \|, \| x_n - t_n \|\},$$
$$\| x_n - p \| + d(p, Sp)) \qquad \dots (2.11)$$

for all n sufficiently large. It follows from $\lim_{n \to \infty} x_n = p$, (1.4), (1.6), (2.7), (2.10) and (2.11) that

$$d(p, Sp)$$
$$\leq \limsup_{n \to \infty} \phi (\max \{\| x_n - p \|, \| t_n - p \|, \| x_n - t_n \|\}, \| x_n - p \| + d(p, Sp))$$
$$\leq \phi (0, d(p, Sp)). \qquad \dots (2.12)$$

From (1.5) and (2.12) we have $p \in Sp$.
In view of (2.5), we know that

$$d(p, Tp) \leq H(Tp, Sp) \leq \psi(d(p, Tp) + d(p, Sp)) = \psi(d(p, Tp)),$$

which implies that $p \in Tp$. This completes the proof.

***Theorem* 2.2** — *Let K be a nonempty closed convex subset of a normed linear space $(X, \|\cdot\|)$ and let $S, T : K \to CB(K)$ be multi-valued mappings. Suppose that the Ishikawa iterative scheme (1.1) with $\{a_n\}_{n\geq 0}, \{b_n\}_{n\geq 0} \subset [0, 1]$ satisfying (2.1), and $\{s_n\}_{n\geq 0}, \{t_n\}_{n\geq 0}$ and $\{\varepsilon_n\}_{n\geq 0}$ satisfying (1.2) and (1.3) converges strongly to a point p. If there exists an $\phi \in \Phi$ and a nonnegative sequence $\{r_n\}_{n\geq 0}$ such that, for all n sufficiently large, S and T satisfy*

$$H(Tx_n, Sy_n) \leq \phi(\|x_n - s_n\|, \max\{\|x_n - t_n\|, \|y_n - t_n\|\}) + r_n, \quad \ldots (2.13)$$

(2.2) and (2.4), then p is a fixed point of S. Furthermore, if there exists an $\psi \in \Psi$ such that (2.5) holds, then p is a common fixed point of S and T.

PROOF : As in the proof of Theorem 2.1, we conclude that (2.7) holds. In view of (1.2), (1.4) and (2.13), we have

$$\begin{aligned}
&\|t_n - p\| \\
&\leq \|t_n - s_n\| + \|s_n - p\| \\
&\leq H(Tx_n, Sy_n) + \varepsilon_n + \|s_n - p\| \\
&\leq \phi(\|x_n - s_n\|, \max\{\|x_n - t_n\|, \|y_n - t_n\|\}) + r_n + \varepsilon_n + \|s_n - p\| \\
&\leq \phi(\|x_n - s_n\|, \max\{\|x_n - p\| + \|t_n - p\|, (1 - b_n)\|x_n - t_n\|\}) \\
&\quad + r_n + \varepsilon_n + \|s_n - p\| \\
&\leq \phi(\|x_n - s_n\|, \max\{\|x_n - p\| + \|t_n - p\|, (1 - b_n)\|x_n - p\| \\
&\quad + (1 - b_n)\|t_n - p\|\}) + r_n + \varepsilon_n + \|s_n - p\| \\
&\leq \phi(\|x_n - s_n\|, \|x_n - p\| + \|t_n - p\|) + r_n + \varepsilon_n + \|s_n - p\| \quad \ldots (2.14)
\end{aligned}$$

for all n sufficiently large. Since $\lim_{n\to\infty} x_n = p$, by (1.3), (1.4), (1.6), (2.2), (2.7) and (2.14) we conclude that

$$\limsup_{n\to\infty} \|t_n - p\| \leq \phi\left(0, \limsup_{n\to\infty} \|t_n - p\|\right).$$

It follows that $\lim_{n\to\infty} \|t_n - p\| = 0$. The rest of the proof is exactly the same as that of Theorem 2.1. This completes the proof.

Remark 2.1 : If $\phi(x, y) = \alpha x + \beta y$, $\psi(x) = \beta x$, where $\alpha > 0$, $\beta \in (0, 1)$, then Theorems 2.1 and 2.2 reduce to the results which extend Theorem 1.1 of Guay-Singh[2], Theorems 1 and 2 of Hu-Huang-Rhoades[3] and Theorem 2.2 of Rashwan-Saddeek[5].

Remark 2.2 : We take this opportunity to point out that Theorems 2.1 and 2.2 are true also if the Ishikawa iterative scheme (1.1) is replaced by the Ishikawa iterative scheme with errors

$$y_n = a_n' x_n + b_n' t_n + c_n' \upsilon_n, \quad t_n \in Tx_n, n \geq 0,$$

$$x_{n+1} = a_n x_n + b_n s_n + c_n u_n, \quad s_n \in Sy_n, n \geq 0,$$

where $\{u_n\}_{n\geq 0}$ and $\{\upsilon_n\}_{n\geq 0}$ are arbitrary bounded sequences in K, $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$, $\{c_n\}_{n\geq 0}$, $\{a_n'\}_{n\geq 0}$, $\{b_n'\}_{n\geq 0}$ and $\{c_n'\}_{n\geq 0}$ are sequences in [0, 1] with

$$a_n + b_n + c_n = a_n' + b_n' + c_n' = 1, \quad n \geq 0.$$

ACKNOWLEDGEMENT

The authors would like to think the referee for his many helpful comments. This work was supported by Korea Research Foundation Grant (KRF-99-005-D00003).

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