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GENERALIZED INDICES OF NON-PRIMITIVE GRAPHS*

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We obtain the maximum values for generalized indices over the class of non-primitive graphs of order n and over the class of non-primitive simple graphs of order n, and determine the generalized index sets for the class of bipartite graphs of order n, the class of non-primitive graphs of order n and the class of non-primitive simple graphs of order n.

Key Words : Graph; Digraph; Index

1. INTRODUCTION

The index and period of a given digraph D are the minimum nonnegative integer $k = k(D)$ and minimum positive integer $p = p(D)$ such that for any ordered pair of vertices x and y, there is a walk of length k from x to y if and only if there is a walk of length $k + p$ from x to y in D. A digraph D is primitive if D is strongly connected and $p(G) = 1$.

Let D be a digraph of order n with period p, and let $x \in V(D)$. The index, $k_D(x)$, of x in D is defined to be the minimum nonegative k such that for each $y \in V(D)$, there is a walk of length k from x to y if and only if there is a walk of length $k + p$ from x to y in D. If we choose to order the vertices of D in such a way that $k_D(v_1) \leq k_D(v_2) \leq \ldots \leq k_D(v_n)$, then we call $k_D(v_i)$ the ith generalized index of D, denoted by $k(D, i)$. It is obvious that $k(D, 1) \leq k(D, 2) \leq \ldots \leq k(D, n) = k(D)$.

Generalized indices have been investigated in [1]. If D is a primitive digraph of order $n \geq 2$, then $k(D, i)$ is just the generalized exponent $\exp_D(i)$ introduced in [2]. Indices of digraphs and generalized exponents of primitive digraphs have been extensively studied.

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A symmetric digraph D is a digraph where for any $x\, y \in V(D)$, (x, y) is an arc if and only if (y, x) is. An (undirected) graph G naturally corresponds to a symmetric digraph D_G by replacing each edge $[x, y]$ by a pair of arcs (x, y) and (y, x). In this paper we will identify the graph G and digraph D_G. Note that any edge of G corresponds to a directed cycle of length 2 in D_G. It follows that (see [1]) for any graph G, $p(G) = 1$ or 2. If G is connected, then G is primitive if and only if $p(G) = 1$, and G is bipartite if and only if $p(G) = 2$.

Let $B(n)$ be the class of all bipartite graphs of order n, $S(n)$ be the class of all simple graphs of order n, $P(n)$ be the class of all primitive graphs of order n, and $N(n)$ be the class of all non-primitive graphs of order n. Note that $B(n) \subseteq S(n) \cap N(n)$.

For a class $D(n)$ of graphs of order n, let $E(D(n), i) = \{k(G, i) \mid G \in D(n)\}$ be the generalized index set of this class, and let $e(D(n), i) = \max\{k(G, i) \mid G \in D(n)\}$ be the largest value in $E(D(n), i)$.

By [2, Theorem 6.2], [3, Lemma 2.1] and [8, Theorem 2], we have for $1 \le i \le n$,

$$e(P(n), i) = n - 2 + i, \qquad \text{... (1.1)}$$

$$e(P(n) \cap S(n), i) = \begin{cases} n-2 & \text{if } i = 1, 2 \text{ and } n \text{ is even,} \\ n-1 & \text{if } i = 1, 2 \text{ and } n \text{ is odd} \\ n-4+i & \text{if } 3 \le i \le n, \end{cases} \qquad \text{... (1.2)}$$

$$e(B(n), i) = \left\lfloor \frac{n+i-3}{2} \right\rfloor . \qquad \text{... (1.3)}$$

The generalized index sets $E(P(n), i), E(P(n) \cap S(n), i)$ have been determined in [4 & 3]. In this paper, we obtain expressions for $e(N(n), i)$, $e(N(n) \cap S(n), i)$ and determine the generalized index sets $E(B(n), i), E(N(n), i)$ for $1 \le i \le n$.

2. Maximum Values

In this section we will determine the value $e(N(n), i)$ and $e(N(n) \cap S(n), i)$ for $1 \le i \le n$.

Let P_n be a path of order n, K_1 be a simple graph of order 1, K_1^0 be a graph of order 1 with a loop. Let mG be the disjoint union of m copies of a graph G.

***Theorem* 2.1** — *For $n \ge 2$,*

$$e(N(n), i) = \begin{cases} \max\left\{\left\lfloor \frac{n}{2} \right\rfloor - 1, 1\right\}, & \text{if } i = 1, \\ \max\{n - 4 + i, 1\} & \text{if } 2 \le i \le n. \end{cases} \qquad \text{... (2.1)}$$

PROOF : Suppose $G \in N'(n)$.

If G is connected, then G is bipartite, and we have by (1.3)

$$k(G, i) \le \left\lfloor \frac{n+i-3}{2} \right\rfloor \le \begin{cases} \lfloor \frac{n}{2} \rfloor - 1 & \text{if } i = 1, \\ n-4+i & \text{if } 2 \le i \le n. \end{cases}$$

Suppose G is not connected. Let $G_1, G_2, \ldots, G_r$ be all the components of G with $r \ge 2$, and let n_j be the order of G_j for $1 \le j \le r$. Suppose $n_{j_1} = \min_{1 \le j \le r} n_j$. Then $n_{j_1} \le n/2$. Let G_j' be the graph obtained by deleting the vertices of G_j from G. For $1 \le j \le r$, choose to order the vertices $v_j^{(1)}, v_j^{(2)}, \ldots, v_j^{(n_j)}$ of G_j such that $k_G(v_j^{(1)}) \le k_G(v_j^{(2)}) \le \ldots \le k_G(v_j^{(n_j)})$. It follows from (1.1) and (1.3) that for $1 \le m \le n_j$,

$$k_G(v_j^{(m)}) = k_{G_j}(v_j^{(m)}) = k(G_j, m) \le \max\ \{n_j - 2 + m, 1\}, \qquad \text{... (2.2)}$$

and hence for $n_j + 1 \le i \le n$,

$$k(G_j', i - n_j) \le n - n_j - 2 + i - n_j. \qquad \text{... (2.3)}$$

Using (2.2) and (2.3), we have

$$k(G, i) \le k(G_{j_i}, i) \le \max\ \{n_{j_1} - 2 + i, 1\} \le \max \left\{ \left\lfloor \frac{n}{2} \right\rfloor - 2 + i, 1 \right\}$$

for $\quad 1 \le i \le n_{j_1}$,

$$k\ (G,\ i) \le \max \left\{ n_{j_1} - 2 + i, k(G_{j_1}', i - n_{j_1}), 1 \right\}$$

$$\le \max \left\{ n_{j_1} - 2 + i, n - n_{j_1} - 2 + i - n_{j_1}, 1 \right\}$$

$$\le \max \left\{ \left\lfloor \frac{n}{2} \right\rfloor - 2 + i, n - 1 - 2 + i - 1, 1 \right\}$$

$$\le \max\ \{n - 4 + i,\ 1\}$$

for $\quad n_{j_1} + 1 \le i \le n.$

Now it follows that

$$k(G, i) \le \begin{cases} \max\left\{\lfloor \frac{n}{2} \rfloor - 1, 1\right\} & \text{if } i = 1, \\ \max(n - 4 + i) & \text{if } 2 \le i \le n. \end{cases} \quad \dots (2.4)$$

Note that $k(2K_1, i) = 1$ for $i = 1, 2$, $k(3K_1, 1) = 1$. Suppose $n \ge 3$. Let P_n^0 be the graph obtained by adding a loop at an endvertex of a P_n. Take $G_1 = P^0_{\lfloor (n+1)/2 \rfloor} \cup P^0_{\lfloor n/2 \rfloor}$, $G_2 = P^0_{n-1} \cup K_1^0$. Then $G_1, G_2 \in N(n)$ and $k(G_1, 1) = \lfloor n/2 \rfloor - 1$, $k(G_2, i) = k(P^0_{n-1}, i-1) = n - 4 + i$ for $2 \le i \le n$. Hence the bound in (2.4) can be attained for any n, i with $1 \le i \le n, n \ge 2$.

□

Theorem 2.2 — *If* $n = 2, 3$, *then* $e(N(n) \cap S(n), i) = 1$. *If* $n \ge 4$, *then*

$$e(N(n) \cap S(n), i) = \begin{cases} \lfloor \frac{n}{2} \rfloor - 1 & \text{if } i = 1, \\ n - 3 & \text{if } i = 2 \text{ and } n \text{ is odd}, \\ n - 2 & \text{if } i = 2 \text{ and } n \text{ is even}, \\ n - 6 + i & \text{if } 3 \le i \le n. \end{cases} \quad \dots (2.5)$$

PROOF : The case $n = 2, 3$ is trivial. Suppose $n \ge 4$ and $G \in N(n) \cap S(n)$. If G is connected, then G is bipartite, and we have by (1.3)

$$k(G, i) \le \left\lfloor \frac{n+i-3}{2} \right\rfloor \le \begin{cases} \lfloor \frac{n}{2} \rfloor - 1 & \text{if } i = 1, \\ n - 3 & \text{if } i = 2 \text{ and } n \text{ is odd}, \\ n - 2 & \text{if } i = 2 \text{ and } n \text{ is even}, \\ n - 6 + i & \text{if } 3 \le i \le n. \end{cases}$$

Suppose G is not connected. Let $G_1, G_2, \dots, G_r$ be all the components of G with $r \ge 2$, and let n_j be the order of G_j for $1 \le j \le r$. Suppose $n_{j_1} = \min_{1 \le j \le r} n_j$. Then $n_{j_1} \le n/2$. Let G_j' be the graph obtained by deleting the vertex of G_j from G. For $1 \le j \le r$, choose to order the vertices $v_j^{(1)}, v_j^{(2)}, \dots, v_j^{(n_j)}$ of G_j such that $k_G(v_j^{(1)}) \le k_G(v_j^{(2)}) \le \dots \le k_G(v_j^{(n_j)})$. It follows from (1.2) and (1.3) that for $1 \le m \le n_j$ and $n_j \ge 2$,

$$k_G(v_j^{(m)}) = k_{G_j}(v_j^{(m)}) = k(G_j, m)$$

$$\leq \begin{cases} n_j - 2 & \text{if } m = 1, 2 \text{and } n_j \textit{ is even}, \\ n_j - 1 & \text{if } m = 1 \text{ and } n_j \textit{ is odd}, \\ n_j - 4 + m & \text{if } 3 \leq m \leq n, \end{cases} \quad \ldots (2.6)$$

and hence for $n_j \leq i \leq n$ and $n_j \geq 1$,

$$k(G_j', i - n_j) \leq \begin{cases} \max\{n - n_j - 2, 1\} & \text{if } i - n_j = 1, 2 \text{ and } n - n_j \text{ is even}, \\ \max\{n - n_j - 1, 1\} & \text{if } i - n_j = 1, 2 \text{ and } n - n_j \text{ is odd}, \\ n - n_j - 4 + i - n_j & \text{if } 3 \leq i - n_j \leq n - n_j \end{cases} \quad \ldots (2.6)$$

$$\leq \begin{cases} n - 3 & \text{if } i + n_j + 1, n_j + 2 \text{ and } n - n_j \text{ is even}, \\ n - 2 & \text{if } i = n_j + 1, n_j + 2 \text{ and } n - n_j \text{ is off}, \\ n - 6 + i & \text{if } n_j + 3 \leq i \leq n. \end{cases} \quad \ldots (2.7)$$

Using (2.6) and (2.7), we have

$$k(G, i) \leq k(G_{j1}, i)$$

$$\leq \begin{cases} n_{j_1} - 2 & \text{if } i = 1, 2 \text{ and } n_{j_1} \text{ is even}, \\ \max(n_{j_1} - 1, 1) & \text{if } i = 1, 2 \text{ and } n_{j_1} \text{ is odd}, \\ n_{j_1} - 4 + 1 & \text{if } 3 \leq i \leq n_{j_1} \end{cases}$$

$$\leq \begin{cases} \lfloor \frac{n}{2} \rfloor - 1 & \text{if } i = 1, \\ n - 3 & \text{if } i = 2 \text{ and } n \text{ is odd}, \\ n - 2 & \text{if } i = 2 \text{ and } n \text{ is even}, \\ n - 6 + i & \text{if } 3 \leq i \leq n_{j_1}. \end{cases}$$

for $1 \leq i \leq n_{j_1}$,

$$k(G, i) \leq \max\left\{ n_{j_1} - 4 + i, k(G_{j_1}', i - n_{j_1}) \right\} \leq n - 6 + i \text{ for } n_{j_1} + 1 \leq i \leq n \text{ and } n_{j_1} \geq 2,$$

$$k(G, i) \leq \max\{1, k(G', i - 1)\} \leq \begin{cases} n - 3 & \text{if } i = 2 \text{ and } n \text{ is odd}, \\ n - 2 & \text{if } i = 2 \text{ and } n \text{ is even}, \\ n - 6 + i & \text{if } 3 \leq i \leq n \end{cases}$$

for $n_{j_1} + 1 \leq i \leq n$ and $n_{j_1} = 1$.

Hence, we have proved that

$$k(G,i) \le \begin{cases} \lfloor \frac{n}{2} \rfloor - 1 & \text{if } i=1, \\ n-3 & \text{if } i=2 \text{ and } n \text{ is odd}, \\ n-2 & \text{if } i=2 \text{ and } n \text{ is even}, \\ n-6+i & \text{if } 3 \le i \le n. \end{cases} \quad \ldots (2.8)$$

Let $G_1 = G^{(1)} \cup K_1$ where $G^{(1)}$ is the graph obtained by identifying an endvertex of P_{n-3} with a vertex of a triangle, $G_2 = C_{n-1} \cup K_1$ where C_{n-1} is a cycle of order $n-1$. Then $P_n, G_1\ G_2 \in N(n) \cap S(n)$ and it is easy to see that $k(P_n, 1) = \lfloor n/2 \rfloor, k(G_1, 2) = n-1-2 = n-3$ if n is odd, $k(G_2, 2) = k(C_{n-1}, 1) = n-2$ if n is even, and $k(G_1, i) = k(G^{(1)}, i-1) = n-1-4+i-1 = n-6+i$ for $3 \le i \le n$. Hence, the bound in (2.8) can be attained for any n, i with $1 < i \le n, n \ge 2$. □

3. Generalized Index Sets

In this section we will determine the sets $E(B(n), i), E(N(n), i)$ and $E(N(n) \cap S(n), i)$ for $1 \le i \le n$ explicitly.

Lemma 3.1[8] — Let G be a connected bipartite graph with $u \in V(G)$ and let $d = \max_{x \in V(G)} d_G(u, x)$, where $d_G(u,x)$ is the distance between u and x in G. Then $k_G(u) = d-1$.

Lemma 3.2 — Suppose $G \in B(n)$ with odd n. Then $k(G, n) \ge 1$.

Proof : Suppose $k(G, n) = 0$. Then G contains at least one edge and no isolated vertex, and $k(G, n) = k(G_j, n_j)$ for any component G_j of G. By Lemma 3.1, $d_G(u, x) = 1$ for all $u, x \in V(G_j)$, i.e., G_j is both complete and bipartite. This implies each component is isomorphic to a P_2, so n is even, a contradiction. □

By Lemma 3.1, we can show the following lemma easily.

Lemma 3.3 — Let $T_{n,j}$ be the graph obtained by identifying an endvertex of P_{j+1} with the center of $K_{1,n-j-1}$ where $1 \le j \le n-2$. Then

$$k(T_{n,j}, i) = \begin{cases} \left\lfloor \frac{i+j-1}{2} \right\rfloor & \text{if } 1 \le i \le j+2, \\ j & \text{if } j+3 \le i \le n. \end{cases}$$

Theorem 3.1 — *For any integers i and n with $1 \le i \le n, n \ge 4$,*

$$E(B(n), i) = \begin{cases} \{1, 2, ..., e(B(n), i)\} & \text{if } n \text{ is odd and } i = n, \\ \{0, 1, ..., e(B(n), i\} & \text{otherwise.} \end{cases} \qquad ... (3.1)$$

PROOF : Take an integer j with max $\{i - 2, 1\} \le j \le n-2$. By Lemma 3.3 we have

$$\left\lfloor \frac{i+j-1}{2} \right\rfloor = k(T_{n,j}\, i) \in E(B(n), i).$$

$$\text{So, } E(B(n), i) \supseteq \begin{cases} \left\{ i-2, i-1, ..., \left\lfloor \frac{n+i-3}{2} \right\rfloor \right\} & \text{if } 3 \le i \le n, \\ \left\{ 1, 2, ..., \left\lfloor \frac{n+i-3}{2} \right\rfloor \right\} & \text{if } i = 2, \\ \left\{ 1, 2, ..., \left\lfloor \frac{n+i-3}{2} \right\rfloor \right\} & \text{if } i = 1. \end{cases} \qquad ... (3.2)$$

For $i > 3$, take an integer j with $1 \le j \le i-3$. By Lemma 3.3,

$$j = k(T_{n,j}, i) \in E(B(n), i).$$

$$\text{So } \{1, 2, ..., i-3\} \subseteq E(B(n), i). \qquad ... (3.3)$$

Note that

$$0 = \begin{cases} k\left(\frac{n}{2} P_2, i\right) \in E(B(n), i) & \text{if } n \text{ is even and } 1 \le i \le n, \\ k\left(\frac{n-1}{2} P_2 \cup K_1, i\right) \in E(B(n), i) & \text{if } n \text{ is odd and } 1 \le i \le n-1. \end{cases} \qquad ... (3.4)$$

By combining (3.2), (3.3) and (3.4), we have

$$E(B(n), i) \supseteq \begin{cases} \{1, 2, ..., e(B(n), i)\} & \text{if } n \text{ is odd and } i = n, \\ \{0, 1, ..., e(B(n), i\} & \text{otherwise.} \end{cases}$$

By (1.3) and Lemma 3.2, (3.1) follows. □

Lemma 3.4[4&5] — For $n \ge 2$,

$$E(p(n), i) = \begin{cases} \{1, 2, ..., n-2+i\} & \text{if } 1 \le i \le n-1, \\ \{1, 2, ..., 2n-2\} \setminus S_1 & \text{if } i = n, \end{cases}$$

where S is the set of odd integers in $\{n, n+1, ..., 2n-3\}$.

Lemma 3.5 — Suppose $G \in N(n)$. Then $k(G, n) \in \{0, 1, ..., 2n-4\}$ S, where S is the set of odd integers in $\{n-1, n, ..., 2n-5\}$.

PROOF : Note that $k(G,n) = k(G_j, n_j)$ for some component G_j of G with order n_j, $1 \le n_j \le n$. If G_j is bipartite, then we have by (1.3) that $k(G) \le n-2$. Suppose G_j is primitive. If $n_j = 1$, then $k(G, n) = 0$; otherwise let S_2 is the setof odd integers in $\{n_j, n_j+1, \ldots, 2n_j-3\}$, we have by Lemma 3.4 that $k(G,n) \in \{1, 2, ..., 2n_j-2\} \backslash S_2 \subseteq \{1, 2, ..., 2n-4\} \backslash S$. □

Theorem 3.2 — *For any integers i and n with* $1 \le i \le n, n \ge 4$,

$$E(n(n), i = \begin{cases} \{0, 1, ..., e(N(n), i)\} & \text{if } 1 \le i \le n-1, \\ \{0, 1, ..., e(N(n), i)\} \backslash S & \text{if } i = n, \end{cases} \quad \text{... (3.5)}$$

where S is the set of odd integers in $\{n-1, n, ..., 2n-5\}$.

PROOF : Note that $E(B(n), 1) \subseteq E(N(n), 1)$ and $e(B(n), 1) = e(N(n), 1)$.

The case $i = 1$ follows from Theorem 3.1.

Suppose $2 \le i \le n$. Let m be any integer (depending on i) with

$$m \in \begin{cases} \{0, 1, ..., e(N(n), i)\} & \text{if } 1 \le i \le n-1, \\ \{0, 1, ..., e(N(n), i)\} \backslash S & \text{if } i = n. \end{cases}$$

We have $0 = k(nK_1^0, i) \in E(N(n), i)$ for $2 \le i \le n$. Suppose $m \ge 1$. Then there is a graph $G' \in P(n-1)$ such that $m = k(G', i-1)$ by Lemma 3.4, and hence $m = k(G', i-1) = k(G' \cup K_1^0, i)$ $\in E(N(n), i)$. Thus, we have proved that

$$E(n(n), i) \supseteq \begin{cases} \{0, 1, ..., e(N(n), i)\} & \text{if } 2 \le i \le n-1, \\ \{0, 1, ..., e(N(n), i)\} \backslash S & \text{if } i = n, \end{cases}$$

Now by Theorem 2.1 and Lemma 3.4, this theorem follows. □

Lemma 3.6[3&6] — For $n \ge 2$,

$$E(P(n) \cap S(n), i) = \begin{cases} \{2, 3, ..., n-2+i\} & \text{if } 1 \le i \le n-1, \\ \{2, 3, ..., 2n-2\} \backslash T_1 & \text{if } i = n, \end{cases}$$

where T_1 is the set of odd integers in $\{n-2, n-1, ..., 2n-5\}$.

Using Lemma 3.6, we can prove the following lemma by similar arguments as in Lemma 3.5.

Lemma 3.7 — Suppose $G \in N(n) \cap S(n)$ and $k(G, n) \neq 0$. Then $k(G, n) \in \{1, 2, ..., 2n - 6\} \backslash T$, where T is the set of odd integers in $\{n - 2, n - 1, ..., 2n - 7\}$.

Lemma 3.8 — Suppose $G \in N(n) \cap S(n)$ with odd n. Then $k(G, n) \geq 1$.

PROOF : If $G \in B(n)$, then we have $k(G, n) \geq 1$ by Lemma 3.2. Suppose $G \notin B(n)$, then G contains a primitive component G_j which is simple. We have $k(G, n) \geq k(G_j) \geq 2$. □

***Theorem* 3.3** — *Denote* $e(n, i) = e(B(n) \cap S(n)\, i)$. *Then for any integers* i *and* n *with* $1 \leq i \leq n, n \geq 4$,

$$E(N(n) \cap S(n), i) = \begin{cases} \{0, 1, ..., e(n, i)\} & \text{if } 1 \leq i \leq n-1, \\ \{0, 1, ..., e(n, i)\} \backslash T & \text{if } n \textit{ is even} \text{ and } i = n, \\ \{1, 2, ..., e(n, i)\} \backslash T & \text{if } n \textit{ is odd} \text{ and } i = n. \end{cases}$$

where T is the set of odd integers in $\{n-2, n-1, ..., 2n-7\}$

PROOF : The case $i = 1$ followd from Theorem 3.1.

Suppose $2 \leq i \leq n$. Let m be any integer (depending on i) with

$$m \in \begin{cases} \{0, 1, ..., e(n, i)\} & \text{if } 2 \leq i \leq n-1, \\ \{0, 1, ..., e(n, i)\} \backslash T & \text{if } n \textit{ is even} \text{ and } i = n, \\ \{1, 2, ..., e(n, i)\} \backslash T & \text{if } n \textit{ is odd} \text{ and } i = n. \end{cases}$$

First we have $1 = k(nK_1, i) \in E(N(n) \cap S(n), i)$, for $2 \leq i \leq n, 0 = k(n/2\, P_2, i) \in E(N(n) \cap S(n), i)$ for $2 \leq i \leq n$ if n is even, and $0 = k((n-1)/P_2 \cup K_1, i) \in E(N(n) \cap S(n), i)$ for $2 \leq i \leq n - 1$ if n is odd. Next suppose $m \geq 2$. Then we have by Lemma 3.6 that $m = k(G', i-1) = k(G' \cup K_1, i) \in E(N(n) \cap S(n), i)$ for some $G' \in P(n-1) \cap S(n-1)$ and $2 \leq i \leq n$. Hence have proved that by Lemma 3.6, we have

$$E(N(n) \cap S(n), i) \supseteq \begin{cases} \{0, 1, ..., e(n, i)\} & \text{if } 2 \leq i \leq n-1, \\ \{0, 1, ..., e(n, i)\} \backslash T & \text{if } n \textit{ is even} \text{ and } i = n, \\ \{1, 2, ..., e(n, i)\} \backslash T & \text{if } n \textit{ is odd} \text{ and } i = n. \end{cases}$$

By combining Theorem 3.2, Lemmas 3.7 and 3.8, this theorem follows. □

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