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ON FULLY ROTUNDITY PROPERTIES AND APPROXIMATIVE COMPACTNESS IN SOME BANACH SEQUENCE SPACES

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A new notion called compactly fully k-rotundity is introduced in the class of real Banach spaces and it is proved that a real Banach space X is fully k-rotund if and only if it is rotund and compactly fully k-rotund. Some other general results concerning compactly fully k-rotundity, rotundity, fully k-rotundity, compactly locally fully k-routin-dity are presented. Next, criteria for these properties as well as for approximative compactness in the class of Musielak-Orlicz sequence spaces are obtained.

Key Words : Rotundity; Banach Spaces; Sequence Spaces; Approximative Compactness

1. INTRODUCTION

Let $\mathcal{N}$, $\mathcal{R}$ and $\mathcal{R}_+$ stand for the sets of natural, real and nonnegative real numbers, respectively. Denote by l^0 the space of all real sequences. Let $(X, \|\ \|)$ be a real Banach spaces and let X^* denote its dual. By $S(X)$ we denote the unit sphere of X.

Let us recall some geometric notions concerning Banach spaces. A Banach space X is said to be uniformly rotund (UR for short) if for any $\varepsilon \in (0, 1)$ there is $\delta(\varepsilon) \in (0, 1)$ such that for any

$x, y \in S(X)$ such that $\|x-y\| \geq \varepsilon$ there holds $\|(x+y)/2\| \leq 1-\delta(\varepsilon)$ (see [2]). A Banach space is said to be rotund (R for short) if for $x, y \in x$ the equations $\|x\| = \|y\| = \|(x+y)/2\|$ imply $x = y$.

Ky Fan and I. Gliksberg[9] introduced the notion of fully rotundity which has (among others) an application in the approximation theory (see [1]). A Banach space X is said to be fully k-rotund ($k \geq 2, k \in \mathcal{N}$) if every sequence (x_n) is $S(X)$ such that $\|x_n^{(1)} + x_n^{(2)} + \ldots + x_n^{(k)}\| \to k$ as $n \to \infty$ for all its subsequences $(x_n^{(1)}), (x_n^{(2)}), \ldots, (x_n^{(k)})$ is a Cauchy sequence.

It is natural to introduce the notion of compactly fully k-rotundity. We say that a Banach space is compactly fully k-rotund if any sequence $(x_n) \in S(X)$ such that $\|x_n^{(1)} + x_n^{(2)} + \ldots + x_n^{(k)}\| \to k$ as $n \to \infty$ for all its subsequences $(x_n^{(1)}), (x_n^{(2)}), \ldots, (x_n^{(k)})$, forms a relatively compact set.

As we will see below a Banach space X is fully k-rotund if and only if it is compactly fully k-rotund and rotund.

Recall following [18] that for any $k \geq 2$ a Banach space X is said to be locally fully k-rotund (LkR for short) if for any sequence (x_n) in $S(X)$ and any $x \in S(X)$ such that $\|x + x_n^{(1)} + x_n^{(2)} + \ldots + x_n^{(k)}\| \to k+1$ as $n \to \infty$ for all its subsequences $(x_n^{(1)}), (x_n^{(2)}), \ldots, (x_n^{(k)})\|$ forms a relatively compact set.

It can be also noted that $CLkR$ of X and rotundity of X give together the LkR property of X.

A convex set C in a Banach space X is said to be approximatively compact if for any $x \in X$ and any sequence (x_n) in C which is minimising for x, i.e. $\|x_n - x\| \to d(x, C) :\equiv \inf\{\|x-y\|, y \in C\}$, it follows that (x_n) has a Cauchy subsequence. X is said to be approximatively compact if any closed convex set in X is approximatively compact.

A Banach space X is said to be a Banach sequence lattice if it is a subspace of l^0 such that

1^o $x \in l^o, y \in X$ and $|x(i)| \leq |y(i)|$ for all $i \in N$ imply that $x \in X$

and $\|x\| \leq \|y\|$,

2^o there is $x \in X$ such that $x(i) \neq 0$ for all $i \in \mathcal{N}$.

We say that a Banach sequence lattice X has the semi-Fatou property if for every $x, x_n \in X, n \in N$, there holds $\|x_n\| \uparrow \|x\|$ whenever $|x_n(i)| \uparrow |x(i)|$ for all $i \in \mathcal{N}$. Sometimes this property is called monotone completeness.

A Banach sequence lattice X is said to be order continuous if for any $x \in X$, $x \geq 0$ and any sequence (x_n) in X such that $0 \leq x_n \uparrow x$ coordinatewise, there holds $\| x_n - x \| \to 0$ as $n \to \infty$. It is known that order continuity of X is equivalent to the fact that $\| (0, \ldots, 0, x(n+1), x(n+2) \ldots) \| \to 0$ as $n \to \infty$ for any $x \in X$.

2. Some General Results

We start with the following

Proposition 1 — If X is compactly fully k-rotund, then X is reflexive.

Proof : By the well-known James result it is enough to show that any $f \in S(X^*)$ attains its norm on $S(X)$. Take any $f \in S(X^*)$ and choose a sequence $(x_n) \in S(X)$ such that $f(x_n) \to 1$. For every k-subsequences $(x_n^{(1)}) .., (x_n^{(k)})$ of (x_n) there holds $f(x_n^{(1)} + \ldots + x_n^{(k)}) \to k$ as $n \to \infty$, whence $\| x_n^{(1)} + \ldots + x_n^{(k)} \| \to k$ as $n \to \infty$. By the assumption that X is compactly fully k-rotund, we conclude that there are a subsequence (x_{n_i}) of (x_n) and some $x \in B(X)$ such that $\| x_{n_i} - x \| \to 0$ as $i \to \infty$. Obviously, $x \in S(X)$ and $f(x) = 1 = \| x \|$, which finishes the proof.

Proposition 2 — A Banach space X is fully k-rotund if and only if it is compactly fully k-rotund and rotund.

Proof : It is obvious that fully k-rotundity implies compactly fully k-rotundity. For the sake of completeness we will give a short proof that it implies rotundity (although this implication is known). Assume that $x, y \in S(X)$ and $\| x + y \| = 2$. Let us define the sequence $(x_n) = (x, y, x, y, \ldots)$. We will show that for every its k-subsequences $(x_n^{(1)}), \ldots, (x_n^{(k)})$ there holds $\| x_n^{(1)} + \ldots + x_n^{(k)} \| \leq k$ for all $n \in N$. It is obvious that for any $n \in N$ the element $x_n^{(1)} + \ldots + x_n^{(k)}$ is of the form $(x + \ldots + x) + (y + \ldots + y)$ where denoting by l the number of x, we have that y appears $k - l$ times. Assume without loss of generality that $l \geq k - l$. Then

$$\| x_n^{(1)} + \ldots + x_n^{(k)} \| = \| \ (x+y) - (2l-k)\, y \| \geq \| \, l\,(x+y) \| - \| (2l-k)\, y \| = 2l - (2l-k) = k.$$

So the equality $\| x_n^{(1)} + \ldots + x_n^{(k)} \| = k$ for any $n \in \mathcal{N}$ is proved. By the assumption that X is fully k-rotund we obtain that (x_n) is a Cauchy sequence. But this implies that $x = y$, which means that X is rotund.

Let us prove now that compactly fully k-rotundity and rotundity of X imply together fully rotundity of X.

Assume that (x_n) is a sequence in $S(X)$ such that $\| x_n^{(1)} + \ldots + x_n^{(k)} \| \to k$ for all its subsequences $(x_n^{(1)}), \ldots, (x_n^{(k)})$. From the assumption that X is compactly fully k-rotund it follows that there is a subsequence (x_{n_i}) of (x_n) and $x \in S(X)$ such that $\| x_{n_i} - x \| \to 0$ as $i \to \infty$. By rotundity of X we conclude that all convergent subsequences of (x_n) have the same limit. To prove this assume that (x_{n_i}) and (x_{n_l}) are subsequences of $(x_n), x, y \in X, \| x_{n_i} - x \| \to 0$ as $i \to \infty$ and $\| x_{n_l} - y \| \to 0$ as $l \to \infty$. It s obvious that $x, y \in S(X)$. By the assumption on (x_n), we have

$$\| \overbrace{(x_{n_i} + \ldots + x_{n_i})}^{(k-1)\text{ times}} + x_{n_l} \| \to k.$$

By $\| x_{n_i} - x \| \to 0$ and $\| x_{n_l} - y \| \to 0$, we get

$$\| \overbrace{(x_{n_i} + \ldots + x_{n_i})}^{(k-1)\text{ times}} \| + x_{n_l} - (k-1)x - y \| \to 0, \text{ whence}$$

$$\| \overbrace{(x_{n_i} + \ldots + x_{n_i})}^{(k-1)\text{ times}} \| + x_{n_l} \| \to \| (k-1)x + y \|.$$

Consequently, $\| (k-1)x + y \| = k$. Note that this yields the equality $\| x + y \| = 2$. Indeed, assuming for the contrary that $\| x + y \| < 2$, we get

$$\| (k-1)x + y \| = \| x + y + (k-2)x \| \leq \| x + y \| + (k-2) \| x \| < 2 + (k-2) = k,$$

a contradiction. By the assumption that X is rotund we get $x = y$.

Proposition 3 — Assume that X is a Banach sequence lattice which has the semi-Fatou property. If X is locally compactly fully k-rotund, then X is order continuous.

PROOF : Under the assumption on X, suppose that X is not order continuous, i.e. there is $x \in S(X)$ such that

$$\lim_{n \to \infty} \| (0, \ldots, 0, x(n+1), x(n+2), \ldots) \| = \alpha > 0.$$

Since X has the semi-Fatou property, there exists a sequence $(m_n) \subset \mathcal{N}$ such that

$$\left\| \sum_{i=m_n+1}^{m_n+1} x(i)\, e_i \right\| \geq \frac{\alpha}{2} \text{ for all } n \in \mathcal{N}. \qquad ..\ (1)$$

Define

$$x_n = (x(1), \ldots, x(m_n), 0, \ldots, 0, x(m_{n+1}+1), x(m_{n+1}+2), \ldots)$$

$$y_n = (x(1), \ldots, x(m_n), 0, \ldots)$$

for $n = 1, 2, \ldots$. Since $0 \leq |y_n| \leq |x_n| \leq |x|$ and $|y_n| \uparrow |x|$ coordinatewise, we get by the semi-Fatou property that $\|x_n\| \to \|x\| = 1$ as $n \to \infty$. We have for every k subsequences $(x_n^{(1)}), \ldots, (x_n^{(k)})$ of (x_n) that

$$\|x + x_n^{(1)} + \ldots + x_n^{(k)}\| \leq k + 1. \qquad \ldots\ (2)$$

On the other hand there is a sequence (l_n) in $\mathcal{N}$ such that $l_n \to \infty$ as $n \to \infty$ and for $p = 1, 2, \ldots, k$ there holds $x_n^{(p)}(i) = x(i)$ for all $i \leq l_n$. Therefore

$$\|x + x_n^{(1)} + \ldots + x_n^{(k)}\| \geq (k+1)\, \|(x(1), \ldots, x(l_n), 0, 0, \ldots)\| \uparrow k + 1 \qquad \ldots\ (3)$$

as $n \to \infty$. Conditions (2) and (3) yield

$$\|x + x_n^{(1)} + \ldots + x_n^{(k)}\| \to k + 1, \text{ as } n \to \infty.$$

On the other hand it follows from condition (1) that the sequence (x_n) does not form a relatively compact set because it is not a Cauchy sequence.

Proposition 4 — A Banach space X is locally fully k-rotund if and only if X is rotund and compactly locally fully k-rotund.

PROOF : It is obvious that if X is locally fully k-rotund, then it is compactly locally fully k-rotund. We will prove that X is then rotund. Assume that $\|x\| = \|y\| = 1$ and $\|x + y\| = 2$. Define $(x_n) = (x, y, x, y, \ldots)$. Then we can prove in the same way as in Proposition 2 that $\|x + x_n^{(1)} + \ldots + x_n^{(k)}\| = k + 1$ for all $n \in \mathcal{N}$ if $(x_n^{(i)})$, $(i = 1, \ldots, k)$ are arbitrary subsequences of (x_n). Since X is locally fully k-rotund, we get $x_n \to x$, Then it must be $y = x$, which means that X is rotund.

Assume now that X is rotund and compactly locally fully k-rotund and that (x_n) is a sequence in $S(X)$ such that for some $x \in S(X)$ there holds $\|x + x_n^{(1)} + \ldots + x_n^{(k)}\| \to k + 1$ as $n \to \infty$, for every subsequences $(x_n^{(i)})$, $(i = 1, \ldots, k)$ of (x_n). By compactly locally fully k-rotundity, for any subsequence (x_{n_k}) of (x_n) there is its subsequence $(x_{n_{k_l}})$ and $y \in S(X)$ such that $\| x_{n_{k_l}} - y \| \to 0$ as $l \to \infty$. In consequence

$$\| kx_{n_{k_l}} + x \| \to \| ky + x \|.$$

On the other hand, $\| k\, x_{n_{k_l}} + x \| \to k+1$. In consequence $\| ky + x \| = k+1$, whence $\| y + x \| = 2$. By rotundity of X, we get $x = y$ and consequently $x_{n_{k_l}} \to x$. By the double extract subsequence theorem, it follows that $x_n \to x$ as $n \to \infty$, which means that X is locally fully k-rotund.

3. Fully Rotundity Properties for Musielak-Orlicz Sequence Spaces

We start with some definitions. A mapping $\Phi : \mathcal{R} \to \mathcal{R}_+$ is said to be an Orlicz function if Φ vanishes at zero, $\Phi(u) \to \infty$ as $|u| \to \infty$ and Φ is even and convex. A sequence $\Phi = (\Phi_i)$ of Orlicz functions Φ_i $(i = 1, 2, \ldots)$ is called a Musielak-Orlicz function. Given a Musielak-Orlicz function Φ the functional I_Φ defined on l^0 by

$$I_\Phi(x) = \sum_{i=1}^{\infty} \Phi_i(x_i)$$

is a convex modular (see [17]). The Musielak-Orlicz space l^Φ generated by Φ is defined by

$$l^\Phi = \{x \in l^0 : I_\Phi(\lambda x) < \infty \text{ for some } \lambda > 0\}.$$

The space l^Φ equipped with the Luxemburg norm $\| \cdot \|_\Phi$ defined by

$$\| x \|_\Phi = \inf \{\lambda : I_\Phi(x/\lambda) \le 1\}$$

is Banach sequence lattice (see [3], [13], [16] and [17]).

We say that a Musielak-Orlicz function Φ satisfies the δ_2-condition ($\Phi \in \delta_2$ for short) if there exist constants $K \ge 2$ and $a > 0$ and a sequence (c_i) in $\mathcal{R}_+$ such that $\sum_{i=1}^{\infty} c_i < \infty$ and the inequality

$$\Phi_i(2u) \le K \Phi_i(u) + c_i$$

holds for any $i \in \mathcal{N}$ and $u \in \mathcal{R}$ whenever $\Phi_i(u) \le a$. We assume that Φ satisfies the δ_2^*-condition ($\Phi \in \delta_2^*$ for short) if $\Psi \in \delta_2$, where ψ is the function complementary to Φ in the sense of Young, i.e., $\Psi(u) = \sup_{\upsilon > 0} \{|u|\upsilon - \Phi(\upsilon)\}$.

We say that a Musielak-Orlicz function Φ satisfies condition (*) if for any $\varepsilon \in (0, 1)$ there is $\delta > 0$ such that $\Phi_i((1+\delta)u) \le 1$ whenever $\Phi_i(u) \le 1 - \varepsilon$ for all $i \in \mathcal{N}$ (see [13]). For more details on Musielak-Orlicz spaces we refer to [3] and [17].

In order to establish our new results, we need to recall some known facts.

Lemma 1 (see [13]) — If a Musielak-Orlicz function $\Phi = (\Phi_n)$ satisfies condition (*) and $\Phi \in \delta_2$, then for each $\varepsilon > 0$ there is $\delta > 0$ such that $\|x\|_\Phi < 1 - \delta$ whenever $I_\Phi(x) < 1 - \varepsilon$.

Lemma 2 (see [13]) — If a Musielak-Orlicz function Φ satisfies condition (*) and $\Phi \in \delta_2$, then for each $\varepsilon > 0$ there is $\sigma > 0$ such that $|I_\Phi(x) - I_\Phi(y)| < \varepsilon$ whenever $I_\Phi(x) \le 1, I_\Phi(y) \le 1$ and $I_\Phi(x - y) \le \sigma$.

Lemma 3 (see [6]) — If a Musielak-Orlicz function Φ satisfies the δ_2-condition, then $\|x\|_\Phi = 1$ if and only if $I_\Phi(x) = 1$.

Lemma 4 (see [6]) — If a Musielak-Orlicz function Φ satisfies δ_2^*-condition, then there are a number $\theta \in (0, 1)$ and a sequence (h_i) in $\mathcal{R}_+$ with $\sum_{i=1}^{\infty} \Phi_i(h_i) < \infty$ such that the inequality

$$\Phi_i\left(\frac{u}{2}\right) \le \frac{(1-\theta)}{2} \Phi_i(u)$$

holds for every $i \in \mathcal{N}$ and $u \ge 0$ satisfying $\Phi_i(h_i) \le \Phi_i(u) \le 1$.

Lemma 5 (see [21]) — If a Musielak-Orlicz function Φ does not satisfy the δ_2^*-condition, then for any $k \in N$ there exist a sequence (I_n) of integers such that $0 = I_0 < I_1 < I_2 < \ldots$ and a family of sequences (u_i^n) in $\mathcal{R}_+$ ($n = 1, 2, \ldots; i = I_{n-1} + 1, \ldots, I_n$) such that

$$\sum_{i=I_{n-1}+1}^{I_n} \Phi_i(u_i^n) > 1 \text{ for each } n \in \mathcal{N} \qquad \ldots (4)$$

$$\Phi_i(u_i^n) \leq \frac{1}{n} \text{ and } \Phi_i\left(\frac{u_i^n}{k}\right) > \left(1 - \frac{1}{n}\right)\frac{\Phi_i(u_i^n)}{k} \qquad \text{... (5)}$$

for every $n \in \mathcal{N}$ and $i \in \{I_{n-1} + 1, \ldots, I_n\}$.

In this section it will be more convenient to denote the subsequences of (x_n) by $(x_{n_1}), \ldots, (x_{n_k})$ in place of $(x_n^{(1)}), \ldots, (x_n^{(k)})$.

Theorem 1 — *Assume that a Musielak-Orlicz function Φ satisfies the condition (*) and $k \geq 2$ is a natural number. Then l^Φ is compactly fully k-rotund if and only if $\Phi \in \delta_2$ and $\Phi \in \delta_2^*$ (i.e. l^Φ is reflexive).*

PROOF : Notice that if a Banach space X is compactly fully 2-rotund, then it is compactly fully k-rotund for $k \geq 2$. Suppose for the contrary that $\Phi \in \delta_2$, $\Phi \in \delta_2^*$ and l^Φ is not compactly fully 2-rotund. Then there exists a sequence (x_n) is $S(l^\Phi)$ such that $\| x_{n_1} + x_{n_2} \|_\Phi \to 2$ as $n_1, n_2 \to \infty$ and (x_n) does not form a relatively compact set, which means that there is $\varepsilon_0 > 0$ such that $\| x_{n_i} - x_{n_j} \|_\Phi > \varepsilon_0$ for all $i, j \in \mathcal{N}$ with $i \neq j$ and some subsequence (x_{n_l}) of (x_n). Hence it follows that there is a sequence (n_m) in $\mathcal{N}$ such that

$$\left\| \sum_{i=m}^{\infty} x_{n_m}(i)\, e_i \right\|_\Phi \geq \frac{\varepsilon_0}{2} \text{ for each } m \in \mathcal{N}. \qquad \text{... (6)}$$

Since $\Phi \in \delta_2$, there exists a number $\eta > 0$ such that

$$I_\Phi\left(\sum_{i=m}^{\infty} x_{n_m}(i)\, e_i \right) \geq \eta \ (m = 1, 2, \ldots). \qquad \text{... (7)}$$

By $\Phi \in \delta_2^*$, there exists $\theta \in (0, 1)$ and a sequence (h_i) in $\mathcal{R}_+$ with $\sum_{i=1}^{\infty} \Phi_i(h_i) < \infty$ such that

$$\Phi_i\left(\frac{u}{2}\right) \leq \frac{(1-\theta)}{2}\, \Phi_i(u) \qquad \text{... (8)}$$

holds for every $i \in \mathcal{N}$ and $u \geq 0$ satisfying $\Phi_i(h_i) \leq \Phi_i(u) \leq 1$. Using $\Phi \in \delta_2$ and condition (*) we conclude that there exists $\sigma > 0$ such that the inequality

$$|I_\Phi(x)-I_\Phi(y)|<\frac{\eta\theta}{8} \quad \text{... (9)}$$

holds whenever $I_\Phi(x)\le 1, I_\Phi(y)\le 1$ and $I_\Phi(x-y)\le\sigma$. Since $I_\Phi\left(\frac{x_{n_{m_1}}+x_{n_{m_2}}}{2}\right)\to 1$ as $m_1, m_2\to\infty$, there exists $m_0\in\mathcal{N}$ such that $I_\Phi\left(\frac{1}{2}(x_{n_{m_1}}+x_{n_{m_2}})\right)\ge 1-\frac{\eta\theta}{8}$ whenever $m_1, m_2\ge m_0$. Let $i_0\in\mathcal{N}$ be such that $i_0\ge m_0$ and

$$\sum_{i=i_0+1}^{\infty}\Phi_i(x_{n_{m_0}}(i))<\sigma \text{ and } \sum_{i=i_0+1}^{\infty}\Phi_i(h_i)<\frac{\eta\theta}{8}. \quad \text{... (10)}$$

Then we get

$$1-\frac{\eta\theta}{8}\le I_\Phi\left(\frac{1}{2}(x_{n_{m_0}}+x_{n_{i_{0+1}}})\right)=\sum_{i=1}^{\infty}\Phi_i\left(\frac{1}{2}(x_{n_{m_0}}(i)+x_{n_{i_{0+1}}}(i))\right)$$

$$\le\sum_{i=1}^{i_0}\Phi_i\left(\frac{1}{2}(x_{n_{m_0}}(i)+x_{n_{i_{0+1}}}(i))\right)+\sum_{i+i_0+1}^{\infty}\Phi_i\left(\frac{1}{2}x_{n_{i_{0+1}}}(i)\right)+\frac{\eta\theta}{8}$$

$$\le\frac{1}{2}\left(\sum_{i=1}^{i_0}\Phi_0(x_{n_{i_{0+1}}}(i))+\sum_{i=1}^{i_0}\Phi_i(x_{n_{m_0}}(i))\right)$$

$$+\sum_{i=i_0+1}^{\infty}\Phi_0\left(\frac{1}{2}x_{n_{i_{0+1}}}(i)\right)+\frac{\eta\theta}{8} \quad \text{... (11)}$$

$$\le\frac{1}{2}\left(\sum_{i=1}^{i_0}\Phi_i(x_{n_{i_{0+1}}}(i))+\sum_{i=1}^{i_0}\Phi_i(x_{n_{m_0}}(i))\right)$$

$$+\frac{1-\theta}{2}\sum_{i=1}^{i_0}\Phi_i\left(x_{n_{i_{0+1}}}(i)\right)+\sum_{i=1}^{i_0}\Phi_i(h_i)+\frac{\eta\theta}{8}$$

$$\le\frac{1}{2}\sum_{i=1}^{i_0}\Phi_i\left(x_{n_{m_0}}(i)\right)+\frac{1}{2}\sum_{i=1}^{\infty}\Phi_i\left(x_{n_{i_{0+1}}}(i)\right)-\frac{\theta}{2}\sum_{i+1}^{i_0}\Phi_i\left(x_{n_{i_{0+1}}}(i)\right)+\frac{\eta\theta}{4}$$

$$\le 1-\frac{\eta\theta}{2}+\frac{\eta\theta}{4}=1-\frac{\eta\theta}{4},$$

a contradiction. Therefore, $\Phi \in \delta_2$ and $\Phi \in \delta_2^*$ imply the compactly fully 2-rotundity and so compactly fully k-rotundity for any natural $k \geq 2$.

Necessity — Assume that l^{Φ} is compactly fully k-rotund for some $k \geq 2, k \in \mathcal{N}$. Then, by Proposition 1, l^{Φ} is reflexive, which means that $\Phi \in \delta_2$ and $\Phi \in \delta_2^*$

This finishes the proof.

Corollary 1 — Assume that Φ is a Musielak-Orlicz function satisfying condition (*) and $k \geq 2$ is a natural number. Then the following are equivalent :

(1) l^{Φ} is fully k-rotund,

(2) l^{Φ} is compactly fully k-rotund and rotund,

(3) (*a*) $\Phi \in \delta_2, \Phi \in \delta_2^*$,

(*b*) If $a_n > 0$ satisfies the equality $\Phi_n(a_n) = \frac{1}{2}$ $(n = 1, 2, \ldots)$, then at most for one $n \in \mathcal{N}$, Φ_n need not be strictly convex on the interval $[0, a_n]$ and

(*c*) If Φ_n is affine on an interval $[b_n, c_n]$ where $0 < b_n < c_n < \infty$ but Φ_n is not affine on the interval $[b_n - \varepsilon, c_n]$ for any $\varepsilon > 0$, then for any $m \in \mathcal{N}$, $n \neq n$, Φ_m is strictly convex on the interval $[0, \Phi_m^{-1}(\Phi_n(b_n))]$.

PROOF : The implication (1) $\Rightarrow$ (2) follows by Proposition 1. Let us prove that (2) $\Rightarrow$ (3). By Theorem 1, compactly fully k-rotundity of l^{Φ} implies that $\Phi \in \delta_2$ and $\Phi \in \delta_2^*$. Moreover, rotundity of l^{Φ} for $\Phi = (\Phi_i)$ with finitely valued Φ_i is equivalent to the conjunction of conditions (b), (c) and $\Phi \in \delta_2$ from condition (3) (see [12]). So, the implication (2) $\Rightarrow$ (3) is proved.

Let us prove that (3) $\Rightarrow$ (1). Condition (a) in (3) implies, by Theorem 1, that l^{Φ} is compactly fully k-rotund. Conditions $\Phi \in \delta_2$ and (b), (c) in (3) imply that l^{Φ} is rotund (see [12]). Therefore, by Proposition 2, l^{Φ} is fully k-rotund.

Theorem 2 — *Let Φ be a Musielak-Orlicz function satisfying condition (*) and $k \geq 2$ be a natural number. Then l^{Φ} is locally compactly fully k-rotund if and only if :*

(*i*) Φ satisfies the δ_2-condition and

(*ii*) $\Phi \in \delta_2^*$ or Φ_i are strictly convex on the intervals $[0, \Phi_i^{-1}(1)]$ for all $i \in \mathcal{N}$.

PROOF : Since order continuity of l^{Φ} is equivalent to $\Phi \in \delta_2$, by Proposition 4, we need only to show that locally compactly fully k-rotundity of l^{Φ} implies condition (ii). If the condition (ii) is not true, then $\Phi \in \delta_2^*$ and there is a number $i \in \mathcal{N}$ such that Φ_i is not strictly convex on the interval $[0, \Phi_i^{-1}(1)]$. We may assume without loss of generality that Φ_1 is the affine on an interval $[a, b] \subset [0, \Phi_1^{-1}(1)]$ with $a < b$. Then, by Lemma 5, there are a sequence (I_n) of integers such that $0 = I_0 < I_1 < I_2 < \ldots$ and a family of sequences (u_i^n) in $\mathcal{R}_+$ $(n = 1, 2, \ldots;$ $i = I_{n-1}+1, \ldots, I_n)$ satisfying conditions (4) and (5) from Lemma 5. Take $t \geq 0$ such that $\Phi_1(b) + \Phi_2(t) = 1$. Choose natural numbers $m_n \in \{I_{n-1}+1, \ldots, I_n\}$ such that

$$\Phi_1(a) + \Phi_2(t) + \sum_{i=I_{n-1}+1}^{m_n} \Phi_i(u_i^n) \leq 1 \qquad \ldots (12)$$

and

$$\Phi_1(a) + \Phi_2(t) + \sum_{i=I_{n-1}+1}^{m_n+1} \Phi_i(u_i^n) > 1 \,. \qquad \ldots (13)$$

Put $x = (b, t, 0, 0, \ldots)$, $x_n = (a, t, 0, \ldots 0, u_{I_{n-1}}+1, \ldots, u_{m_n}, \ldots)$ where $u_{I_{n-1}}+1$ and u_{m_n} are located on the $(I_{m-1}+3)$th and (m_r+2) the coordinates rest. for each $n \in \mathcal{N}$. It is clear that $\|x\|_{\Phi} = 1$ and $1 - 1/n \leq I_{\Phi}(x_n) \leq 1$, whence $1 - 1/n \leq \|x_n\|_{\Phi} \leq 1$. Moreover,

$$I_{\Phi}\left(\frac{x_{n_1} + \ldots + x_{n_k} + x}{k+1}\right) = \Phi_1\left(\frac{ka+b}{k+1}\right) + \Phi_2(t) + \sum_{j=1}^{k} \sum_{i=I_{n_j}-1+1}^{m_{n_j}} \Phi_i\left(\frac{u_i^{n_j}}{k+1}\right)$$

$$\geq \frac{k}{k+1}\Phi_1(a) + \frac{1}{k+1}\Phi_1(b) + \Phi_2(t) + \sum_{j=1}^{k} \frac{1-\frac{1}{n_j}}{k+1} \sum_{i=I_{n_j}-1+1}^{m_{n_j}} \Phi_i(u_i^{n_j})$$

$$= \frac{k}{k+1}\Phi_1(a) + \frac{1}{k+1}(\Phi_1(b) + \Phi_2(t)) + \frac{k}{k+1}\Phi_2(t) + \sum_{j=1}^{k} \frac{1-\frac{1}{n_j}}{k+1} \sum_{i=I_{n_j-1}+1}^{m_{n_j}} \Phi_i(u_i^{n_j})$$

$$> \frac{1}{k+1}(k\,\Phi_1(a) + 1 + k\,\Phi_2(t)) + \sum_{j=1}^{k} \frac{1-\frac{1}{n_j}}{k+1} \sum_{i=I_{n_j-1}+1}^{m_{n_j}} \Phi_i(u_i^{n_j})$$

$$> 1-\frac{1}{k+1}\sum_{j=1}^{k}\frac{1}{n_j}\to 1$$

as $n_j\to\infty$ for $j = 1, 2, ..., k$. Hence,

$$\left|\left|\frac{1}{k+1}(x_{n_1}+x_{n_2}+\ldots+x_{n_k}+x)\right|\right|_{\Phi}\to 1$$

as $n_j\to\infty$ for $j = 1, 2, ..., k$. On the other hand conditions (12), (13), the equality $\Phi_1(b)+\Phi_2(t)=1$ and the inequality $\Phi_i(u_i^n)\le\frac{1}{n}$ imply that

$$\Phi_1(a)+\Phi_2(t)+\sum_{i=I_{n-1_j}+1}^{m_{n+2}}\Phi_i(u_i^{n_j})\to 1+\Phi_1(b)-\Phi_1(a)$$

as $n_j\to\infty$. Therefore, $\displaystyle\sum_{i=I_{n-1_j}+1}^{m_{n_j}}\Phi_i(u_i^{n_j})\to 1+\Phi_1(b)-\Phi_1(a)$

as $n_j\to\infty$. Consequently,

$$\sum_{i=I_{n-1_j}+1}^{m_{n+2}}\Phi_i(u_i^{n_j})>\frac{1}{2}+(\Phi_1(b)-\Phi_1(a)),$$

for n_j large enough, whence we get that

$$I_{\Phi}(x_{n_l}-x_{n_m})>\frac{1}{2}(\Phi_1(b)-\Phi_1(a))$$

for $m\ne l$ and l, m large enough. Therefore, $\| x_{n_l}-x_{n_m}\|_{\Phi}>\frac{1}{2}(\Phi_1(b)-\Phi_1(a))$ for $m\ne l$ and l, m large enough. This means that the sequence (x_n) does not form a relatively compact set. Consequently, l^{Φ} is not locally compactly fully k-rotund. This finishes the proof of the necessity.

Sufficiency — Assume first that $\Phi\in\delta_2$ and $\Phi\in\delta_2^*$. Then, by Theorem 1, l^{Φ} is compactly fully k-rotund, so locally compactly fully k-rotund as well. Assume now that $\Phi\in\delta_2$ and Φ_i are strictly convex on the interval $[0, \Phi_i^{-1}(1)]$ for is $i\in\mathcal{N}$. We will show that in this case l^{Φ} is locally uniformly rotund, so locally compactly fully k-rotund as well. First, we will show that $x_n\to x$, as

$n \to \infty$, coordinatewise whenever $\| x_n \|_\Phi = \| x \|_\Phi = 1$ for all $n \in \mathcal{N}$ and $\| x + x_n \|_\Phi \to 2$ as $n \to \infty$. If not, we may assume without loss of generality that there exist $i_0 \in \mathcal{N}$ and $\varepsilon_0 > 0$ such that $| x_n(i_0) - x(i_0) | \geq \varepsilon_0$ for all $n \in \mathcal{N}$. Since Φ_{i_0} is strictly convex on $[0, \Phi_{i_0}^{-1}(1)]$ there exists $\theta \in (0, 1)$ such that

$$\Phi_{i_0}\left(\frac{x_n(i_0) + x(i_0)}{2}\right) \leq \frac{1-\theta}{2}(\Phi_{i_0}(x_n(i_0)) + \Phi_{i_0}(x(i_0)))$$

for all $n \in \mathcal{N}$ (see [3]). Hence, we get

$$1 \leftarrow I_\Phi\left(\frac{x_n + x}{2}\right) = \sum_{i \neq i_0} \Phi_i\left(\frac{x_n(i) + x(i)}{2}\right) + \Phi_{i_0}\left(\frac{x_n(i_0) + x(i_0)}{2}\right)$$

$$\leq \frac{1}{2} \sum_{i \neq i_0} (\Phi_i(x_n(i)) + \Phi_i(x(i))) + \frac{1-\theta}{2}(\Phi_{i0}(x_n(i_0)) + \Phi_{i_0}(x(i_0)))$$

$$= \frac{1}{2} \sum_{i=1}^{\infty} (\Phi_i(x_n(i)) + \Phi_i(x(i))) - \theta\, \Phi_{i_0}\left(\frac{x_n(i_0) + x(i_0)}{2}\right)$$

$$\leq 1 - \theta\, \Phi_{i_0}\left(\frac{\varepsilon_0}{2}\right),$$

a contradiction which proves that $x_n \to x$ coordinatewise under the above assumptions.

Now, we will prove that the sequence (x_n) has equi-absolutely continuous norm. For any $\varepsilon > 0$ there exists $i_1 \in \mathcal{N}$ such that $\sum_{i=i_1+1}^{\infty} \Phi_i(x(i)) < \varepsilon$, whence $\sum_{i=1}^{i_1} \Phi_i(x(i)) > 1 - \varepsilon$. Since $x_n \to x$ coordinatewise, we conclude that there is $n_0 \in \mathcal{N}$ such that $\sum_{i=1}^{i_1} \Phi_i(x(i)) > 1 - \varepsilon$ for all $n \geq n_0$. Hence,

$$\sum_{i=i_1+1}^{\infty} \Phi_i(x_n(i)) = 1 - \sum_{i=1}^{i_1} \Phi_i(x_n(i)) < 1 - (1 - \varepsilon) = \varepsilon$$

for all $n \geq n_0$. Since $\Phi \in \delta_2$ we easily get from the last condition that (x_n) has the desired property. Consequently $\| x_n - x \|_\Phi \to 0$ as $n \to \infty$.

Remark 1 : It follows from the proof of Theorem 2 that for any Musielak-Orlicz function $\Phi = (\Phi_i)$, the space l^{Φ} is locally uniformly rotund if and only if it is locally compactly fully k-rotund.

Corollary 2 — Let Φ be Musielak-Orlicz functions satisfying condition (*). Then l^{Φ} is approximatively compact if and only if $\Phi \in \delta_2$ and $\Phi \in \delta_2^*$ (i.e. l^{Φ} is reflexive).

PROOF : Every approximatively compact Banach space is reflexive (see [1]). So, the necessity is obvious. On the other hand, every Banach space which is compactly fully k-rotund for some natural $k \geq 2$ is approximatively compact (see [11] Lemma 1). So, by Theorem 1, the proof is complete.

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