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ON $\lambda(P)$-STRONG BASIS

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This article introduces the notion of a $\lambda(P)$-strong basis. Efforts have been directed to characterize a $\lambda(P)$-strong basis analytically. It has been established that a Schauder basis $\{x_i, f_i\}$ in a barrelled space E is $\lambda(P)$-strong iff $\{x_i, f_i\}$ is an ∞-absolute semi-$\lambda(P)$-basis in E which in turn, is equivalent to the fact that $\{f_i\}$ is a semi-$\lambda(P)$-basis for E^*_β and $l^1[E] = l^1(E)$ where $\lambda(P) \subseteq l^1$. It is observed that a Schauder basis $\{x_i, f_i\}$ in a Frechet space E is a $\lambda(P)$-strong basis iff $\{f_i, jx_i\}$ is a $\lambda(P)$-strong basis E^*_β, for a Köthe space $\lambda(P)$ with $\lambda(P) \subseteq l^1$. As an application of $\lambda(P)$-strong basis it is proved that a Schauder basis $\{y_i, g_i\}$ in a barrelled space having a strong basis $\{x_i, f_i\}$, is a fully $\lambda(P)$-basis for E iff $\{g_i, Jy_i\}$ is a fully-$\lambda(P)$-basis for E^*_β provided-$\lambda(P) \subseteq l^1$. Further, the discussion adds that for a Köthe space $\lambda(P)$ with $\lambda(P) \subseteq l^1$, each ∞-absolute basis in a barrelled space having a semi-$\lambda(P)$-basis; is a $\lambda(P)$-strong basis which is proved by showing that if a barrelled space admits a semi-absolute basis and an ∞-absolute basis then both these bases are strong bases.

Key Words : $\lambda(P)$-Strong Basis; Kothe Space; Sequence Spaces; Topological Vector Spaces

INTRODUCTION

In order to appreciate the subject matter of this article one needs to have familiarity with (*i*) classical theory of locally convex spaces (cf. [3], [4], [11]) in general and nuclear spaces and their structures (cf. [14], [17]) in particular, (*ii*) a general study of sequence spaces (cf. [6], [16]) and (*iii*) Schauder bases and their types (cf. [7], [8]). However, for the strong basis and its application related aspects we request the reader to refer [10] and [2]. Lastly, the generalized bases, namely fully-$\lambda(P)$-bases and the related study in details can be obtained from [8] and [9].

For an l.c. TVS E, D_E denotes the fundamental system of semi-norms generating the topology of E while B_E is a fundamental system of closed, absolutely convex and bounded subsets of E.

Let E be an l.c.TVS having a Schauder basis $\{x_i, f_i\}$ and λ be a perfect sequence space. Then

(*i*) $\{x_i, f_i\}$ is called a *semi-λ-basis* if for all $x \in E$ and $p \in D_E$, $\{f_i(x)\, p(x_i)\} \in \lambda$

(*ii*) $\{x_i, f_i\}$ is called a *fully-λ-basis* if for each $p \in D_E$ the mapping $\psi_p : E \to \lambda$: $x \to \{f_i(x)\, p(x_i)\}$ is continuous (for $\lambda = l^1$, we get *semi-absolute* basis from (*i*) while (*ii*) yields *absolute basis*) (*iii*) $\{x_i, f_i\}$ is called an ∞-*absolute* basis if whenver $\{\alpha_i x_i\}$ is bounded and $\{a_i\} \in c_0$, then $\Sigma\, \alpha_i a_i x_i$ converges in E.

Finally, we call $\{x_i, f_i\}$ a *strong basis* if for each $B \in B_E$ and $p \in D_E$, $\{p_B(f_i)\, p(x_i)\} \in l^1$.

We recall from [7] that each ∞-absolute basis in a sequentially complete barrelled space is an u-Schauder basis.

Following [14], by $l^1[E]$ we mean the collection of all weakly summable sequences in E while $l^1(E)$ stands for the collection of all absolutely summable sequences in E.

In the first section we make use of the notion of a simple set which was introduced by Köthe in [12]. A subset B of a Köthe space $\lambda(P)$ is called simple if the sequence $\left(\sup_{\beta \in B} |\beta_i|\right)$ is an element of $\lambda(P)$. Köthe proves that in a nuclear space $\lambda(P)$ every bounded subset is simple while the inverse of this statement is true whenever $\lambda(P)$ is a Frechet space or a DF-space (cf. [13], [11]).

2. $\lambda(P)$-STRONG BASIS

In this section we introduce the notion of a $\lambda(P)$-strong basis and try to construct a $\lambda(P)$-strong basis $\{x_i, f_i\}$ from a strong basis $\{x_i, f_i\}$ via the notion of semi-$\lambda(P)$-basis. Further, the relationship of $\lambda(P)$-strong basis $\{x_i, f_i\}$ with the fully-$\lambda(P)$-basis $\{x_i, f_i\}$, is discussed.

We begin the discussion with the introduction of a $\lambda(P)$-strong basis contained in Definition 1.1. A Schauder basis $\{x_i, f_i\}$ in an l.c. TVS E is said to be a $\lambda(P)$-*strong basis* if for each $B \in B_E$ and $p \in D_E$ $\{p_B(f_i)\, p(x_i)\} \in \lambda(P)$.

Clearly, each $\lambda(P)$-strong basis in an l.c. TVSE is a semi-$\lambda(P)$-basis. Further, if E is a Mackey space with a weakly sequentially complete dual then each $\lambda(P)$-strong basis in E is a fully-$\lambda(P)$--basis in view of Proposition 1.1.[9]. Thus, in a barrelled space each $\lambda(P)$-strong basis becomes a fully-$\lambda(P)$-basis. Observe that each $\lambda(P)$-strong basis in an l.c. TVS is a strong basis if $\lambda(P) \subsetneq l^1$.

Example 1.2 — Let $\lambda(P)$ be a barrelled G_∞-space (cf. [17]) in which bounded sets are simple. Then $\{e_i, e_i\}$ is a $\lambda(P)$-strong basis for $\lambda(P)$. For this note that for each $a \in P, B \in B\lambda(p)$ and $b \in P$ we have

$$\sum_i p_a(e_i)\, p_B(e_i)\, b_i = \Sigma\, a_i \sup_{\beta \in B} |\beta_i|\, b_i$$

$$\leq \Sigma \sup_{\beta \in B} |\beta_i| c_i < \infty,$$

where $a_i b_i \leq c_i$ for some $c \in P$ due to the G_∞-character, as bounded sets are simple in $\lambda(P)$.

In fact, one can have

Lemma 1.3 — Each fully-$\lambda(P)$-basis in an l.c. TVS E is a $\lambda(P)$-strong basis provided bounded sets are simple in $\lambda(P)$.

PROOF : By the definition of a fully-$\lambda(P)$-basis, for each $p \in D_E$ the mapping

$$\phi_p : E \to \lambda(P) : x \to (f_i(x)\, p(x_i))$$

is continuous. Thus, if $B \subset E$ is bounded then the set $\phi_p(B)$ is bounded in $\lambda(P)$ and by the assumption on $\lambda(P)$ it follows that

$$(p_B(f_i)\, p(x_i)) = \left(\sup_{x \in B} |f_i(x)|\, p(x_i) \right) \in \lambda(P)$$

Therefore, $\{x_i, f_i\}$ is a $\lambda(P)$-strong basis for E.

This yields

Proposition 1.4 — Let E be a Mackey space whose dual E^* is weakly sequentially complete. Then a Schauder basis $\{x_i, f_i\}$ in E is a $\lambda(P)$-strong basis iff $\{x_i, f_i\}$ is a fully-$\lambda(P)$-basis if bounded sets in $\lambda(P)$ are simple.

Note — Observe that each semi-$\lambda(P)$-basis in a barrelled space E is a $\lambda(P)$-strong basis if bounded subsets of $\lambda(P)$ are simple.

Lemma 1.5 — Let $\{x_i, f_i\}$ be a strong basis for an l.c.TVS (E, T). Then $\{x_i, f_i\}$ is a $\lambda(P)$-strong basis if either $\{x_i\}$ is a fully-$\lambda(P)$-basis for E or $\{f_i\}$ is a fully-$\lambda(P)$-basis for the strong dual E^*_β.

PROOF : We outline the proof of one part. Similarly, the other will follow.

So let $\{x_i, f_i\}$ be a fully-$\lambda(P)$-basis for E. Thus, for each $p \in D_E$ and $a \in P$ there exists a $q \in D_E$ such that

$$\sum_i |f_i(x)|\, p(x_i)\, a_i \leq q(x) \qquad (x \in E)$$

This in particular, yields

$$p(x_i)\, a_i \leq q(x_i)$$

(+)

Now, for any $B \in B_E, p \in D_E$ and $a \in P$ we have the inequality :

$$\Sigma p_B(f_i) p(x_i) a_i \leq \Sigma p_B(f_i) q(x_i) < \infty$$

in view of (+) as $\{x_i, f_i\}$ is a strong basis. This says that $\{x_i, f_i\}$ is $\lambda(P)$-strong.

In the case of a semi $\lambda(P)$-basis one obtains

Lemma 1.6 — Let E be a locally convex space with a strong basis $\{x_i, f_i\}$; whose dual E^* is weakly sequentially complete. Suppose $\{x_i, f_i\}$ is a semi-$\lambda(P)$-basis. Then $\{x_i, f_i\}$ is a $\lambda(P)$-strong basis.

PROOF : By the application of Lemma 1.1[9] for each $p \in D_E$ and $a \in P$ we get a $f \in E^*$ such that

$$p(x_i) a_i = f(x_i) \ , \ i = 1, 2, \ldots .$$

For this f there exist $p_0 \in D_E$ and $k > 0$ such that

$$f(x_1) \leq k p_0(x_1), \ i = 1, 2, \ldots$$

Consequently, for each $B \in B_E, p\ D_E$ and $a \in P$ we have the following inequality :

$$\Sigma p_B(f_i) p(x_i) a_i \leq k \Sigma p_B(f_i) p_0(x_i) < \infty$$

as $\{x_i, f_i\}$ is a strong basis. So $\{x_i, f_i\}$ is a $\lambda(P)$-strong basis.

This gives us easily

Proposition 1.7 — Let E be a locally convex space having a Schauder basis $\{x_i, f_i\}$. Suppose E^* is weakly sequentially complete. If $\lambda(P) \subsetneq l^1$ then $\{x_i, f_i\}$ is a $\lambda(P)$-strong basis iff $\{x_i, f_i\}$ is a strong basis which is a semi-$\lambda(P)$-basis as well.

Remarks 1.8 : If $\lambda(P) \subsetneq l^1$, then a Schauder basis $\{x_i, f_i\}$ in a barrelled space E is a $\lambda(P)$-strong basis iff $\{x_i\}$ is a strong basis which is also a $\lambda(P)$-Köthe basis (cf. [8]).

2. Analytical Characterization of $\lambda(P)$-Strong Basis

This section deals with the analytical characterization of a $\lambda(P)$-strong basis. The results reveal that the structure of $\lambda(P)$ plays a crucial role.

To start with we have an interesting characterisation of a $\lambda(P)$-strong basis contained in

Proposition 2.1 — Let $\{x_i, f_i\}$ be a Schauder basis in an l.c. TVS E. Suppose $\lambda(P)$ is a Köthe space such that $\lambda(Q)$ is nuclear where

$$Q = \{p(x_i) a_i : p \in D_E, a \in P\}$$

Then $\{x_i, f_i\}$ is a $\lambda(P)$-strong basis iff for each $B \in B_E, p \in D_E$ and $a \in P$

$$\{p_B(f_i)\, p(x_i)\, a_i\} \in l^\infty \text{ or } (c_0)$$

PROOF : ($\Rightarrow$) This is clear by the definition.

Suppose (+) holds. Take $B \in B_E, p \in D_E$ and $a \in P$ arbitrarily. Since $\lambda(Q)$ is nuclear, for this p and a in view of Grothendieck-Pietsch criteria (cf. Theorem 3.6.4[17]) there exist $q \in D_E$ and $b \in P$ such that

$$\left\{\frac{p(x_i)\, a_i}{q(x_i)\, b_i}\right\} \in l^1$$

Consequently, we have the following inequality :

$$\Sigma\, p_B(f_i)\, p(x_i)\, a_i \leq \sup\ \{p_B(f_i)\, q(x_i)\, b_i\} \sum_i \frac{p(x_i)\, a_i}{q(x_i)\, b_i} < \infty$$

in view of (+). Thus, $\{x_i, f_i\}$ is $\lambda(P)$-strong.

Since nuclearity of $\lambda(P)$ always yields the nuclearity of $\lambda(Q)$ (Q is as in above), Proposition 2.1 results in

Corollary 2.2 — Let $\lambda(P)$ be a nuclear Köthe space. Then a Schauder basis $\{x_i, f_i\}$ in an l.c.TVS E is a strong basis iff for each $B \in B_E$, $p \in D_E$ and $a \in P$

$$\{p_B(f_i)\, p(x_i)\, a_i\} \in l^\infty \text{ OR } \{p_B(f_i)\, p(x_i)\, a_i\} \in c_0$$

Also, Proposition 2.1 makes the way for

Corollary 2.3 — Let $\{x_i, f_i\}$ be an equicontinuous basis for a nuclear space E. Then $\{x_i, f_i\}$ is a $\lambda(P)$-strong basis iff for each $B \in B_E, p \in D_E$ and $a \in P$

$$\{p_B(f_i)\, p(x_i)\, a_i\} \in l^\infty \text{ OR } \{p_B(f_i)\, p(x_i)\, a_i\} \in c_0$$

PROOF : In this case, $\lambda(Q)$ that occurs in Proposition 2.1 becomes nuclear in view of Proposition 3.1 [5].

Taking $\lambda(P)$ to be a nuclear G_∞-space (cf. [17]) in Corollary 2.2 one arrives at

Corollary 2.4 — Let $\lambda(P)$ be a nuclear G_∞-space. Then a Schauder basis $\{x_i, f_i\}$ in an l.c.TVS (E, T) is a $\lambda(P)$-strong basis iff for each $B \in B_E, p \in D_E, k \geq 1$ and $a \in P$

$$\left\{(i+1)^{2^k}\, p_B(f_i)\, p(x_i)\, a_i\right\} \in c_0$$

PROOF : This follows from Corollary 2.2 by the application of Proposition 3.6.12[17].

If this nuclear G_∞-space is chosen to be a nuclear power series space $\Lambda(\alpha)$ of infinite type (cf. [17]) then we have

Corollary 2.5 — A Schauder basis $\{x_i, f_i\}$ in an l.c. TVS E is a $\Lambda(\alpha)$-strong basis iff for each $k \geq 1, B \in B_E$ and $p \in D_E$

$$\left\{ ((i+1)^{2^k} p_B(f_i)\, p(x_i))^{1/\alpha_i} \right\} \in c_0.$$

PROOF : Follows from Corollary 2.4 in view of Lemma 2[15] as $\Lambda(\alpha)$ is given to be nuclear.

Note — The proof of the above result can also be given via the fact that $x \in \Lambda(\alpha)$ iff $\left((i+1)^{2^k} x_i \right\} \in \Lambda(\alpha)$ for each $k \geq 1$ (cf. [17]) and Lemma 2[15].

If the nuclear G_∞-space $\lambda(P)$ is taken to be a FK-space then one can have the following from Corollary 2.4.

Corollary 2.6 — Let $\lambda(P)$ be a nuclear G_∞-space which is also a FK-space. Then a Schauder basis $\{x_i, f_i\}$ in an l.c.TVS (E, T) is a $\lambda(P)$-strong basis iff for each $k \geq 1$, $B \in B_E$, $p \in D_E$ and $a \in P$

$$\left\{ (i+1)^{2^k} p_B(f_i)\, p(x_i)\, (\pi(i))^{-1} a_i \right\} \in c_0$$

for some permutation π on the set of integers.

PROOF : It follows from Corollary 2.4 by the help of Corollary 3.8[16].

The analytical characterization of $\lambda(P)$-strong bases ends in

Corollary 2.7 — Let $\lambda(P)$ be a nuclear space which is also a G_∞-space. Suppose $\Lambda(\alpha)$ is a nuclear power series space of infinite type. Then $\{e_i, e_i\}$ is a $\Lambda(\alpha)$-strong basis for $\lambda(P)$ iff for each B $B_{\lambda(P)}, a \in P$ and $k \geq 1$

$$\{ / (p_B(e_i)\, (i+1)^{2^k} (\pi(i))^{-1} a_i)^{1/\alpha_1} \} \in c_0$$

for some permutation π on the set of integers.

PROOF : This follows from Corollary 2.5 in view of Corollary 3.8[16]

3. Some Criteria for $\lambda(P)$-Strong Basis

In this section we try to characterize a $\lambda(P)$-strong basis $\{x_i, f_i\}$ in terms of the semi-$\lambda(P)$-character of $\{x_i\}$ and $\{f_i\}$ which in turn is used to study the implication :

$$\{x_i, f_i\} \text{ is a } \lambda(P)\text{-strong basis in } E \Rightarrow \{f_i, J x_i\} \text{ is a } \lambda(P)\text{-strong basis in } E^*_\beta$$

(the reverse implication is also discussed)

This section is initiated by the following in which the weak sequential completeness of the dual of the space plays a crucial role.

Proposition 3.1 — Let E be an l.c. TVS with a Schauder basis $\{x_i, f_i\}$. Suppose E^* is weakly sequentially complete. If $\lambda(P) \subseteq l^1$ then $\{x_i, f_i\}$ is a $\lambda(P)$-strong basis for E iff one of the following holds -

(*i*) $\{x_i, f_i\}$ is a semi-$\lambda(P)$-basis for E and $\{f_j, Jx_i\}$ is a semi-$\lambda(P)$-basis for E^*_β.

(*ii*) $\{x_i, f_i\}$ is a semi-$\lambda(P)$-basis for E and $\{f_i, Jx_i\}$ is a semi-absolute basis for E^*_β.

(*iii*) $\{x_i, f_i\}$ is a semi-absolute basis for E and $\{f_i, Jx_i\}$ is a semi-$\lambda(P)$-basis for E^*_β.

PROOF : First of all we show that (ii) is equivalent to the $\lambda(P)$-strong character of $\{x_i, f_i\}$.

Suppose $\{x_i, f_i\}$ is a $\lambda(P)$-strong basis. Then obviously $\{x_i, f_i\}$ is a semi-$\lambda(P)$-basis. Since $\lambda(P) \subseteq l^1$, $\{x_i, f_i\}$ is in particular a strong basis. Therefore, $\{f_i, Jx_i\}$ is a semi-absolute basis for E^*_β in view of Proposition 1[12].

Conversely, suppose (ii) holds. Take any $B \in B_E, p \in D_E$ and $a \in P$ arbitrarily. By making use of Lemma 1.1[9] we get a $f \in E^*$ ($f = f(p, a)$) such that $p(x_i)\, a_i = f(x_i)$, for all $i \geq 1$. But then we arrive at

$$\Sigma\, p_\beta(f_i)\, p(x_i)\, a_i = \Sigma\, p_B(f_i) f(x_i) < \infty$$

because $\{f_i\}$ is a semi-absolute basis for E^*_β. This says that $\{x_i, f_i\}$ is a $\lambda(P)$-strong basis.

Next we prove that $\lambda(P)$-strong character of $\{x_i, f_i\}$ is equivalent to (iii).

If $\{x_i, f_i\}$ is $\lambda(P)$-strong then $\{x_i, f_i\}$ is a strong basis in particular. So by making use of Proposition 1[2] we find that $\{x_i\}$ is a semi-absolute shrinking basis for E. For $f \in E^*$ there always exists $p \in D_E$ such that

$$|f(x_i)| \leq p(x_i), \; i = 1, 2, \ldots$$

Let $B \in B_E$ and $a \in P$, then

$$\sum_i p_B(f_i)\, |f(x_i)|\, a_i \leq \Sigma\, p_B(f_i)\, p(x_i)\, a_i < \infty$$

as $\{x_i, f_i\}$ is $\lambda(P)$-strong. Thus, $\{f_i, Jx_i\}$ is a semi-$\lambda(P)$-basis for E^*_β.

Conversely, suppose (iii) remains true. Then, for each $B \in B_E, f \in E^*$ and $a \in P$, the sequence

$$\left\{ p_B(f_i) f(x_i) a_i \right\} \in l^1$$

Now, take any $B \in B_E, p \in D_E$ and $a \in P$. Taking $y = (y_i) = (1, 1, \ldots) \in l^\infty$ and proceeding as in the proof of Proposition 1[10], we get $f \in E^*$ such that

$$p(x_i) y_i = f(x_i), \text{ for all } i \geq 1$$

Consequently, $\left\{ p_B(f_i) p(x_i) a_i \right\} \in l^1$

Equivalently, $\{x_i, f_i\}$ is a $\lambda(P)$-strong basis for E.

Since $\lambda(P \subseteq l^1$, now the equivalence of (*i*) with $\lambda(P)$-strong character of $\{x_i, f_i\}$ follows from (*ii*) and (*iii*).

This finishes the proof.

The above result paves the way for

Corollary 3.2 — Let E be a barrelled space with a Schauder basis $\{x_i, f_i\}$. Suppose $\lambda(P) \subseteq l^1$. Then $\{x_i, f_i\}$ is a $\lambda(P)$-strong basis for E iff

(*i*) $\{x_i, f_i\}$ is an ∞-absolute semi-$\lambda(P)$-basis for E, iff

(*ii*) $l^1[E] = l^1\ [E]$ and $\{f_i\}$ is a semi-$\lambda(P)$-basis for E^*_β.

PROOF : First of all we show that $\lambda(P)$-strong character of $\{x_i, f_i\}$ is equivalent to (i).

So let $\{x_i, f_i\}$ be $\lambda(P)$-strong in E. Then clearly $\{x_i\}$ is a semi-$\lambda(P)$-basis for E. Since $\lambda(P) \subseteq l^1$, $\{x_i, f_i\}$ is a strong basis in particular. Thus, by proposition 1[2] $\{f_i\}$ is a semi-absolute basis for E^*_β. Since a barralled space with an absolute basis is complete (cf. [8]) by making use of Theorem 9.5.4[7] we find that $\{x_i\}$ is an ∞-absolute basis for E.

Conversely, suppose (*i*) holds. Since a barrelled space with an absolute basis is always complete, invoking Theorem 9.5.4[7] we get that $\{f_i\}$ is a semi-absolute basis for E^*_β. Thus, $\{x_i, f_i\}$ is a $\lambda(P)$-strong basis in view of Proposition 3.1.

For the second part, if $\{x_i, f_i\}$ is a $\lambda(P)$-strong basis, then by Proposition 3.1 $\{f_i\}$ is a semi-$\lambda(P)$-basis for E^*_β. Further, $\{x_i\}$ is a strong basis. So by making use of Proposition 5[10] we find that $l^1[E] = l^1(E)$.

Conversely, suppose (ii) holds. Since $\{f_i\}$ is a semi-$\lambda(P)$-basis for E^*_β, the sequence

$$\left\{p_B(f_i)\, f(x_i)\, a_i\right\}$$

is in l^1 for every $B \in B_E$, every $f \in E^*$ and every $a \in P$. Thus, the sequence

$$\left\{x_i\, p_B(f_i)\, a_i\right\} \in l^1[E] = l^1(E)$$

Equivalently, $\Sigma\, p_B(f_i)\, p(x_i)\, a_i < \infty$

for all $B \in B_E$, all $p \in D_E$ and all $a \in P$. This says that $\{x_i, f_i\}$ is a $\lambda(P)$-strong basis for E. This concludes the proof.

Since a barrelled space having a strong basis is a Montel space by the Corollary to Proposition 4[10], a close look at the proof of the previous two results reveals that the following is true :

A Schauder basis $\{x_i, f_i\}$ in a barrelled space E is a $\lambda(P)$-strong basis for E iff $\{x_i, f_i\}$ is a fully-$\lambda(P)$-basis for E and $\{f_i, Jx_i\}$ is a fully-$\lambda(P)$-basis for E^*_β wherein $\lambda(P) \subseteq l^1$.

However, in a semi-reflexive space we obtain

Proposition 3.3 — Let E be a semi-reflexive space with a Schauder basis $\{x_i, f_i\}$. Suppose $\lambda(P)$ is a Köthe space such that for each $a \in P$ there exists $b \in P$ such that $a_i \le b_i^2$, for each $i \ge 1$. Then $\{x_i, f_i\}$ is a $\lambda(P)$-strong basis for E iff $\{x_i, f_i\}$ is a semi-$\lambda(P)$-basis for E and $\{f_i, Jx_i\}$ is a semi-$\lambda(P)$-basis for E^*_β.

PROOF : Since a Schauder basis in a semi-reflexive space is always shrinking (cf. [1]) the part namely, ($\Rightarrow$) follows by the analysis of Proposition 3.1 ($\{x_i, f_i\}$ is $\lambda(P)$-strong implies (*iii*)).

Conversely, suppose $\{x_i, f_i\}$ is a semi-$\lambda(P)$-basis for E and $\{f_i, Jx_i\}$ is a semi-$\lambda(P)$-basis for E^*_β. Now, take any $B \in B_E$, $p \in D_E$ and $a \in P$ arbitrarily. By our hypothesis we get a b depending on a in P such that $a_i \le b_i^2$. Then by making use of Lemma 1.1^9^ we obtain some

$f = f(p, b)$ in E^* with

$p(x_i)\, b_i = f(x_i)$, $i = 1, 2, \ldots$

But then we have $\Sigma\, p_B(f_i)\, p(x_i)\, a_i \le \Sigma\, p_B(f_i)\, p(x_i)\, b_i\, b_i$

$$= \Sigma p_B (f_i) f(x_i) b_i < \infty$$

as $\{f_i, Jx_i\}$ is a semi-$\lambda(P)$-basis for E^*_β. So $\{x_i, f_i\}$ is a $\lambda(P)$-strong basis for E.

For the next result we make use of —

Lemma 3.4 — Let E be a DF-nuclear space. Then each semi-$\lambda(P)$-basis $\{x_i, f_i\}$ in E is a $\lambda(P)$-strong basis for any Köthe space $\lambda(P)$.

PROOF : By making use of Proposition 3[2] we find that $\{x_i, f_i\}$ is a strong basis for E. Observe that a DF-nuclear space is always reflexive in view of the Theorem p. 83[14]. Now the required assertion follows from Lemma 1.6.

This results in the following

Proposition 3.5 — Let E be a Frechet space having a Schauder basis $\{x_i, f_i\}$. Suppose $\lambda(P) \subseteq l^1$. Then $\{x_i, f_i\}$ is a $\lambda(P)$-strong basis for E iff $\{f_i, Jx_i\}$ is a $\lambda(P)$-strong basis for E^*_β.

PROOF : ($\Rightarrow$) : Since $\lambda(P) \subseteq l^1$ it follows that $\{x_i, f_i\}$ is a strong basis which makes the space E nuclear by Mertin's result (cf. [13]) i.e. Remark 2[10]. But a Frechet nuclear space is co-nuclear by the Theorem on p.8 [14] and hence E^*_β is a DF-nuclear space.

Further, by Proposition 3.1 $\{f_i, Jx_i\}$ is a semi-$\lambda(P)$-basis for E^*_β. Thus, by Lemma 3.4 $\{f_i, Jx_i\}$ is a $\lambda(P)$-strong basis for E^*_β.

Conversely, suppose that $\{f_i, Jx_i\}$ is a $\lambda(P)$-strong basis for the sequentially complete DF-space E^*_β. Therefore, in particular it is a strong basis for the strong dual. Since separable DF-spaces are infrabarrelled (cf. 11]) it follows from Proposition 3^2^ that E^*_β is nuclear. Thus, $l^1[E] = l^1(E)$ by Proposition 4.2.8[14]. So the desired conclusion follows from corrollay 3.2.

Remarks 3.6 : Observe that making use of Proposition 3.1 one can easily establish that a Schauder basis $\{x_i, f_i\}$ in a reflexive space E is a $\lambda(P)$-strong basis iff $\{f_i, Jx_i\}$ is a $\lambda(P)$-strong basis for E^*_β provided $\lambda(P) \subseteq l^1$. Even Corollary 3.2 can be used to prove this result.

4. Applications of $\lambda(P)$-Strong Basis

This is the concluding section dealing with the application aspects of $\lambda(P)$-strong basis. The discussion underscores that $\lambda(P)$-strong basis plays a prominant role in taking the fully-$\lambda(P)$-character of a basis $\{x_i, f_i\}$ in E into the fully-$\lambda(P)$-character of $\{f_i, Jx_i\}$ in the strong dual. Interestingly, in this section it is also observed that if a barrelled space admits a semi-absolute basis and an ∞-absolute basis then both these bases are strong bases.

The section begins with

Proposition 4.11 — Let E be a barrelled space with a strong basis $\{x_i, f_i\}$. Suppose $\lambda(P)$ is any Köthe space. Then $\{x_i, f_i\}$ is a fully-$\lambda(P)$-basis for E iff $\{f_i, Jx_i\}$ is a fully-$\lambda(P)$-basis for the strong dual E^*_β.

PROOF : ($\Rightarrow$) In view of Lemma 1.5 $\{x_i, f_i\}$ is a $\lambda(P)$-strong basis for E. But a barrelled space having a strong basis is always Motnel by the Corollary to Proposition 4[10] and so in particular E is a reflexive space. Thus, $\{x_i, f_i\}$ is shrinking (cf. [1]) and E^*_β is reflexive (cf. [3]). Now for $f \in E^*$ there always exists $p \in D_E$ such that

$$|f(x_i)| \leq p(x_i), \; i = 1, 2, \ldots$$

Let $B \in B_E$ and $a \in P$, then

$$\Sigma\, p_B(f_i)\, |f(x_i)|\, a_i \leq \Sigma\, p_B(f_i)\, p(x_i)\, a_i < \infty$$

as $\{x_i, f_i\}$ is $\lambda(P)$-strong. Therefore, $\{f_i, Jx_i\}$ is a semi-$\lambda(P)$-basis for E^*_β. So it becomes a fully-$\lambda(P)$-basis for E^*_β in view of Proposition 1.1.[9]

Conversely, since $\{x_i, f_i\}$ is a strong basis for E resorting to Proposition 5[10] we find that $l^1[E] = l^1(E)$. For each $B \in B_E, f \in E^*$ and $a \in P$ the sequence $\{p_B(f_i) f(x_i)\, a_i\} \in l^1$ as $\{f_i\}$ is semi-$\lambda(P)$-basis. Thus,

$$\left\{ x_i\, p_B(f_i)\, a_i \right\} \in l^1[E] = l^1(E)$$

i.e. $\Sigma\, p_B(f_i) p(x_i)\, a_i < \infty$ for all $B \in B_E$, all $p \in D_E$ and all $a \in P$.

Remarks 4.2 : The proof of the above converse part (namely, ($\Leftarrow$) can also be given by directly appealing to Lemma 1.5.

However, for a Köthe space $\lambda(P) \subseteq l^1$ we have an interesting result contained in

Theorem 4.3 — *Let E be a barrelled space with a strong basis $\{x_i, f_i\}$. Then a Schauder basis $\{y_i, g_i\}$ in E is a fully-$\lambda(P)$-basis iff $\{g_i, Jy_i\}$ is a fully-$\lambda(P)$-basis for E^*_β if $\lambda(P) \subseteq l^1$.*

PROOF ($\Rightarrow$) : Suppose $\{y_i, g_i\}$ is a fully-$\lambda(P)$-basis for E. Thus, in particular $\{y_i\}$ is an absolute basis for E. Therefore, invoking Theorem 10.1.4[14] E can be topologically identified with a Köthe space $\lambda(L)$; $L = \{q(y_i) : q \in D_E\}$. Further, as $\{x_i\}$ is in particular absolute, resorting to Theorem 10.1.4[14] we can identify E topologically with the Köthe space $\lambda(M) : M = \left\{p(x_i) : p \in D_E\right\}$. But $\{x_i, f_i\}$ is a strong basis for E and hence $\{e_i, e_i\}$ is a strong basis for $\lambda(M)$. Now by the Proposition in p.650[10] $\{e_i, e_1\}$ is a strong basis for $\lambda(L)$. Thus, $\{y_i, g_i\}$ is a strong basis for E. Consequently $\{y_i, g_i\}$ is a $\lambda(P)$-strong basis for E by Lemma 1.5. Now, appealing to Proposition 3,.1 we find that $\{g_i, Jy_i\}$ is a semi-$\lambda(P)$-basis for E^*_β which is Montel as a barrelled space having a strong basis is always Montel by the corollary to Propsition 4[10]. Hence $\{g_i, Jy_i\}$ is a fully-$\lambda(P)$-basis for E^*_β in view of Proposition 1.1[9].

For the converse part observe that by making use of Proposition 5[10], $l^1[E] = l^1(E)$ because $\{x_i, f_i\}$ is given to be a strong basis. Using Corollary 3.2 we find that $\{y_i, g_i\}$ is a $\lambda(P)$-strong basis for E. So by making use of Proposition 1.1[9] we conclude that $\{y_i, g_i\}$ is a fully-$\lambda(P)$-basis for E.

The article concludes with

Proposition 4.4 — Let E be a barrelled space having two bases $\{x_i, f_i\}$ and $\{y_i, g_i\}$ such that one of them is ∞-absolute and the other is a semi-$\lambda(P)$-basis where $\lambda(P) \subseteq l^1$. Then both bases are $\lambda(P)$-strong.

PROOF : Let $\{x_i, f_i\}$ be ∞-absolute and $\{y_i, g_i\}$ be a semi-$\lambda(P)$-basis for E. Since a barrelled space with an absolute basis is always complete (cf. [8]) and each ∞-absolute basis in a sequenially complete barrelled space is u-Schauder (cf. [7]), by making use of Theorem 6.4.4[7] we conclude that $\{x_i, f_i\}$ is an absolute basis (in fact, it is a Köthe basis)/. Now resorting to Theorem 9.5.4[7] we obtain that $\{f_i\}$ is a semi-absolute basis for E^*_β. Thus, by Proposition 1[2]. $\{x_i\}$ becomes a strong basis. Since $\{x_i, f_i\}$ is strong and $\{y_i, g_i\}$ is a semi-$\lambda(P)$-basis for E, the analysis involved in the proof of Theorem 4.3 ($\Rightarrow$) suggests that $\{y_i, g_i\}$ is $\lambda(P)$-strong. Similarly, using the same analysis one can easily see that $\{x_i, f_i\}$ is a $\lambda(P)$-strong basis for E.

Remarks 4.5 : Notice that the fact which holds the key to the proof of Proposition 4.4 is that if a barrelled space contains a ∞-absolute basis and an absolute basis then both bases are strong.

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