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EXISTENCE OF POSITIVE SOLUTIONS FOR *P*-LAPLACIAN SINGULAR BOUNDARY VALUE PROBLEMS*

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By using Leray-Schauder degree theory, positive solutions are established for p-Laplacian singular second-order boundary value problem, singularities at (i) $u' = 0$ but not $u = 0$, (*ii*) $u = 0$, and $u' = 0$ are 1 discussed, respectively.

Key Words : Singular Boundary Value Problem; Leray-Schauder Degree; P-Laplacian; Positive Solutions

1 INTRODUCTION AND PRELIMINARIES

Boundary value problems for ordinary diffferential with p-Laplacian arise in the search of radial solutions of nonlinear partial differential equations. The types of nonlinear partial differential equations we have in mind arise in a multitude of applied areas, such as the study of porous media, elasticity theory, plasma problems, astrophysics and etc. we can see these in [1] and the references therein. In addition, in the study of nonlinear phenomena, many mathematical models also give rise to the singular boundary value problems. They have been studied by a number of authors. We can erfer to Ravi P. Agarwal and O'Regan[2&3], Wang Hongzhou *et al.*[4], Yang Zuodong[5] and their references. However, almost all papers in the literature discussed the case that the system don't include p-laplacian. The aim of this paper is to extend the results in [2] [3] [4] and [5] to the following p-Laplacian singular boundary value problem :

$$\begin{cases} (\phi_p(u'))' + q(t) f(t, u, u') = 0, \quad 0 < t < 1, \\ u(0) = u'(1) = 0 \end{cases} \quad \text{... (1)}$$

where $\phi_p(u) = |u|^{p-2} u, p > 1$, and our nonlinear term f may be singular at (*i*) $u' = 0$ but not $u = 0$, (*ii*) $u = 0$ and $u' = 0$. Sufficient conditions are established.

Let $C_B[0, 1] = \{u \in C[0, 1], \phi_p(u') \in C[0, 1], u(0) = a, u'(1) = b$ with norm $|u|_1\}$ and

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define $|u|_1 = \max\{|u|_0, |\phi_p(u')|_0\}$ where $|u|_0 = \sup_{t \in (0,1)} |u(t)|$. Then $C_B[0, 1]$ is a normed linear space.

In this section, we want to establish some important lemmas. First considering

$$\begin{cases} (\phi_p(u'))' + q(t) F(t, u, u') = 0 \quad 0 < t < 1 \\ u(0) = a, u'(1) = b \end{cases} \qquad \dots (2)$$

we have

Lemma 1 — Suppose

$$F : [0, 1] \times \mathcal{R}^2 \to \mathcal{R} \text{ is continuous}$$

and $\quad q \in C(0, 1)$ with $q > 0$ on $(0, 1)$ and $q \in L^1[0, 1]$

In addition, assume that there is a constant M independent of λ such that

$$|u|_1 \leq M$$

for any solution $u \in C^1[0, 1]$, $\phi_p(u') \in C^1[0, 1]$ to

$$\begin{cases} (\phi_p(u'))' + \lambda q(t) F(t, u, u') = 0, \quad 0 < t < 1, \\ u(0) = u'(1) = 0 \end{cases} \qquad \dots (3)_\lambda$$

for each $\lambda \in (0, 1)$. Then (2) has a solution $u \in C^1[0, 1]$, $\phi_p(u') \in C^1[0, 1]$.

PROOF : Solving $(3)_\lambda$ is equivalent to finding a $u \in C^1[0, 1]$ with $\phi_p(u') \in C^1[0, 1]$ which satisfies

$$u(t) = a + \int_0^t \phi_p^{-1}\left[\phi_p(b) + \lambda \int_s^1 q(x) F(x, u(x), u'(x))\, dx\right] ds, \qquad \dots (4)$$

where $\phi_p^{-1}(w) = |w|^{\frac{1}{p-1}} sgn(w)$ is the inverse function of $\phi_p(u)$.

Define the operator $N_\lambda : C_B[0, 1] \to C_B[0, 1]$ by setting

$$N_\lambda u(t) = a + \int_0^t \phi_p^{-1}\left[\phi_p(b) + \lambda \int_s^1 q(x) F(x, u(x), u' x))\, dx\right] ds.$$

Consequently, $(3)_\lambda$ is equivalent to the fixed point problem $N_\lambda u = u$ in $C_B[0, 1]$.

Next we will prove $N_\lambda : C_B[0,1] \to C_B[0,1]$ is completely continuous. To see this, let $\Omega \subseteq C_B[0,1]$ be bounded, i.e. there exists a constant $M_0 > 0$, with $|u|_1 \le M_0$ for each $u \in \Omega$, so we have

$$|N_\lambda u| \le |a| + \int_0^1 G(s)\,ds$$

here

$$G(s) = \max$$

$$\left\{ \left| \phi_p^{-1}\left(|\phi_p(b)| + M_1 \int_s^1 q(x)\,dx\right) \right|, \left| \phi_p^{-1}\left(-|\phi_p(b)| - M_1 \int_s^1 q(x)\,dx\right) \right| \right\},$$

$$M_1 = \sup |F(t,x,y)| \text{ for } (t,x,y) \in [0,1] \times [-M_{0,} M_0] \times [-M_0, M_0]$$

and $$|\phi_p(N_\lambda u)'(t)| = \left| \phi_p(b) + \lambda \int_t^1 q(s) F(s, u(s), u'(s))\,ds \right| \le |\phi_p(b)| + M_1 \left| \int_0^1 q(s)\,ds \right|.$$

So we obtain the boundedness of $N_\lambda \Omega$. Next consider $u \in \Omega$ and $s, t \in [0,1]$, then since

$$|N_\lambda u(t) - N_\lambda u(s)| \le \left| \int_s^t G(\upsilon)\,d\upsilon \right|$$

and $$|\phi_p(N_\lambda u)'(t) - \phi_p(N_\lambda u'(s)| \le \left| \int_s^t q(x) F(x, u(x), u'(x))\,dx \right| \le M_1 \left| \int_s^t q(x)\,dx \right|$$

the equicontinuity of $N_\lambda \Omega$ follows from the above inequalities. Consequently, the Arzela-Ascoli theorem implies that $N_\lambda : C_B[0,1] \to C_B[0,1]$ is completely continuous. Let

$$U = \{u \in C_B[0,1] : |u|_1 < 2M + M_1 + 2\}$$

so for $u \in \partial U$, we have $(1 - N_\lambda)(u) \ne 0$. then by using Leray-Schauder degree theory [6] [7], we obtain

$$deg\ \{I - N_1, U, 0\} = deg\ \{I - N_0, U, 0\}$$

where $N_0 = a + bt = \theta(t)$ (let $|a|, |b| < M + 1$), we have $|N_0 u| < 2M + 2$, so $N_0 u = \theta(t) \in U$, then we get

$$deg\ \{I - N_1, U, 0\} = deg\ \{I, U, \theta(t)\} = 1 \neq 0$$

and we deduce that $N_1 u = u$ has a fixed point in U, i.e. (2) has a solution $u \in C^1 [0, 1]$ with $\phi_p(u') \in C^1 [0, 1]$.

Lemma 2 — Suppose $u \in C^1 [0, 1]$ with $\phi_p(u') \in C^1 [0, 1]$ and satisfies

$$\begin{cases} -[\phi_p(u'(t))]' > 0 & 0 < t < 1 \\ u(0) = 0, u'(1) = a \geq 0 \end{cases} \quad \dots (5)$$

Then we have $u(t) \geq tu(1) = t \sup_{t \in [0, 1]} |u(t)|$.

PROOF : Since $-[\phi(u'(t))]' > 0$, we have $(\phi_p(u'))' < 0$, it implies that $\phi_p(u')$ is decreasing on (0, 1) and hence $u'(t)$ is decreasing. Because of $u'(1) = a \geq 0, u(0) = 0$, we have $u'(t) \geq 0$ and $u(t) \geq 0$ for $t \in [0, 1]$, also from (5) and u is concave, then we get $u(t) \geq tu(1) = t \sup_{t \in [0, 1]} |u(t)|$.

2. SINGULARITY AT $u' = 0$ BUT NOT AT $u = 0$

In this section we discuss (1). Our nonlinearity f may be singular at $u' = 0$ but not at $u = 0$. Throughout this section, we always assume that

(H_1) $q \in C(0, 1)$ with $q > 0$ on (0, 1) and $q \in L^1 [0, 1]$

(H_2) $f: [0, 1] \times [0, +\infty) \times (0, +\infty) \to [0, +\infty)$

is continuous with $f(t, x, y) > 0$, $\lim_{y \to 0+} f(t, x, y) = \infty$, for $(t, x, y) \in [0, 1] \times (0, +\infty) \times (0, +\infty)$

(H_3) $f(t, x, y) \leq h(x)[g(y) + r(y)]$ on $[0, 1] \times (0, +\infty) \times (0, +\infty)$

with $g > 0$ continuous and nonincreasing on $(0, +\infty)$, and $h \geq 0, r \geq 0$ continuous and nondecreasing on $(0, +\infty)$.

$$(H_4) \quad \sup_{c \in (0, +\infty)} \frac{c}{\phi_p^{-1}\left(I^{-1}\left(h(c) \int_0^1 q(s)\, ds\right)\right)} > 1$$

where $$I(z) = \int_0^z \frac{du}{g(\phi_p^{-1}(u)) + r(\phi_p^{-1}(u))}$$

for $z > 0$, $I(\infty) = \infty$.

(H_5) for constants $H > 0$, $L > 0$, there exists a function $\Psi_{H,L}$ continuous on [0, 1] and positive on (0, 1), and a constant $\gamma, 0 \le \gamma < 1, f(t,x,y) \ge \Psi_{H,L}(t)\, x^{\gamma}$ on $[0,1] \times [0,H] \times (0,L]$.

$$(H_6) \int_0^1 q(t)\, g\left(k_0\, \phi_p^{-1}\left(\int_t^1 s^{\gamma}\, \Psi_{H,L}(s)\, q(s)\, ds\right)\right) dt < \infty, \text{ for any constant } k_0 > 0.$$

We have

Theorem 1 — *Suppose (H_1) – (H_6) hold. Then (1) has a solution $u \in C^1[0,1]$, $\phi_p(u') \in C^1[0,1]$ with $u > 0$ on (0, 1].*

PROOF : Choose $M > 0$ with

$$\frac{M}{\phi_p^{-1}\left(I^{-1}\, h(M) \int_0^1 q(s)\, ds\right)} > 1$$

Next choose $\varepsilon > 0$ and $\varepsilon < M$ with

$$\frac{M}{\phi_p^{-1}\left(I^{-1}\left(h(M) \int_0^1 q(s)\, ds + I(\phi_p(\varepsilon)\right)\right)} > 1. \qquad \text{... (6)}$$

Take $n_0 \in \{1, 2, ...\}$ such that $\frac{1}{n_0} < \varepsilon$ and let $N_0 = \{n_0, n_0 + 1, ...\}$. We first show that

$$\begin{cases} (\phi_p(u'))' + q(t) f^*(t,u,u') = 0 \quad 0 < t < 1 \\ u(0) = 0,\ u'(1) = \frac{1}{m} \end{cases} \qquad \text{... } (7)^m$$

has a solution for each $m \in N_0$; here

$$f^*(t,x,y) = \begin{cases} f(t,x,y), & x \ge 0,\ y \ge \frac{1}{m} \\ f\left(t, x, \frac{1}{m}\right), & x \ge 0,\ y < \frac{1}{m} \\ f(t,0,y), & x < 0\ \ y \ge \frac{1}{m} \\ f\left(t, 0, \frac{1}{m}\right), & x < 0,\ y < \frac{1}{m} \end{cases}$$

and from (H_2), we have $f^*(t,x,y) \ge 0$.

To show $(7)^m$ has a solution, we consider the family of problems

$$\begin{cases} (\phi_p(u'))' + \lambda q(t) f^*(t, x, y) = 0 & 0 < t < 1 \\ u(0) = 0,\ u'(1) = \dfrac{1}{m} & m \in N_0 \end{cases} \qquad \dots (8)_\lambda^m$$

for $0 < \lambda < 1$. Let $u' \in C^1[0, 1]$, $\phi_p(u') \in C^1[0, 1]$ be any solution of $(8)_\lambda^m$.

The differential equation and $(H_1)(H_2)$ immediately imply that $(\phi_p(u'))' \le 0$ on $(0, 1)$, $u'(t) \ge \frac{1}{m}$, on $[0, 1]$, and $u(t) \ge \frac{1}{m}$ on $[0, 1]$, also from (H_3) we have

$$-(\phi_p(u'))' = \lambda q(t) f^*(t, u, u') = \lambda q(t) f(t, u, u')$$

$$\le \lambda q(t) h(u) [g(u') + r(u')]$$

$$\le q(t) h(u(1)) [g(u') + r(u')]$$

for $t \in (0, 1)$, and we have

$$\frac{-(\phi_p(u'))'}{g(u') + r(u')} \le q(t) h(u(1))$$

for $t \in (0, 1)$. Integration from t to 1 yields

$$\int_{\phi_p\left(\frac{1}{m}\right)}^{\phi_p(u'(t))} \frac{dx}{g(\phi_p^{-1}(x)) + r(\phi_p^{-1}(x))} \le h(u(1)) \int_0^1 q(s)\, ds$$

i.e.,

$$I(\phi_p(u'(t))) - I\left(\phi_p\left(\frac{1}{m}\right)\right) \le h(u(1)) \int_0^1 q(s)\, ds$$

and so

$$u'(t) \le \phi_p^{-1}(I^{-1}(h(u(1)) \int_0^1 q(s)\, ds + I(\phi_p(\varepsilon)))$$

Now integrate from 0 to 1 to obtain

$$\frac{u(1)}{\phi_p^{-1}\left(I^{-1}(h(u(1)) \int_0^1 q(s)\, ds + I(\phi_p(\varepsilon))\right)} \le 1 \qquad \dots (9)$$

now (6) together with (9) implies

$$|u|_0 = u(1) \ne M. \qquad \dots (10)$$

Notice any solution u of $(8)_\lambda^m$ that satisfies $0 \le u(t) < M$ for $t \in [0, 1]$, also satisfies

$$\frac{1}{m} \le u'(t) < \phi_p^{-1}\left(I^{-1}\left(h(M) \int_0^1 q(s)\,ds + I(\phi_p(\varepsilon))\right)\right) + 1 \equiv M_1 \qquad \text{... (11)}$$

for $t \in [0, 1]$. Let $M_0 = \max\{M, \phi_p(M_1)\}$ in lemma 1, and from (10) and (11) that

$$|u|_1 = \max\left\{|u|_0, |\phi_p(u')|_0\right\} < M_0 \qquad \text{... (12)}$$

thus lemma 1 implies $(7)^m$ has a solution u_m. In fact

$$0 \le u_m(t) < M, \quad \frac{1}{m} \le u_m'(t) < M_1 \qquad \text{... (13)}$$

for $t \in [0, 1]$, and u_m satisfies

$$\begin{cases} (\phi_p(u'))' + q(t) f(t, u, u') = 0 & 0 < t < 1 \\ u(0) = 0,\ u'(1) = \frac{1}{m} \end{cases}$$

The condition (H_5) guarantees the existence of a function $\Psi_{M, M_1}(t)$ continuous on [0, 1] and positive on (0, 1) and a constant γ, $0 \le \gamma < 1$, with

$$f(t, u_m(t), u_m'(t)) \ge \Psi_{M, M_1}(t)\,[u_m(t)]^\gamma$$

for $(t, u_m(t), u_m'(t)) \in [0, 1] \times [0, M] \times (0, M_1]$, we claim

$$u_m'(t) \ge \left[\int_0^1 \phi_p^{-1}\left(\int_t^1 q(s)\,\Psi_{M, M_1}(s)\, s^\gamma\, ds\right) dt\right]^{\frac{\gamma}{p-1-\gamma}}$$

$$\phi_p^{-1}\left(\int_t^1 q(s)\,\Psi_{M, M_1}(s)\, s^\gamma\, ds\right) \qquad \text{... (14)}$$

from (H_5) we have

$$-(\phi_p(u_m'))' = q(t) f(t, u_m, u_m') \ge q(t)\,\Psi_{M, M_1}(t)\,[u_m(t)]^\gamma$$

Integration from t to 1 yields

$$\phi_p(u_m'(t)) \geq \phi_p\left(\frac{1}{m}\right) + \int_t^1 q(s)\,\Psi_{M,M_1}(s)\,[u_m(s)]^\gamma ds$$

$$> \int_t^1 q(s)\,\Psi_{M,M_1}(s)\,[u_m(s)]^\gamma ds$$

and from lemma 2, we get

$$u_m'(t) > \phi_p^{-1}\left(\int_t^1 q(s)\,\Psi_{M,M_1}(s)\,[u_m(s)]^\gamma ds\right)$$

$$> \phi_p^{-1}\left(\int_t^1 q(s)\,\Psi_{M,M_1}(s)\,u_m^\gamma(1)\,s^\gamma ds\right) \qquad \ldots (15)$$

Now integrate from 0 to t to obtain

$$u_m(t) > \int_0^1 \phi_p^{-1}\left(\int_t^1 q(s)\,\Psi_{M,M_1}(s)\,u_m^\gamma(1)\,s^\gamma ds\right)dt$$

and so
$$u_m(1) > \int_0^t \phi_p^{-1}\left(\int_t^1 q(s)\,\Psi_{M,M_1}(s)\,u_m^\gamma(1)\,s^\gamma ds\right)dt$$

$$= \int_0^1 \phi_p^{-1}(u_m^\gamma(1)) \int_t^1 q(s)\,\Psi_{M,M_1}(s)\,s^\gamma ds)\,dt$$

$$= [u_m(1)]^{\frac{\gamma}{p-1}} \int_0^1 \left(\int_t^1 q(s)\,\Psi_{M,M_1}(s)\,s^\gamma ds\right)dt$$

and so
$$u_m(1) > \left[\int_0^1 \phi_p^1\left(\int_t^1 q(s)\,\Psi_{M,M_1}(s)\,s^\gamma ds\right)dt\right]^{\frac{p-1}{p-1-\gamma}} = a_0.$$

from (15) we have

$$u_m'(t) > \phi_p^{-1}(u_m^\gamma(1))\,\phi_p^{-1}\left(\int_t^1 q(s)\,\Psi_{M,M_1}(s)\,s^\gamma ds\right)$$

$$> \phi_p^{-1}(a_0^{\gamma})\,\phi_p^{-1}\left(\int_t^1 q(s)\,\Psi_{MM_1}(s)\,s^{\gamma}\,ds\right)$$

$$= a_0^{\frac{\gamma}{p-1}}\,\phi_p^{-1}\left(\int_t^1 q(s)\,\Psi_{M,M_1}(s)\,s^{\gamma}\,ds\right).$$

so (14) is true.

Next we show that $\left\{u_m\right\}_{m\in N_0}$ and $\left\{\phi_p(u_m')\right\}_{m\in N_0}$ are bounded, equicontinuous families on [0, 1].

We need only check equicontinuity since (13) holds. Of course for $t\in(0,1)$, we have

$$0\le -(\phi_p(u_m'(t)))'\le h(M)\,[g(u_m'(t))+r(M_1)]\,q(t)$$

$$\le h(M)\left[g\left(a_0^{\frac{\gamma}{p-1}}\,\phi_p^{-1}\int_t^1 s^{\gamma}q(s)\,\Psi_{M,.M_1}(s)\,ds\right)+r(M_1)\right]q(t)$$

now equicontinuity comes immediately from the above (H_6) and (13).

The Arzela-Ascoli theorem guarantees the existence of a subsequence N of N_0 and a function $u\in C^1[0,1]$, $\phi_p(u')\in C^1[0,1]$ with u_m converging uniformly on [0, 1] to u and $\phi_p(u_m')$ to $\phi_p(u')$ as $m\to\infty$ through N; Also $u(0)=0=u'(1)$. In addition, since

$$\phi_p(u_m'(t))\ge a_0^{\gamma}\int_t^1 s^{\gamma}q(s)\,\Psi_{M,M_1}(s)\,ds$$

for $t\in[0,1]$. We have

$$\phi_p(u'(t))\ge a_0^{\gamma}\int_t^1 s^{\gamma}q(s)\,\Psi_{M,M_1}(s)\,ds$$

for $t\in[0,1]$, and so $u'>0$ on [0, 1) and $u>0$ on (0, 1]. Now u_m, $m\in N$ satisfies

$$\phi_p(u_m'(t))=\int_t^1 q(s)f\Big(s,u_m(s),u_m'(s)\Big)ds+\frac{1}{m}$$

for $t\in[0,1]$. Fix $t\in[0,1]$, let $m\to\infty$ through N in the above equality to obtain

$$\phi_p(u'(t))=\int_t^1 q(s)f(s,u(s),u'(s))\,ds$$

for $t \in [0, 1]$, and we deduce immediately that $u \in C^1 (0, 1]$, $\phi_p (u') \in C^1 [0, 1]$,

and $$(\phi_p (u'))' = - q (t) f (t, u, u')$$

for $t \in (0, 1)$.

Remark : In theorem 1, if we let $p = 2$, then the result is the same result as [1].

Example 1 — Consider the boundary value problem

$$\begin{cases} (\phi_p (u'))' + \mu (u')^{-\alpha} [u^{\beta} + 1] = 0 \quad 0 < t < 1 \\ u (0) = u' (1) = 0 \end{cases} \qquad \text{... (16)}$$

with $0 < \alpha < 1, \beta \geq 0$ and $\mu > 0$. If

$$\mu < \frac{p-1}{\alpha + p - 1} \left(\sup_{c \in (0, \infty)} \frac{c^{\alpha + p - 1}}{(c^{\beta} + 1)} \right) \qquad \text{... (17)}$$

then (16) has a solution $u \in C^1 [0, 1]$, $\phi_p (u') \in C^1 [0, 1]$ with $u > 0$ on $(0, 1]$.

To see that (16) has a solution, we will apply theorem 3 with $q \doteq 1$, $g (u) = u^{-\alpha}$, $r = 0$, and $h (u) = \mu [u^{\beta} + 1]$. Clearly, $(H_1) (H_2) (H_3) (H_5)$ (with $\Psi_{H, L} = L^{-\alpha}$ and $\gamma = 0$) and (H_6) (since $0 < \alpha < 1$) are satisfied.Next notice that $I (z) = \frac{p-1}{\alpha + p - 1} z^{\frac{\alpha + p - 1}{p - 1}}$. Also

$$\sup_{c \in (0, \infty)} \frac{c}{\phi_p^{-1} \left(I^{-1} \left(h (c) \int_0^1 q (s)\, ds \right) \right)} = \sup_{c \in (0, \infty)} \frac{c}{\left[\frac{\alpha + p - 1}{p - 1} \mu (c^{\beta} + 1) \right]^{\frac{1}{\alpha + p - 1}}}$$

so (17) guarantees that (H_4) holds. Theorem 3 now establishes the result.

3. Singularity at $u' = 0$ and $u = 0$

In this section our nonlinearity f may be singular at $u' = 0$ and $u = 0$. Throughout this section we will assume that the following conditions hold ;

(G_1) $q \in C [0, 1]$ with $q > 0$ on $(0, 1)$;

(G_2) $f \in C ([0, 1] \times (0, + \infty) \times (0, + \infty), (0, + \infty))$; and

(G_3) $f (t, x, y) \leq [h (x) + w (x)] [g (y) + r (y)]$ on $[0, 1] \times (0, + \infty) \times (0, + \infty)$

with $w > 0$, $g > 0$ continuous and nonincreasing on $(0, +\infty)$, and $h \geq 0, r \geq 0$ continuous and nondecreasing on $[0 + \infty)$.

$$(G_4) \quad \sup_{c \in (0,+\infty)} \frac{c}{\phi_p^{-1}\left(I^{-1}\left[|q|_0\, ch(c) + |q|_0 \int_0^c w(x)\, dx \right]\right)} > 1$$

where $$I(z) = \int_0^z \frac{\phi_p^{-1}(u)\, du}{g(\phi_p^{-1}(u)) + r(\phi_p^{-1}(u))}$$

for $$z > 0,\ |q|_0 = \sup_{t \in [0,1]} |q(t)|,\ I(\infty) = \infty.\ \int_0^a w(x)\, dx < \infty, \text{ for any } a > 0;$$

(G_5) for constants $H > 0, L > 0$, there exists a function $\Psi_{H,L}(t)$ continuous on [0, 1] and positive on (0, 1), such that $f(t, x, y) \geq \Psi_{H,L}(t)$ on $[0, 1] \times (0, H] \times (0, L]$; and

$$(G_6) \int_0^1 q(t)\, w(k_0 t) dt < \infty, \int_0^1 q(t)\, g\left(\phi_p^{-1}\left(\int_t^1 \Psi_{H,L}(s)\, q(s) ds\right) dt\right) < \infty,$$

for any constant $k_0 > 0$.

We have

***Theorem* 2** — *Suppose* (G_1)–(G_6) *hold. Then (1) has a solution* $u \in C^1$ *[0, 1],* $\phi_p(u') \in C^1$ *[0, 1] with* $u(t) > 0$ *on (0, 1].*

PROOF : Choose $M > 0$, and $\varepsilon > 0$ with $\varepsilon < \frac{M}{2}$ and with

$$\frac{M}{\phi_p^{-1}(\varepsilon) + \phi_p^{-1}\left\{ I^{-1}\left[Mh(M)\, |\phi|_q + |\phi|_q \int_0^M w(x) dx + I(\varepsilon)\right]\right\}} > 1 \qquad \text{... (18)}$$

Choose $n_0 \in \{1, 2, ...\}$ with $\phi_p\left(\frac{1}{n_0}\right) < \varepsilon$ and let $N_0 = \{n_0, n_0 + 1, ...\}$

Consider the following system

$$\begin{cases} (\phi_p(u'))' + q(t) f^*(t, u, u') = 0 \quad 0 < t < 1 \\ u(0) = u'(1) = \frac{1}{m} \end{cases} \qquad \text{... (19)}m$$

We first prove that (19)m has a solution for each $m \in N_0$; here

$$f^*(t,x,v)=\begin{cases} f(t,x,v), & u\geq \frac{1}{m},\ v\geq\frac{1}{m} \\ f\left(t,u,\frac{1}{m}\right), & u\geq\frac{1}{m},\ v<\frac{1}{m} \\ f\left(t,\frac{1}{m,}v\right), & u<\frac{1}{m}\ \ v\geq\frac{1}{m} \\ f\left(t,\frac{1}{m},\frac{1}{m}\right), & u<\frac{1}{m},\ v<\frac{1}{m} \end{cases}$$

Consider the family of problems

$$\begin{cases} (\phi_p(u'))'+\lambda q(t)f^*(t,u,u')=0 \quad 0<t<1 \\ u(0)=u'(1)=\frac{1}{m} \qquad m\in N_0 \end{cases} \qquad \ldots (20)_\lambda^m$$

for $0<\lambda<1$. Let $u\in C^1$ [0, 1], $\gamma_\pi(u')\in C^1$ [0, 1] be any solution of $(20)_\lambda^m$. Then according to the definition of f^* and (G_2), we get: $f^*\in C([0,1]\times\Re^2,\Re)$ and $f^*(t,u,v)>0$, for $(t,u,v)\in[0,1]\times\Re^2$. We have $(\phi_p(u'))'=-\lambda q(t)f^*(t,u,u')\leq 0$, so $\phi_p(u')$ nonincreasing, also since ϕ_p increasing, we obtain u' nonincreasing. It is immediate that $u'(t)\geq u'(1)=\frac{1}{m}$ on $t\in[0,1]$, and

$u(t)=\int_0^t u'(t)dt+u(0)\geq\frac{1}{m}t+\frac{1}{m}\geq\frac{1}{m}$ on $t\in[0,1]$. That is

$$u'(t)\geq\frac{1}{m},\ u(t)\geq\frac{1}{m}\quad t\in[0,1] \qquad \ldots (21)$$

by the definition of f^*, we have

$$-(\phi_p(u'))'=\lambda q(t)f^*(t,u,u')=\lambda q(t)f(t,u,u')$$

$$\leq q(t)[h(u)+w(u)]\,[g(u')+r(u')] \qquad \ldots (22)$$

and $$\frac{-(\phi_p(u'))'}{[g(u')+r(u')]}\leq q(t)\,[h(u)+w(u)] \qquad \ldots (23)$$

and so $$\frac{-u'(\phi_p(u'))'}{[g(u')+r(u')]}\leq|q|_0\,[h(u)+w(u)]u'$$

Integration from t to 1 yields

$$\int_t^1 \frac{-u'd(\phi_p(u'))}{g(u')+r(u')} \le |q|_0\, h(u(1))u(1) + |q|_0 \int_{u(t)}^{u(1)} w(x)dx.$$

Notice that

$$\int_t^1 \frac{-u'd(\phi_p(u'))}{g(u')+r(u')} = \int_{\phi_p(u'(1))}^{\phi_p(u'(t))} \frac{\phi_p^{-1}(z)dz}{g[\phi_p^{-1}(z)]+r[\phi_p^{-1}(z)]}$$

$$= I(\phi_p(u'(t))) - \phi_p(u'(1)))$$

we have

$$\phi_p(u'(t)) \le I^{-1}\,[I(\phi_p(u'(1))) + |q|_0\, h(u(1))u(1) + |q|_0 \int_{u(t)}^{u(1)} w(x)dx]$$

$$\le I^{-1}\,[I(\varepsilon) + |q|_0\, h(u\ 1))u(1) + |q|_0 \int_0^{u(1)} w(x)dx].$$

Now integrate from 0 to 1 to obtain

$$u(1) \le \frac{1}{m} + \phi_p^{-1}\left\{ I^{-1}[I(\varepsilon) + |q|_0 h(u(1))u(1) + |h|_0 \int_0^{u(1)} w(x)dx] \right\}$$

and so,

$$\frac{u(1)}{\phi_p^{-1}(\varepsilon + \phi_p^{-1}\left\{ I^{-1}\,[I(\varepsilon) + |q|_0 h(u\ 1))u(1) + |q|_0 \int_0^{u(1)} w(x)dx] \right\}} \le 1 \qquad \text{... (23)}$$

Now (G_4) together with (23) implies

$$|u|_0 = u(1) < M. \qquad \text{... (24)}$$

Next notice any solution u of $(20)_\lambda^m$ that satisfies $\frac{1}{m} \le u(t) < M$ for $t \in [0, 1]$ also satisfies

$$\frac{1}{m} \le u'(t) < \phi_p^{-1}\left\{ I^{-1}\,[|q|_0 h(M)M + |q|_0 \int_0^M w(x)dx + I(\varepsilon)] \right\} + 1 := M_1 \qquad \text{... (25)}$$

Let $M_0 = \max\{M, \phi_p(M_1)\}$ in Lemma 1, notice from (24) and (25) that

$$|u|_1 = \max\{|u|_0, |\phi_p(u')|_0\} < M_0. \qquad \text{... (26)}$$

So Lemma 1 implies (19)m has a solution u_m with

$$\frac{1}{m} \le u_m(t) < M \quad \frac{1}{m} \le u_m'(t) < M_1 \qquad \text{... (27)}$$

and u_m satisfies

$$\begin{cases} (\phi_p(u_m'))' + q(t)f(t, u, u') = 0, \quad 0 < t < 1 \\ u(0) = u'(1) = \dfrac{1}{m} \end{cases} \qquad \text{... (28)}$$

Next notice (G_5) guarantees the existence of a function $\Psi_{M,M_1}(t) \in C[0, 1]$ and positive on (0, 1), such that $f(t, u_m(t), u_m'(t)) \ge \Psi_{m,M_1}(t)$ for $(t, u_m(t), u_m'(t)) \in [0, 1] \times (0, M] \times (0, M_1]$. We have

$$-(\phi_p(u_m'))' \ge q(t)\, \Psi_{M,M_1}(t)$$

Integrate from t to 1 to obtain

$$\phi_p(u_m'(t)) \ge \phi_p(u_m'(1)) + \int_t^1 q(s)\, \Psi_{M,M_1}(s)ds$$

$$\ge \int_t^1 q(s)\, \Psi_{M,M_1}(s)ds$$

and so

$$u_m'(t) \ge \phi_p^{-1}\left(\int_t^1 q(s)\, \Psi_{M,M_1}(s)ds\right), \quad t \in [0, 1] \qquad \text{... (29)}$$

Integrate from 0 to t, we have

$$u_m(t) \ge u_m(0) + \int_0^t \phi_p^{-1}\left(\int_x^1 q(s)\, \Psi_{M,M_1}(s)ds \right)dx$$

$$\ge \int_0^t \phi_p^{-1}\left(\int_x^1 q(s)\, \Psi_{M,M_1}(s)ds \right)dx \qquad \text{... (30)}$$

$$:= t\, \Omega_{M,M_1}(t)$$

here $\Omega_{M,M_1}(t) = \dfrac{1}{t} \displaystyle\int_0^t \phi_p^{-1}\left(\int_x^1 \phi(s)\ \Psi_{M,M_l}(s)ds \right)dx$. Now since

$$\lim_{t\to 0+} \Omega_{M,M_1}(t) = \lim_{t\to 0^+} \frac{\int_0^t \phi_p^{-1}\left(\int_x^1 q(s)\Psi_{M,M_1}(s)ds\right)dx}{t}$$

$$= \phi_p^{-1}\left(\int_0^1 q(s)\,\Psi_{M,M_1}(s)ds\right).$$

Let $\Omega_{M,M_1}(t)$ extend to a continuous function on [0, 1]. Consequently, there exists a $k_0 > 0$ with $\Omega_{M,M_1}(t) \geq k_0 > 0$ for $t \in [0, 1]$. This together with (30) implies

$$u_m(t) \geq k_0 t, \quad t \in [0, 1] \qquad \dots (31)$$

also from (29) and (30) we have that

$$0 \leq -(\phi_p(u_m'(t)))' \leq [h(M) + w(u_m(t))]\,[g(u_m'(t)) + r(M_1)]q(t)$$

$$\leq [h(M) + w(k_0 t)]$$

$$\left[g\left(\phi_p^{-1}\left(\int_t^1 q(s)\,\Psi_{M,M_1}(s)ds\right) + r(M_1)\right]q(t)$$

for $t \in (0, 1)$. From (G_6) we get:

$$\{u_m\}_{m\in N_0},\ \{\phi_p(u_m')\}_{m\in N_0} \text{ is a bound, equicontinuous family on } [0, 1].$$

The Arzela-Ascoli Theorem guarantees the existence of a subsequence N of N_0, and a function $u \in C^1[0, 1]$, $\phi_p(u' \in C^1[0, 1],$, with $u_m(t)$ and $\phi_p(u_m'(t))$ converging uniformly on [0, 1] to $u(t)$ and $\phi_p(u'(t))$ as $m \to \infty$ through N, respectively. Also $u(0) = 0 = u'(1)$ with $u(t) \geq k_0 t$ for $t \in [0, 1]$, and

$$\phi_p(u'(t)) \geq \int_t^1 q(s)\,\Psi_{M,M_1}(s)ds, \quad t \in [0, 1]$$

In addition $u_m(t)$, $M \in N$ satisfies

$$\phi_p(u_m'(t)) = \int_t^1 q(s) f(s, u_m(s), u_m'(s))ds + \frac{1}{m}, \quad t \in [0, 1]$$

fix $t \in [0, 1]$ and let $m \to \infty$ through N to deduce that

$$\phi_p(u'(t)) = \int_t^1 q(s) f(s, u, (s), u'(s)) ds$$

i.e. $$(\phi_p(u'(t)))' = -\phi(t) f(t, u(t), u'(t))$$

That is to say that $u(t)$ is a solution of (1).

Example 2: Consider the boundary value problem

$$\begin{cases} (\phi_p(u'(t))' + \mu(u')^{-\alpha}[u^{-\beta} + \eta_0 u^\gamma + \eta_1] = 0 \quad 1 < t < 1 \\ u(0) = u'(1) = 0 \end{cases} \quad \dots (32)$$

with $0 < \alpha < 1$, $0 < \beta < 1, \eta_0 \geq 0, \eta_1 \geq 0, \gamma \geq 0, \mu > 0$. If

$$\mu < \sup_{c \in (0, \infty)} \frac{c}{\left[c(\eta_0 c^\gamma + \eta_1) + \frac{c^{\beta+1}}{1+\beta}\right][(p-1)(1+\alpha)+1}$$

Then (32) has a solution $u \in C^1[0, 1]$, $\phi_p(u') \in C^1[0, 1]$ with $u(t) > 0$ on $(0, 1]$.

PROOF : Let $q = \mu$, $g(u) = u^{-\alpha}, r = 0, h(u) = \eta_0 u^\gamma + \eta_1$, $w(u) = u^{-\beta}$. Clearly, $(G_1), (G_2)$, $(G_3), (G_5)$ (with $\Psi_{H,L} = H^{-\beta} L^{-\alpha}$) and (G_6) (Since $0 < \alpha < 1$, $0 < \beta < 1$) are satisfied. Also

$$I(z) = \int_0^z \frac{\phi_p^{-1}(u) du}{(\phi_p^{-1}(u))^{-\alpha}} = \frac{z^{(p-1)(1+\alpha)+1}}{(p-1)(1+\alpha)+1},$$

$$I^{-1}(u) = \{u(p-1)(1+\alpha)+1]\}^{\frac{1}{(p-1)(1+\alpha)+1}}$$

$$\int_0^c w(u) du = \int_0^c u^\beta du = \frac{c^{\beta+1}}{1-\beta},$$

and $$\sup_{c \in (0, \infty)} \frac{c}{\phi_p^{-1}\left\{I^{-1}\left[|q|_0 ch(c) + |q|_0 \int_0^c w(x) dx\right]\right\}}$$

$$= \sup_{c \in (0, \infty)} \frac{c}{\phi_p^{-1}\left\{\mu\left[c(\eta_0 c^\gamma + \eta_1) + \frac{c^{\beta+1}}{1-\beta}\right][(p-1)(1+\alpha)+1]\right\}^{\frac{1}{(p-1(1+\alpha)+1(p-1)}}}$$

So (G_4) holds. Theorem 2 now establishes the result.

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