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TWO POSITIVE SOLUTIONS FOR BOUNDARY VALUE PROBLEMS OF A KIND OF STURM-LIOUVILLE FUNCTIONAL DIFFERENTIAL EQUATIONS*

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In this paper, by using of a fixed point theorem in cones, we investigate the existence of two positive solutions for boundary value problem (BVP, in short) of a kind of Sturm-Liouville functional differential equations (FDE, in short) of the form :

$$\left.\begin{aligned} &(p(t)u')' + q(t)u + f(t, u^t) = 0 \quad \text{for} \quad 0 \le t \le 1, \\ &\alpha_1 u(t) - \beta_1 u'(t) = \mu(t) \quad \text{for} \quad -\tau \le t \le 0, \\ &\alpha_2 u(t) + \beta_2 u'(t) = \nu(t) \quad \text{for} \quad 1 \le t \le 1 + h, \end{aligned}\right\}$$

where $f(t, u^t) = \sum_{i=1}^{n} a_i(t)\, u(t + \tau_i))^{\gamma_i}$.

Key Words : Existence; Functional Differential Equations; Boundary Value Problems; Fixed-Point Theorems

1. INTRODUCTION

In recent years, the increasing applications get arise to the increasing interests in BVP of the second-order FDE[1, 2&4-6]. Motivated by [3, 5, 6], this article investigates the existence of two positive solutions for BVP of the form :

$$(p(t)u')' + q(t)u + f(t, u^t) = 0 \text{ for } 0 \le t \le 1, \qquad \dots (1.1)$$

$$\left.\begin{aligned} &\alpha_1 u(t) - \beta_1 u'(t) = \mu(t) \quad \text{for} \quad -\tau \le t \le 0, \\ &\alpha_2 u(t) + \beta_2 u'(t) = \nu(t) \quad \text{for} \quad 1 \le t \le 1 + h, \end{aligned}\right\} \qquad \dots (1.2)$$

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where $f(t, u^t) = \sum_{i=1}^{n} a_i(t)(u(t+\tau_i))^{\gamma_i}$, both τ_i and γ_i are real numbers together with $\gamma_i > 0$ for $i = 1, \ldots, n$ and $\tau = \max\{0, -\tau_1, \ldots, -\tau_n\}$, $h = \max\{0, \tau_1, \ldots, \tau_n\}$ such that $\tau + h < 1$ and $u^t(\theta) = u(t+\theta)$ for $\theta \in [-\tau, h]$.

Let $\mathbb{C} = C([-\tau, h] R)$ be a space equipped with a norm

$$\|\phi\|_{\mathbb{C}} = \sup_{-\tau \le \theta \le h} |\phi(\theta)|$$

and $$\mathbb{C}^+ = \{\phi \in \mathbb{C}; \phi(\theta) \ge 0 \text{ for } \theta \in [-\tau, h]\}.$$

It is obvious that f is nonnegative continuous functional defined on $[-0, 1+h] \times \mathbb{C}^+$.

In this paper, we shall use the following assumptions

(H_1) $p(t) \in C^1[0, 1], q(t) \in C[0, 1]$ and $p(t) > 0, q(t) \le 0$, for $t \in [0, 1]$;

(H_2) $\alpha_i, \beta_i \ge 0, \alpha_i^2 + \beta_i^2 > 0$ for $i = 1, 2$ and BVP:

$$\left.\begin{array}{ll} (p(t)u')' + q(t)u = 0 & \text{for } 0 \le t \le 1, \\ \alpha_1 u(0) - \beta_1 u'(0) = 0, & \\ \alpha_2 u(1) + \beta_2 u'(1) = 0 & \end{array}\right\} \quad \ldots (1.3)$$

has only the zero solution;

(H_3) $\mu(t)$ and $v(t)$ are both continuous functions defined, respectively, on $[-\tau, 0]$ and $[1, b]$, where $b = 1 + h$, $\mu(0) = v(1) = 0$, $\mu(t) \ge 0$ as $\beta_1 = 0$, $\int_t^0 e^{-(\alpha_1/\beta_1)s}\mu(s)\,ds \ge 0$ as $\beta_1 > 0$, $v(t) \ge 0$ as $\beta_2 = 0$, $\int_1^t e^{(\alpha_2/\beta_2)s} v(s)\,ds \ge 0$ as $\beta_2 > 0$.

By solving the linear equations in (1.2), we see that (1.2) is equivalent to

$$u(t) = u(-\tau, t) := \begin{cases} e^{(\alpha_1/\beta_1)t}\left(\dfrac{1}{\beta_1}\displaystyle\int_t^0 e^{-(\alpha_1/\beta_1)s}\mu(s)\,ds + u(0)\right) & \text{for } \beta_1 > 0, \\ \dfrac{1}{\alpha_1}\mu(t) & \text{for } \beta_1 = 0 \end{cases} \quad \ldots (1.4)$$

for $t \in [-\tau, 0]$ and

$$u(t) = u(b;t) := \begin{cases} e^{-(\alpha_2/\beta_2)t}\left(\frac{1}{\beta_2}\int_1^t e^{-(\alpha_2/\beta_2)s}\, v(s)\, ds + e^{\alpha_2/\beta_2} u(1)\right) & \text{for} \quad \beta_2 > 0, \\ \frac{1}{\alpha_2} v(t) & \text{for} \quad \beta_2 = 0 \end{cases}$$

for $t \in [1, b]$

... (1.5)

We say a function in $C[-\tau, b] \cap C^2[0, 1]$ is a *solution* of BVP (1.1)-(1.2) if it satisfies the eq. (1.1) and the boundary condition (1.2).

2. Some Lemmas

Lemma 1 — Let $Q = [0, 1] \times [0, 1]$, $Q_1 = \{(t, s) \in Q \mid 0 \le t \le s \le 1\}$ and $Q_2 = \{(t, s) \in Q \mid 0 \le s \le t \le 1\}$. If (H_1) and (H_2) are satisfied, then there uniquely exists a function $k(t, s)$ for $(t, s) \in Q$ (which is called the green function for (1.3)) which satisfies the following properties:

(*i*) $k(t, s)$ is nonnegative and continuous for $(t, s) \in Q$ and $k(t, s) > 0$ for

$(t, s) \in (0, 1) \times (0, 1)$;

(*ii*) $k(t, s)$ is symmetrical, i.e., $k(t, s) = k(s, t)$ for $(t, s) \in Q$;

(*iii*) $k(t, s)$ has continuous partial derivatives k_t' and k_{tt}'' in Q_1 and Q_2;

and (*iv*) $k(t, s)$ satisfies, for a fixed $s \in [0, 1]$,

$$Lk(t, s) = 0 \text{ as } t \ne s, t \in [0, 1]$$

and

$$\alpha_1 k(0, s) - \beta_1 k_t'(0, s) = 0,$$

$$\alpha_2 k(1, s) + \beta_2 k_t'(1, s) = 0$$

for $s \in (0, 1)$,

where L is the operator defined by $L\,u = (p(t)\,u')' + q(t)\,u$;

(*v*) k_t' has the first type of noncontinuous points as $t = s$ and

$$k_t'(s + 0, s) - k_t'(s - 0, s) = -\frac{1}{p(s)} \text{ for } s \in (0, 1);$$

(*vi*) there exists a constant ε satisfying $0 < \varepsilon < 1$ such that $k(t, s) \ge \varepsilon k(r, s)$ for $t \in J$, $s, r \in [0, 1]$, where J is any interval in $(0, 1)$.

The proof of the above lemma can follow [3, Lemmas 3.1.2-3.1.4]

Lemma 2 — Under the hypotheses in Lemma 1, if the function $u(t)$ is decided by the integral equation :

$$u(t)=\begin{cases} u(-\tau; t) & \text{for} \quad t\in[-\tau,0] \\ \int_0^1 k(t,s)f(s,u^s)\,ds & \text{for} \quad t\in[0,1], \\ u(b;t) & \text{for} \quad t\in[1,b], \end{cases} \qquad \text{... (2.1)}$$

then we have $u(t)\in C^2[0,1]$ and is a solution of BVP (1.1)-(1.2)

PROOF : It is easy for one to see that (2.1) can be rewritten as

$$u(t)=\int_0^t k(t,s)f(s,u^s)\,ds+\int_t^1 k(t,s)f(s,u^s)\,ds \text{ for } t\in[0,1].$$

Taking derivative of the above, we have

$$u'(t)=k(t,t)f(t,u^t)+\int_0^t k_t'(t,s)f(s,u^s)\,ds$$

$$-k(t,t)f(t,u^t)+\int_t^1 k_t'(t,s)f(s,u^s)\,ds$$

$$=\int_0^1 k_t'(t,s)f(s,u^s)\,ds. \qquad \text{... (2.2)}$$

Furtherly, by noting (v) of Lemma 1, we have

$$u''(t)=k_t'(t+0,t)f(t,u^t)+\int_0^t k_{tt}''(t,s)f(s,u^s)\,ds$$

$$-k_t'(t-0,t)f(t,u^t)+\int_t^1 k_{tt}''(t,s)f(s,u^s)\,ds \qquad \text{... (2.3)}$$

$$=\int_0^1 k_{tt}''(t,s)f(s,u^s)\,ds-\frac{f(t,u^t)}{p(t)}.$$

From (2.2), (2.3) and (iv) of Lemma 1, we have

$$-Lu=-[p(t)u''(t)+p'(t)u'(t)+q(t)u(t)]$$

$$= -\int_0^1 [p(t)\, k''_{tt}(t, s) f(s, u^s) + p'(t)\, k'_t(t, s) f(s, u^s)$$

$$+ q(t)\, k(t, s) f(s, u^s)]\, ds + f(t, u^t)$$

$$= -\int_0^1 L\, k(t, s) f(s, u^s)\, ds + f(t, u^t)$$

$$= f(t, u^t),$$

which shows that $u(t)$ satisfies (1.1) in [0, 1].

Furthermore, from again (iv) of Lemma 1, we get

$$\alpha_1 u(0) - \beta_1 u'(0) = \alpha_1 \int_0^1 k(0, s) f(s, u^s)\, ds - \beta_1 \int_0^1 k'_t(0, s) f(s, u^s)\, ds$$

$$= \int_0^1 [\alpha_1 k(0, s) - \beta_1 k'_t(0, s)] f(s, u^s)\, ds = 0.$$

Similarly, we can get $\alpha_2 u(1) + \beta_2 u'(1) = 0$.

Cosequently, $u(t)$ is a solution of BVP (1.1)-(1.2). The proof is complete.

Remark : Suppose that $u_0(t)$ is the solution of BVP (2.1) with $f \equiv 0$. Then it can be expressed as

$$u_0(t) = \begin{cases} u(-\tau; t) & \text{for} \quad t \in [-\tau, 0], \\ 0 & \text{for} \quad t \in [0, 1], \\ u(b; t) & \text{for} \quad t \in [1, b]. \end{cases}$$

We write

$$N = \| u_0(t) \|_{[-\tau, b]} = \max_{t \in [-\tau, b]} | u_0(t) |.$$

Lemma 3 (Krasnoselskii's fixed point theorem in cones) — Assume that X is a Banach space and $P \subset X$ is a cone in X, Ω_1, Ω_2 are open subsets of X and $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$. Furthermore, let $\Phi : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator satisfying one of the following conditions:

(*i*) $\Phi u \not\geq u$ for $u \in P \cap \partial \Omega_1$ and $\Phi u \not\leq u$ for $u \in P \cap \partial \Omega_2$;

(*ii*) $\Phi u \leq u$ for $u \in P \cap \partial \Omega_1$ and $\Phi u \not\geq u$ for $u \in P \cap \partial \Omega_2$.

Then there is a fixed point of Φ in $P \cap (\Omega_2 \setminus \overline{\Omega}_1)$.

3. MAIN THEOREM

The following theorem is our main result in the present paper :

***Theorem* 1** — *Assume that* $(H_1)-(H_3)$ *hold. Then BVP (1.1)-(1.2) has at least two nonnegative solutions which are not identically to zero if the following conditions are satisfied,*

(i) $a_i(t)$ $(i = 1, ..., n)$ is a nonnegative continuous function and there exist $i_0, i_1 \in \{1, ..., n\}$ such that $\gamma_{i_0} < 1, \gamma_{i_1} > 1$ and

$$q_0 = \inf_{t \in [0,1]} a_{i_0}(t) > 0 \text{ and } q_1 = \inf_{t \in [0,1]} a_{i_1}(t) > 0;$$

(ii) $\sum_{i=1}^{n} \int_0^1 a_i(t)\, dt < \frac{1}{M(1+N)^\gamma}$, where $M = \max_{(t,s) \in Q} k(t,s)$, $k(t,s)$ is the green function of BVP (1.3) and $\gamma = \max\{\gamma_1, ..., \gamma_n\}$

PROOF : From $0 \le \tau + h < 1$, there exist two positive constants δ_1 and δ_2 such that $\delta_1 < \tau + \delta_1 < 1 - h - \delta_2 < 1 - \delta_2$.

Let $J_1 = [\delta_1, 1 - \delta_2]$, $J_2 = [\tau + \delta_1, 1 - h - \delta_2]$ and $d = (1 - h - \delta_2) - (\tau + \delta_1)$. Then we get from ($i$) and ($vi$) of Lemma 1 that $k(t, s)$ is nonnegative symmetrical continuous and there exist constants $q > 0$ and $0 < \varepsilon_0 < 1$ such that

$$k(t,s) \ge q \text{ for } (t,s) \in J_1 \times J_1 \quad \text{... (3.1)}$$

and

$$k(t,s) \ge \varepsilon_0 k(r,s) \text{ for } t \in J_1, s \in [0,1] \text{ and } r \in [0,1]. \quad \text{... (3.2)}$$

For the sake of simplicity, we shall show the conclusion of our theorem only for the case $\beta_1 > 0$ and $\beta_2 = 0$, since the other cases can be similarly discussed. Then $u_0(t)$ can be expressed as

$$u_0(t) = \begin{cases} \frac{1}{\beta_1} e^{(\alpha_1/\beta_1)t} \int_t^0 e^{-(\alpha_1/\beta_1)s} \mu(s)\, ds & \text{for } -\tau \le t \le 0, \\ 0 & \text{for } 0 \le t \le 1, \\ \frac{1}{\alpha_2} v(t) & \text{for } 1 \le t \le b. \end{cases} \quad \text{... (3.3)}$$

Let $u(t)$ be a solution of BVP (2.1) and $\upsilon(t) = u(t) - u_0(t)$. Noting that $\upsilon(t) \equiv u(t)$ for $0 \le t \le 1$, we have from (2.1) that

$$v(t) = \begin{cases} e^{(\alpha_1/\beta_1)t} \int_0^1 k(0,s) f(s, u_0^s + v^s)\, ds & \text{for } -\tau \le t \le 0, \\ \int_0^1 k(t,s) f(s, u_0^s + v^s)\, ds & \text{for } 0 \le t \le 1, \\ 0 & \text{for } 1 \le t \le b. \end{cases} \quad \dots (3.4)$$

Let P be a cone in the Banach space $X = C[-\tau, b]$ defined by

$$P = \left\{ v \in C[-\tau, b],\ v(t) \ge 0,\ \min_{t \in J_1} v(t) \ge \varepsilon_0 \| v \|_{[-\tau, b]} \right\},$$

where $\| v \|_{[-\tau, b]} = \sup_{-r \le t \le b} | v(t) |.$

Define Φ as

$$(\Phi v)(t) = \begin{cases} e^{(\alpha_1/\beta_1)t} \int_0^1 k(0,s) f(s, u_0^s + v^s)\, ds & \text{for } -\tau \le t \le 0, \\ \int_0^1 k(t,s) f(s, u_0^s + v^s)\, ds & \text{for } 0 \le t \le 1, \\ 0 & \text{for } 1 \le t \le b. \end{cases} \quad \dots (3.5)$$

It is not difficult to prove that Φ is completely continuous. Next, we shall show that $\Phi(P) \subset P$.

In fact, for $-\tau \le t \le 0$, we have $0 \le (\Phi v)(t) \le (\Phi v)(0)$. Thus we get

$$\| \Phi v \|_{[-\tau, b]} = \| \Phi v \|_{[0, 1]}.$$

While $v \in P$, we get from (3.2) that

$$(\Phi v)(t) \ge \varepsilon_0 \int_0^1 k(r,s) f(s, u_0^s + v^s)\, ds = \varepsilon_0 \Phi v(r) \text{ for } t \in J_1 \text{ and } r \in [0, 1].$$

Hence, $\min_{t \in J_1} \Phi v(t) \ge \varepsilon_0 \| \Phi v \|_{[0,1]} = \varepsilon_0 \| \Phi v \|_{[-\tau, b]},$

which follows that $\Phi v \in P$ and $\Phi(P) \subset P$.

Let $T_l = \left\{ v \in C[-\tau, b] : \| v \|_{[-\tau, b]} \right\}$ $(l > 0)$. Select real numbers R_1 and R_2 such that

$$0 < R_1 < \min \left\{ 1, (dqq_0)^{\frac{1}{1-\gamma_{i_0}}} \varepsilon_0^{\frac{\gamma_{i_0}}{1-\gamma_{i_0}}} \right\} \quad \dots (3.6)$$

and $$R_2 > \max\left\{1, (dqq_1)^{\frac{1}{1-\gamma_{i_1}}} \varepsilon_0^{\frac{\gamma_{i_1}}{1-\gamma_{i_1}}}\right\}. \quad \text{... (3.7)}$$

In what follows, we shall prove that

$$\Phi v \not\leq v \text{ for } v \in P \cap \partial T_{R_1} \quad \text{... (3.8)}$$

and $$\Phi v \not\leq v \text{ for } v \in P \cap \partial T_{R_2}. \quad \text{... (3.9)}$$

In fact, if there exists a $v_0 \in P \cap \partial T_{R_1}$ such that $\Phi v_0 \leq v_0$, then when $t \in J_2 \subset J_1$ we have

$$v_0(t) \geq \Phi v_0(t) \geq \int_0^1 k(t, s)\, a_{i_0}(s)\, (u_0(s - \tau_{i_0}) + v_0(s - \tau_{i_0}))^{\gamma_{i_0}} ds$$

$$\geq \int_{J_2} k(t, s)\, a_{i_0}(s)\, (v_0(s - \tau_{i_0}))^{\gamma_{i_0}} ds$$

$$\geq dqq_0\, \varepsilon_0^{\gamma_{i_0}} \| v_0 \|_{[-\tau, b]}^{\gamma_{i_0}}.$$

Hence, $R_1 = \| v_0 \|_{[-\tau, b]} \geq dqq_0\, \varepsilon_0^{\gamma_{i_0}} R_1^{\gamma_{i_0}}$, which contradicts (3.6).

Similarly, if there exists a $v_1 \in P \cap \partial T_{R_2}$ such that $\Phi v_1 \leq v_1$, then when $t \in J_2 \subset J_1$ we have

$$v_1(t) \geq (\Phi v_1)(t) \geq \int_{J_2} k(t, s)\, a_{i_1}(s)\, (v_1(s - \tau_{i_1}))^{\gamma_{i_1}} ds$$

$$\geq dqq_1\, \varepsilon_0^{\gamma_{i_1}} \| v_1 \|_{[-\tau, b]}^{\gamma_{i_1}}.$$

Therefore, $R_2 = \| v_1 \|_{[-\tau, b]} \geq dqq_1\, \varepsilon_0^{\gamma_{i_1}} R_2^{\gamma_{i_1}}$, which contradicts (3.7).

Finally, we claim that

$$\Phi v \not\geq v \text{ for } v \in P \cap \partial T_1. \quad \text{... (3.10)}$$

Otherwise, if there exists a $v_2 \in P \cap \partial T_1$ such that $\Phi v_2 \geq v_2$, then

$$1 = \| v_2 \|_{[-\tau, b]} \leq \| \Phi v_2 \|_{[-\tau, b]} = \| \Phi v_2 \|_{[0, 1]}$$

$$= \left\| \left| \int_0^1 k(t,s) f(s, u_0^s + v_2^s)\, ds \right| \right\| \le M \sum_{i=1}^{n} \| a_i \|_L \| u_0 + v_2 \|_{[-\tau, b]}^{\gamma_i}$$

$$\le M (1+N)^{\gamma} \sum_{i=1}^{n} \| a_i \|_L,$$

where $\| a_i \|_L = \int_0^1 a_i(s)\, ds$. The above is a contradiction with (ii) of the theorem.

From (3.8)-(3.10) and Lemma 3, we know that Φ at least have two fixed points v^* and v^{**}, respectively, in $P \cap (T_{R_2} \setminus \bar{T}_1)$ and $P \cap (T_1 \setminus \bar{T}_{R_1})$.

It is obvious that $v^*(t)$ and v^{**} satisfy (3.4). By the fact that $u_0(t) \ge 0$ and $v^*(t) > 0$, $v^{**}(t) > 0$ for $t \in H_1$, we conclude that $u^*(t) = u_0(t) + v^*(t)$, $u^{**}(t) = u_0(t) + v^{**}(t)$ are two nonnegative solutions of BVP (2.2), which are not identically to zero. By virtue of Lemma 2, we complete the proof.

4. An Example

Let us now introduce an example to illustrate the usage of our theorem.

Example 1 — Consider BVP

$$\left.\begin{array}{ll} u''(t) + \frac{1}{6} u^{\frac{1}{2}}\left(t - \frac{1}{2}\right) + \frac{1}{5} u^{\frac{3}{2}}\left(t + \frac{1}{3}\right) = 0 & \text{for } 0 \le t \le 1, \\ u(t) = -\sin \pi t & \text{for } -\frac{1}{2} \le t \le 0, \\ u'(t) = 0 & \text{for } 1 \le t \le \frac{4}{3}. \end{array}\right\}$$

It is easy to show that the Green function $k(t, s) = \min\{t, s\} = \begin{cases} t & \text{for } t \le s, \\ s & \text{for } t > s. \end{cases}$ Hence, $M = 1$, $N = 1$ and all conditions of the Theorem hold for BVP above. Then there are at least two nonnegative solutions which are not identically to zero.

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