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FINSLER SPACE WITH THE GENERAL APPROXIMATE MATSUMOTO METRIC

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We define the general approximate Matsumoto metric, and find the conditions for a Finsler space with general approximate Matsumoto metric to be a Berwald space and a Douglas space.

Key Words : Finsler Space; General Approximate Matsumoto; Randers Space; Berwald Space; Douglas Space

1. INTRODUCTION

A Finsler metric $L(x, y)$ is a differentiable manifold M^n is called as an (α, β)-metric, if L is a positively homogeneous function of degree one of a Riemannian metric $\alpha = (a_{ij}(x)\, y^i y^j)^{1/2}$ and a one-form $\beta = b_i(x)\, y^i$ on M^n. The interesting and important examples of an (α, β)-metric are Randers metric, Kropina metric and Matsumoto metric. The notion of an (α, β)-metric was introduced by Matsumoto[10] and has been studied by many authors. Specially the Matsumoto metric is an exact formulation of the model of Finsler space. A Finsler space is called as a Berwald space if the Berwald connection is linear. In the Matsumoto metric, the 1-form $\beta = b_i(x)\, y^i$ was originally induced by earth's gravity[8]. Hence we could regard $b_i(x)$ as the infinitesimals. The notion of a Douglas space has been introduced by S. Bácsó and M. Matsumoto[2] as a generalization of a Berwald space from the viewpoint of geodesic equation. It is remarkable that a Finsler space is a Douglas space, if and only if the Douglas tensor is vanishes identically.

Recently, M. Matsumoto has found the conditions for the Finseler space with some (α, β)-metrics to be a Douglas space and Berwald space[11]. The first author of present paper and Choi[15&16] have investigated the Finsler spaces with the first approximate Matsumoto metric in which all powers greater than 2 of $b_i(x)$ are neglected, and the second approximate Matsumoto metric in which all powers greater than 3 of $b_i(x)$ are neglected, to be a Berwald space and a Douglas space.

The present paper is the generalization of the above notion. We are devoted to finding the conditions for the Finsler space with the general approximate Matsumoto metric in which all powers greater than r (r is arbitrary integer greater than 3) of $b_i(x)$ are neglected to be a Berwald space and a Douglas space. Much long complicated calculations in this paper are used as computer by the third author.

2. PRELIMINARIES

The Matsumoto metric $L=\alpha^2/(\alpha-\beta)$ is expressed as the form

$$L=\lim_{r\to\infty}\alpha\sum_{k=0}^{r}\left(\frac{\beta}{\alpha}\right)^k \qquad \dots (2.1)$$

for $|\beta|<|\alpha|$. We regard $b_i(x)$ as very small numerically. If we neglect all the powers greater than r of $b_i(x)$ in (2.1), then the (α,β)-metric L.

$$L=\alpha\sum_{k=0}^{r}\left(\frac{\beta}{\alpha}\right)^k \qquad \dots (2.2)$$

is an approximate metric of the Matsumoto metric. If $r=0$, then $L=\alpha$ is a Riemannian metric. If $r=1$, then $L=\alpha+\beta$ is a Randers metric. The conditions for the Randers space to be a Berwald space and a Douglas space are found in [11]. If $r=2$, and $r=3$, then L is the first and second approximate Matsumoto metric and they are investigated in [15] and [16]. We shall deal with arbitrary integer r greater than 3 in this paper. We shall call the (α,β)-metric (2.2) the general approximate Matsumoto metric or *the rth approximate Matsumoto metric.*

On the other hand, the geodesics of a Finsler space $F^n=(M^n,L)$ are given by the system of differential equations including the function

$$4G^i(x,y)=g^{ij}(y^r\partial_j\partial_r L^2-\partial_j L^2)=2\gamma^i_{jk}y^jy^k,$$

where γ^i_{jk} are Christoffel symbols constructed from the Finsler metric tensor $g_{ij}(x,y)$ with respect to x^i. The space $R^n=(M^n\,\alpha)$ is called the associated Riemannian space with $F^n=(M^n,L(\alpha,\beta))$[1&7] The covariant differentiation with respect to the Levi-Civita Connection $\{{}_j{}^i{}_k\}(x)$ of R^n is denoted by (;). We use the symbols as follows :

$$r_{ij}=\frac{1}{2}(b_{i;j}+b_{j;i}),\ s_{ij}=\frac{1}{2}(b_{i;j}-b_{j;i}),\ s^i_j=a^{ir}s_{rj}\ \text{and}\ s_j=b_rs^r_j.$$

According to [5] and [9], the function $G^i(x,y)$ of F^n with an (α,β)-metric are written in the form

$$2G^i=\left\{{}_0{}^i{}_0\right\}+2B^i$$

and $$B^i = \frac{\alpha L_\beta}{L_\alpha} s_0^i + C^* \left\{ \frac{\beta L_\beta}{\alpha L} y^i - \frac{\alpha L_{\alpha\alpha}}{L_\alpha} \left(\frac{1}{\alpha} y^i - \frac{\alpha}{\beta} b^i \right) \right\}, \quad \text{... (2.3)}$$

where $L_\alpha = \partial L / \partial \alpha$, $L_\beta = \partial L / \partial \beta$, $L_{\alpha\alpha} = \partial^2 L / \partial \alpha \partial \alpha$, the subscript 0 means contraction by y^i and we put

$$C^* = \frac{\alpha \beta (r_{00} L_\alpha - 2 s_0 \alpha L_\beta)}{2 (\beta^2 L_\alpha + \alpha \gamma^2 L_{\alpha\alpha})}, \; b^i = a^{ij} b_j,$$

$$\gamma^2 = b^2 \alpha^2 - \beta^2, \; b^2 = a^{ij} b_i b_j.$$

We shall denote the homogeneous polynomials in (y^i) of degree r by $hp(r)$ for brevity.

From $(2.3)_1$ the Berwald connection $B\Gamma = (G^i_j k, G^i_j, 0)$ of F^n with an (α, β)-metric is given by

$$G^i_j = \partial_j G^i = \left\{ {}_0{}^i{}_j \right\} + B^i_j,$$

and $$G^i_{jk} = \partial_k G^i_j = \left\{ {}_j{}^i{}_k \right\} + B^i_{jk},$$

where we put $B^i_j = \partial_j B^i$ and $B^i_j k = \partial_k B^i_j$. On account of [10], $B^i_j k$ are determined by

$$L_\alpha B^t_j i y^j y_t + \alpha L_\beta (B^t_j i b_t - b_{j,i}) y^j = 0. \quad \text{... (2.4)}$$

The Finsler space F^n with an (α, β)-metric is a Douglas space, if and only if $B^{ij} \equiv B^i y^j - B^j y^i$ are $hp(3)$[2]. From (2.3) this equation is written as follows :

$$B^{ij} = \frac{\alpha L_\beta}{L_\alpha} (s_0^i y^j - s_0^j y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} C^* (b^i y^j - b^j y^i). \quad \text{... (2.5)}$$

We shall state the following lemma for later[3];

Lemma — If $\alpha^2 \equiv 0 \pmod{\beta}$, that is, $a_{ij}(x) y^i y^j$ contain $b_i(x) y^i$ as a factor, then the dimension is equal to two and b^2 vanishes. In this case we have $\delta = d_i(x) y^i$ satisfying $\alpha^2 \equiv \beta \delta$ and $d_i b^i = 2$.

3. Berwald Space

In the n-dimensional Finsler space F^n with the rth $(r \geq 3)$ approximate Matsumoto metric (2.2), we have

$$L_\alpha = -\sum_{k=0}^{rk} (k-1)\left(\frac{\beta}{\alpha}\right)^k, \quad L_\beta = \sum_{k=0}^{r} \left(\frac{\beta}{\alpha}\right)^{k-1},$$

$$L_{\alpha\alpha} = \frac{1}{\alpha}\sum_{k=0}^{r} (k-1)\,k\left(\frac{\beta}{\alpha}\right)^k. \qquad \text{... (3.1)}$$

Substituting (3.1) into (2.4), we have

$$-\sum_{k=0}^{r} (k-1)\left(\frac{\beta}{\alpha}\right)^k B_j^t{}_i y^j y_t + \alpha \sum_{k=0}^{r} k\left(\frac{\beta}{\alpha}\right)^{k-1} (B_j{}^t{}_i b_t - b_{j;i})\, y^j = 0. \qquad \text{... (3.2)}$$

Suppose that F^n is a Berwald space, that is $B_j{}^i{}_k$ is a function of position alone.

We shall divide our consideration into two cases of which r is even or odd.

(*i*) *Case of $r = 2h$ (h is A positive integer)*

Since
$$\sum_{k=0}^{r} (k-1)\left(\frac{\beta}{\alpha}\right)^k = \frac{1}{\alpha^{2h}} \sum_{k=0}^{2h} (2h-k-1)\,\alpha^k \beta^{2h-k},$$

and
$$\sum_{k=0}^{r} k\left(\frac{\beta}{\alpha}\right)^{k-1} = \frac{\alpha}{\alpha^{2h}} \sum_{k=0}^{2h} (2h-k)\,\alpha^k \beta^{2h-k-1},$$

(3.2) is written as the following

$$\sum_{k=0}^{2h} (2h-k-1)\,\alpha^k \beta^{2h-k} B_j{}^t{}_i y^j y_t$$

$$-\alpha^2 \sum_{k=0}^{2h} (2h-k)\,\alpha^k \beta^{2h-k-1} (B_j{}^t{}_i b_t - b_{j;i})\, y^j = 0. \qquad \text{... (3.3)}$$

Separating (3.3) in the rational and irrational terms in y^i, we rewrite (3.3) as the following form

$$\sum_{k=0}^{h} (2h-2k-1)\,\alpha^{2k} \beta^{2h-2k} B_j{}^t{}_i y^j y_t$$

$$-\alpha^2 \sum_{k=0}^{h} (2h-2k)\,\alpha^{2k} \beta^{2h-2k-1} (B_j{}^t{}_i b_t - b_{j;i})\, y^j$$

$$+ \alpha [\{ \sum_{k=0}^{h-1} (2h-2k-2)\, \alpha^{2k} \beta^{2h-2k-1} B_j{}^t{}_i y^j y_t$$

$$\left. - \alpha^2 \sum_{k=0}^{h} (2h-2k-1)\, \alpha^{2k} \beta^{2h-2k-2} (B_j{}^t{}_i b_t - b_{j;\,i}) y^j \right] = 0,$$

that is,

$$AB_{j\,i}^{\ t} y^j y_t - \alpha^2 B (B_j{}^t{}_i b_t - b_{j;\,i}) y^j + \alpha \left\{ CB_j{}^t{}_i y^j y_t - \alpha^2 D (B_j{}^t{}_i b_t - b_{j;\,i}) y^j \right\} = 0,$$

where

$$A = \sum_{k=0}^{h} (2h-2k-1)\, \alpha^{2k} \beta^{2h-2k}, \quad B = \sum_{k=0}^{h} (2h-2k)\, \alpha^{2k} \beta^{2h-2k-1},$$

$$C = \sum_{k=0}^{h-1} (2h-2k-2)\, \alpha^{2k} \beta^{2h-2k-1}, \quad D = \sum_{k=0}^{h-1} (2h-2k-1)\, \alpha^{2k} \beta^{2h-2k-2}. \quad \text{... (3.4)}$$

Since A, B, C, D, $B_j{}^t{}_i y^j y_t$, $(B_j{}^t{}_i b_t - b_{j;\,i}) y^j$ are rational polynomials and α is an irrational polynomial of y^i, we have

$$AB_j{}^t{}_i y^j y_t - \alpha^2 B (B_j{}^t{}_i b_t - b_{j;\,i}) y^j = 0,$$

$$CB_j{}^t{}_i y^j y_t - \alpha^2 D (B_j{}^t{}_i b_t - b_{j;\,i}) y^j = 0,$$

from which

$$(AD - BC)\, B_j{}^t{}_i y^j y_t = 0, \quad (AD - BC)(B_j{}^t{}_i b_t - b_{j;\,i}) y^j = 0. \quad \text{... (3.5)}$$

If $AD - BC = 0$, that is,

$$\sum_{k=0}^{h} (2h-2k-1)\, \alpha^{2k} \beta^{2h-2k} \sum_{k=0}^{h-1} (2h-2k-1)\, \alpha^{2k} \beta^{2h-2k-2}$$

$$- \sum_{k=0}^{h} (2h-2k)\, \alpha^{2k} \beta^{2h-2k-1} \sum_{k=0}^{h-1} (2h-2k-2)\, \alpha^{2k} \beta^{2h-2k-1} = 0, \quad \text{... (3.6)}$$

then the term of (3.6) which does not contain α^2 is β^{4h-2}. Therefore, there exists $hp(4h - 4)$: V_{4h-4} such that

$$\beta^{4h\ 2} = \alpha^2 V_{4h-4}.$$

For the dimension n greater than 2, we have $\alpha^2 \not\equiv 0$ (mod. β) from Lemma and $V_{4h-4}=0$. That is a contradiction. Thus from (3.5) we have

$$B_j{}^t{}_i y^j y_t = 0, \quad (B_j{}^t{}_i b_t - b_{j;i}) y^j = 0.$$

These show $B_j{}^t{}_i y^j a_{th} + B_j{}^t{}_i y^j a_{tj} = 0$, $B_j{}^t{}_i b_t - b_{j;i} = 0$.

The former yields $B_j{}^t{}_i = 0$ and hence we get $b_{j;i} = 0$ from the latter.

(*ii*) ***Case of*** $r = 2h + 1$ (h is A Positive Integer)

Eq (3.2) is written as the following form

$$\sum_{k=0}^{2h+1} (2h-k)\,\alpha^k \beta^{2h-k+1} B_j{}^t{}_i y^j y_t$$

$$-\alpha^2 \sum_{k=0}^{2h+1} (2h-k+1)\,\alpha^k \beta^{2h-k} (B_j{}^t{}_i b_t - b_{j;i}) y^j = 0.$$

which implies $\beta^2 B B_j{}^t{}_i y^j y_t - \alpha^2 E (B_j{}^t{}_i b_t - b_{j;i}) y^j$

$$+ \alpha\left\{ A B_j{}^t{}_i y^j y_t - \alpha^2 B (B_j{}^t{}_i b_t - b_{j;i}) y^j \right\} = 0,$$

where $$E = \sum_{k=0}^{h} (2h-2k+1)\,\alpha^{2k} \beta^{2h-2k}. \qquad \text{... (3.7)}$$

Thus we have

$$\beta^2 B B_j{}^t{}_i y^j y_t - \alpha^2 E (B_j{}^t{}_i b_t - b_{j;i}) y^j = 0,$$

and $$A B_j{}^t{}_i y^j y_t - \alpha^2 B (B_j{}^t{}_i b_t - b_{j;i}) y^j = 0.$$

From the above two equations, we have

$$(\beta^2 B^2 - AE) B_j{}^t{}_i y^j y_t = 0, \quad (\beta^2 B^2 - AE)(B_j{}^t{}_i b_t - b_{j;i}) y^j = 0.$$

If $\beta^2 B^2 - AE = 0$, that is,

$$\beta^2 \sum_{k=0}^{h} (2h-2k)\,\alpha^{2k} \beta^{2h-2k-1} \sum_{k=0}^{h} (2h-2k)\,\alpha^{2k} \beta^{2h-2k-1}$$

$$-\sum_{k=0}^{h} (2h-2k-1)\,\alpha^{2k} \beta^{2h-2k} \sum_{k=0}^{h} (2h-2k+1)\,\alpha^{2k} \beta^{2h-2k} = 0, \qquad \text{... (3.8)}$$

then the term of (3.8) which seemingly does not contain α^2 is β^{4h}. Therefore, there exists $hp\,(4h-2): V_{4h-2}$ such that $\beta^{4h} = \alpha^2 V_{4h-2}$. From $\alpha^2 \not\equiv 0$ (mod. β), we have a contradiction. Hence, we have $B_j{}^t{}_i y^j y_t = 0$, $B_j{}^t{}_i b_t - b_{j;i} = 0$. Thus we get $b_{j;i} = 0$.

Conversely, if $b_{j;i} = 0$, then it is known[4] that F^n with an (α, β)-metric is a Berwald space. Consequently, we have

***Theorem* 3.1** — *The Finsler space F^n $(n > 2)$ with a general approximate Matsumoto metric (2.2) is a Berwald space if and only if $b_{j;i} = 0$.*

4. Douglas Space

The present section is devoted to finding the condition for a Finsler space F^n with the general approximate Matsumoto metric (2.2) to be a Douglas space. Substituting (3.1) in to (2.5), we have

$$-2\beta\left\{\sum_{k=0}^{r}(k-1)\left(\frac{\beta}{\alpha}\right)^{k}\right\}$$

$$\left\{\beta^2\sum_{k=0}^{r}(k-1)\left(\frac{\beta}{\alpha}\right)^{k}-(b^2\alpha^2-\beta^2)\sum_{k=0}^{r}(k-1)\,k\left(\frac{\beta}{\alpha}\right)^{k}\right\}B^{ij}$$

$$=2\alpha^2\sum_{k=0}^{r}k\left(\frac{\beta}{\alpha}\right)^{k}$$

$$\left\{\beta^2\sum_{k=0}^{r}(k-1)\left(\frac{\beta}{\alpha}\right)^{k}-(b^2\alpha^2-\beta^2)\sum_{k=0}^{r}(k-1)\,k\left(\frac{\beta}{\alpha}\right)^{k}\right\}(s^i_0\,y_j-s^j_0\,y^i) \quad \ldots (4.1)$$

$$+\ \alpha\sum_{k=0}^{r}(k-1)\,k$$

$$\left(\frac{\beta}{\alpha}\right)^{k}\left\{\alpha\beta r_{00}\sum_{k=0}^{r}(k-1)\left(\frac{\beta}{\alpha}\right)^{k}+2s_0\,\alpha^3\sum_{k=0}^{r}k\left(\frac{\beta}{\alpha}\right)^{k}\right\}(b^i y^j-b^j y^i).$$

We consider the two cases of which r is even or odd.

(*i*) *Case of* $r = 2h$ (h is a positive integer)

Eq. (4.1) is rewritten as follows :

$$2\beta \sum_{k=0}^{2h} (2h-k-1)\, \alpha^k \beta^{2h-k} \Bigg[\beta^2 \sum_{k=0}^{2h} (2h-k-1)\, \alpha^k \beta^{2h-k}$$

$$- (b^2 \alpha^2 - \beta^2) \sum_{k=0}^{2h} (2h-k-1)(2h-k)\, \alpha^k \beta^{2h-k} \Bigg] B^{ij}$$

$$+ 2\alpha^2 \sum_{k=0}^{2h-1} (2h-k)\, \alpha^k \beta^{2h-k} \Bigg[\beta^2 \sum_{k=0}^{2h} (2h-k-1)\, \alpha^k \beta^{2h-k}$$

$$- (b^2 \alpha^2 - \beta^2) \sum_{k=0}^{2h} (2h-k-1)(2h-k)\, \alpha^k \beta^{2h-k} \Bigg] (s_0^i y^j - s_0^j y^i)$$

$$+ \alpha \sum_{k=0}^{2h} (2h-k-1)(2h-k)\, \alpha^k \beta^{2h-k} \Bigg[\alpha \beta r_{00} \sum_{k=0}^{2h} (2h-k-1)\, \alpha^k \beta^{2h-k}$$

$$+ 2 s_0 \alpha^3 \sum_{k=0}^{2h} (2h-k)\, \alpha^k \beta^{2h-k} \Bigg] (b^i y^j - b^j y^i) = 0.$$

This equation is written as the following form

$$[2\beta^2 (A^2 + \alpha^2 C^2 + 2\alpha AC) - 2(b^2\alpha^2 - \beta^2)\ \{(AF + \alpha^2 CG)$$

$$+ \alpha(AG + CF)\}]\ B^{ij} + [2\alpha^2 \beta^2\ \{(BA + \alpha^2 DC)$$

$$+ \alpha(BC + DA)\}\ -\ 2\alpha^2 (b^2\alpha^2 - \beta^2)\ \{(BF + \alpha^2 DG)$$

$$+ \alpha(BG + DF)\}]\ (s_0^i y^j - s_0^j y^i) \qquad \text{... (4.2)}$$

$$+ [\alpha^2 r_{00}\ \{(AF + \alpha^2 CG) + \alpha(AG + CF)\}$$

$$+ 2 s_0 \alpha^4\ \{(BF + \alpha^2 DG) + \alpha(BG + DF)\}]\ (b^i y^j - b_j y^i) = 0,$$

where A, B, C, D are defined in (3.4) and

$$F = \sum_{k=0}^{h} (2h-2k-1)(2h-2k)\, \alpha^{2k} \beta^{2h-2k} \qquad \text{... (4.3)}$$

and $$G = \sum_{k=0}^{h-1} (2h-2k-2)(2h-2k-1)\,\alpha^{2k}\beta^{2h-2k-1}.$$

Suppose that F^n is a Douglas space, that is, B^{ij} are $hp(3)$. Separating (4.2) in the rational and irrational terms of y^i, we have

$$\begin{aligned}
&2\ \{\beta^2 (A^2 + \alpha^2 C^2) - (b^2\alpha^2 - \beta^2)(AF + \alpha^2 CG)\}\ B^{ij} \\
&+ 2\alpha^2\{\beta^2 (BA + \alpha^2 DC) - (b^2\alpha^2 - \beta^2)(BF + \alpha^2 DG)\}\ (s^i_0 y^j - s^j_0 y^i) \\
&+ \alpha^2\ \{r_{00}(AF + \alpha^2 CG) + 2s_0\alpha^2 (BF + \alpha^2 DG)\}\ (b^i y^j - b_j y^i) \qquad \ldots (4.4)\\
&+ \alpha[2\ \{2\beta^2 AC - (b^2\alpha^2 - \beta^2)(AG + CF)\}\ B^{ij} \\
&+ 2\alpha^2\ \{\beta^2 (BC + DA) - (b^2\alpha^2 - \beta^2)(BG + DF)\}\ (s^i_0 y^j - s^j_0 y^i) \\
&+ \alpha^2\ \{r_{00}(AG + CF) + 2s_0\alpha^2 (BG + DF)\}\ (b^i y^j - b_j y^i)] = 0,
\end{aligned}$$

because α is irrational and A, B, C, D, E, F, G are rational polynomials of y^i. Hence, (4.4) is divided into two equations as follows :

$$\begin{aligned}
&2\ \{\beta^2 (A^2 + \alpha^2 C^2) - (b^2\alpha^2 - \beta^2)(AF + \alpha^2 CG)\}\ B^{ij} \\
&+ 2\alpha^2\ \{\beta^2 (BA + \alpha^2 DC) - (b^2\alpha^2 - \beta^2)(BF + \alpha^2 DG)\}\ (s^i_0 y^j - s^j_0 y^i) \qquad \ldots (4.5)\\
&+ \alpha^2\ \{r_{00}(AF + \alpha^2 CG) + 2s_0\alpha^2 (BF + \alpha^2 DG)\}\ (b^i y^j - b_j y^i) = 0,
\end{aligned}$$

$$\begin{aligned}
&2\ \{2\beta^2 AC - (b^2\alpha^2 - \beta^2)(AG + CF)\}\ B^{ij} \\
&+ 2\alpha^2\ \{\beta^2 (BC + DA) - (b^2\alpha^2 - \beta^2)(BG + DF)\}\ (s^i_0 y^j - s^j_0 y^i) \qquad \ldots (4.6)\\
&+ \alpha^2\ \{r_{00}(AG + CF) + 2s_0\alpha^2 (BG + DF)\}\ (b^i y^j - b_j y^i) = 0.
\end{aligned}$$

Eliminating B^{ij} from (4.5) and (4.6), we obtain

$$P\,(s^i_0 y^j - s^j_0 y^i) + Q\,(b^i y^j - b^j y^i) = 0, \qquad \ldots (4.7)$$

where
$$\begin{aligned}
P = {} & 2\left\{\beta^2 (BA + \alpha^2 DC) - (b^2\alpha^2 - \beta^2)(BF + \alpha^2 DG)\right\} \\
&\times\left\{2\beta^2 AC - (b^2\alpha^2 - \beta^2)(AG + CF)\right\} \\
&- 2\left\{\beta^2 (BC + DA) - (b^2\alpha^2 - \beta^2)(BG + DF)\right\} \\
&\times\left\{\beta^2 (A^2 + \alpha^2 C^2) - (b^2\alpha^2 - \beta^2)(AF + \alpha^2 CG)\right\}, \qquad \ldots (4.8)
\end{aligned}$$

$$Q=\Big\{ r_{00}\,(AF+\alpha^2\,CG)+2s_0\,\alpha^2\,(BF+\alpha^2\,DG)\Big\}$$
$$\times\Big\{2\,\beta^2 AC-(b^2\,\alpha^2-\beta^2)\,(AG+CF)\Big\}$$
$$-\Big\{ r_{00}\,(AG+CF)+2s_0\,\alpha^2\,(BG+DF)\Big\}$$
$$\times\Big\{\beta^2\,(A^2+\alpha^2\,C^2)-(b^2\,\alpha^2-\beta^2)\,(AF+\alpha^2\,CG)\Big\}.$$

Transvection of (4.7) by $b_i\,y_j$ leads to

$$Ps_0\,\alpha^2+Q\,(b^2\,\alpha^2-\beta^2)=0. \qquad \text{... (4.9)}$$

The terms of (4.9) which does not contain α^2 are found in $-Q\beta^2$, and from (4.8) it is found in

$$-\beta^4\,r_{00}\,A^2\,(CF-AG).$$

Therefore, from (3.4) and (4.3), the term of (4.9) which seemingly does not contain α^2 is $-(2h-1)^3\,(2h-2)\,r_{00}\,\beta^{8h+3}$. Hence, there exists hp $(8h+3)$: V_{8h+3}. such that

$$(2h-1)^3\,(2h-2)\,r_{00}\,\beta^{8h+3}=\alpha^2\,V_{8h+3}. \qquad \text{... (4.10)}$$

If $h=1$, then $L=\alpha+\beta+\beta^2/\alpha$. In this case, we have investigated it in [14] already. Therefore, we deal with this paper for $h>1$ and $n>2$. From Lemma, we assumed that $\alpha^2\not\equiv 0$ (mod. β).

It will be better to divide our consideration into two cases as follows :

$$(1^0)\;\; V_{8h+3}=0,\ (2^0)\; V_{8h+3}\neq 0.$$

The case of (1^0) leads to $r_{00}=0$, that is, $r_{ij}=0$. Hence (4.9) is reduced to

$$s_0\Big\{P\,\alpha^2+2Q_1\,(b^2\,\alpha^2-\beta^2)\Big\}=0, \qquad \text{... (4.11)}$$

where
$$Q_1=(BF+\alpha^2\,DG)\Big\{2\,\beta^2\,AC-(b^2\,\alpha^2-\beta^2)\,(AG+CF)\Big\}$$
$$-(BG+DF)\Big\{\beta^2\,(A^2+\alpha^2\,C^2)-(b^2\,\alpha^2-\beta^2)\,(AF+\alpha^2\,CG)\Big\}.$$

If $P\alpha^2+2Q_1\,(b^2\,\alpha^2-\beta^2)=0$. in (4.11), then the term of this equation which does not contain α^2 is $-(2h-1)\,(2h-2)\,\beta^{8h+3}$. Therefore, there exists $hp\,(8h+1):V_{8h+1}$ such that

$$-(2h-1)\,(2h-2)\,\beta^{8h+3}=\alpha^2\,V_{8h+1}.$$

From $\alpha^2 \not\equiv 0$, we have $V_{8h+2} = 0$, which leads a contradiction. Therefore, $P\alpha^2 + 2Q_1 (b^2\alpha^2 - \beta^2) \neq 0$. Thus $s_0 = 0$ from (4.11). Substituting $s_0 = 0$ and $r_{00} = 0$ into (4.7), we have

$$P(s^i_0 y^j - s^j_0 y^i) = 0 \qquad \text{... (4.12)}$$

If $P = 0$, then we have from (4.8)

$$\begin{aligned}&\left\{\beta^2(BA + \alpha^2 DC) - (b^2\alpha^2 - \beta^2)(BF + \alpha^2 DG)\right\}\\ &\times\left\{(2\beta^2 AC - (b^2\alpha^2 - \beta^2)(AG + CF)\right\}\\ &-\left\{\beta^2(BC + DA) - (b^2\alpha^2 - \beta^2)(BG + DF)\right\}\\ &\times\left\{\beta^2(A^2 + \alpha^2 C^2) - (b^2\alpha^2 - \beta^2)(AF + \alpha^2 CG)\right\} = 0.\end{aligned} \qquad \text{... (4.13)}$$

The term of (4.13) which does not contain α^2 is $-(2h-1)^2(2h+1)^2\beta^{8h+2}$. Thus there exists $hp(8h) : V_{8h}$ such that

$$-(2h-1)^2(2h+1)^2\beta^{8h+2} = \alpha^2 V_{8h},$$

from which we have $V_{8h} = 0$. Thus we have a contradiction. Therefore, $P \neq 0$, fromwhich $s^i_0 y^j - s^j_0 y^i = 0$ in (4.12). Transvection of this equation by y_j gives $s^i_0 = 0$. Finally $r_{ij} = s_{ij} = 0$ are concluded, that is, $b_{i,j} = 0$.

In the case of (2^0), (4.10) shows that there exists a function $f = f(x)$ satisfying

$$r_{00} = f\alpha^2. \qquad \text{... (4.14)}$$

Substituting (4.14) into (4.9) and using (4.8), we have

$$\begin{aligned}s_0 P + \Big[&\left\{f(AF + \alpha^2 CG) + 2s_0(BF + \alpha^2 DG)\right\}\\ &\times\left\{2\beta^2 AC - (b^2\alpha^2 - \beta^2)(AG + CF)\right\}\\ &-\left\{f(AG + CF) + 2s_0(BG + DF)\right\}\\ &\times\left\{\beta^2(A^2 + \alpha^2 C^2) - (b^2\alpha^2 - \beta^2)(AF + \alpha^2 CG)\right\}\Big](b^2\alpha^2 - \beta^2) = 0.\end{aligned} \qquad \text{... (4.15)}$$

The term of (4.15) which seemingly does not contain α^2 is included in the following terms

$$\{2(BC + BG - AD - DF)s_0 - f(CF - AG)\}\ A^2$$

and it is obtained as follows :

$$-2(2h-1)^2\beta^{8h-2}\left\{(4h^2-2h+1)s_0+f\beta(2h^2-4h+1)\right\}.$$

Therefore, there exists $hp(8h-3): V_{8h-3}$ such that

$$-2(2h-1)^2\beta^{8h-2}(c_1 s_0+c_2 f\beta)=\alpha^2 V_{8h-3},$$

where $c_1=4h^2-2h+12$, $c_2=2h^2-4h+1$. From $\alpha^2 \not\equiv 0$ (mod. β), it follows that V_{8h-3} must vanish and hence we have

$$c_1 s_0+c_2 f\beta=0, \qquad \text{... (4.16)}$$

where $c_1\neq 0, c_2\neq 0$ for a positive integer h. Therefore, we get $c_1 s_i+c_2 f b_i=0$. Transvection of this equation by b^i gives $c_2 f b^2=0$, which implies $fb^2=0$. In the case of $f=0$ we get $s_0=0$ from (4.16) and $r_{00}=0$ from (4.14), that is, $r_{ij}=0$.

On the other hand, in the case of $b^2=0$, (4.15) is reduced to

$$\{2(BC+BG-AD-DF)s_0-f(CF-AG)\}\ (A^2-\alpha^2C^2)=0. \qquad \text{... (4.17)}$$

If $A^2-\alpha^2C^2=0$ in (4.17), there exists hp $(4h-2)$: V_{4h-2} such that $(2h-1)^2\beta^{4h}=\alpha^2 V_{4h-2}$, which leads a contradiction. Therefore, we get $A^2-\alpha^2C^2\neq 0$, hence we have

$$2(BC+BG-AD-DF)s_0-f(CF-AG)=0. \qquad \text{... (4.18)}$$

From (4.18) we get

$$-2(2h-1)^2\beta^{4h-2}(c_1 s_0+c_2 f\beta)+\alpha^2\beta^{4h-4}$$

$$\{-4h(2h+1)s_0-f\beta(8h^2-20h+6)\}\ +\ (*)\ =\ 0, \qquad \text{... (4.19)}$$

where (*) is the term containing powers greater than 3 of α. From (4.16), (4.19) and $\alpha^2\not\equiv 0$ (mod. β), there exists $hp(4h-6): V_{4h-6}$ such that

$$\beta^{4h-4}(c_3 s_0+c_4 f\beta)=\alpha^2 V_{4h-6},$$

where $c_3=4h(2h+1)$, $c_4=2(4h^2-10h+3)$. It follows

$$c_3 s_0+c_4 f\beta=0. \qquad \text{... (4.20)}$$

Since $c_1c_4-c_2c_3=4(6h^4-17h^3+42h^2-64h+18)\neq 0$ for a positive integer h, from (4.16) and (4.20) we have $s_0=0$ and $f\beta=0$. The latter implies $f=0$ and $r_{00}=0$. Therefore, either case of $f = 0$ or $b^2 = 0$, we have $s_0 = 0$ and $r_{00} = 0$, that is, $r_{ij} = 0$. Hence, $Q = 0$ and $P \neq 0$, we have $s_{ij} = 0$. Thus $b_{ij} = 0$ is cncluded.

Conversely, if $b_{i;j} = 0$, then $B^{ij} = 0$ from (2.5), Hence F^n is a Douglas space.

(*ii*) *Case of* $r = 2h + 1$ (h is a positive integer)

We find the condition for the odd general approximate Matsumoto metric to be a Douglas space as same as the case of $r = 2h$.

Eq. (4.1) is written as the following form

$$2\beta \sum_{k=0}^{2h+1} (2h-k)\,\alpha^k \beta^{2h-k+1} \left[\beta^2 \sum_{k=0}^{2h+1} (2h-k)\,\alpha^k \beta^{2h-k+1} \right.$$

$$\left. - (b^2\alpha^2 - \beta^2) \sum_{k=0}^{2h+1} (2h-k)(2h-k+1)\,\alpha^k \beta^{2h-k+1} \right] B^{ij}$$

$$+ 2\alpha^2 \sum_{k=0}^{2h+1} (2h-k+1)\,\alpha^k \beta^{2h-k+1} \left[\beta^2 \sum_{k=0}^{2h+1} (2h-k)\,\alpha^k \beta^{2h-k+1} \right.$$

$$\left. - (b^2\alpha^2 - \beta^2) \sum_{k=0}^{2h+1} (2h-k)(2h-k+1)\,\alpha^k \beta^{2h-k+1} \right] (s^i_0 y^j - s^j_0 y^i)$$

$$+ \alpha \sum_{k=0}^{2h+1} (2h-k)(2h-k+1)\,\alpha^k \beta^{2h-k+1} \left[\alpha\beta r_{00} \sum_{k=0}^{2h+1} (2h-k)\,\alpha^k \beta^{2h-k+1} \right.$$

$$\left. + 2s_0\alpha^3 \sum_{k=0}^{2h+1} (2h-k+1)\,\alpha^k \beta^{2h-k+1} \right] (b^i y^j - b^j y^i) = 0,$$

which implies the following form

$$2\beta\,[\beta^2 (\beta^4 B^2 + \alpha^2 A^2 + 2\alpha\beta^2 AB)$$

$$\left. - (b^2\alpha^2 - \beta^2) \left\{ \beta^2 BK + \alpha^2 AF + \alpha(\beta^2 BF + AK) \right\} \right] B^{ij}$$

$$+ 2\alpha^2 \Big[\beta^2 \left\{ \beta^2 BJ + \alpha^2 \beta AB + \alpha(\beta JA + \beta^3 B^2) \right\}$$

$$\left. - (b^2\alpha^2 - \beta^2) \left\{ JK + \alpha^2 \beta BF + \alpha(\beta BK + JF) \right\} \right] (s^i_0 y^j - s^j_0 y^i)$$

$$+ \alpha \Big[\alpha\beta r_{00} \left\{ \beta^2 BK + \alpha^2 AF + \alpha(\beta^2 BF + AK) \right\}$$

$$+ \; 2s_0\, \alpha^3 \left\{ JK + \alpha^2 \beta BF + \alpha(\beta BK + JF) \right\} \Big] (b^i y^j - b^j y^i) = 0,$$

where A, B, C and F are defined in (3.4), (4.3) and

$$K = \sum_{k=0}^{h} (2h-2k)(2h-2k+1)\, \alpha^{2k} \beta^{2h-2k+1},$$

and

$$J = \sum_{k=0}^{h} (2h-2k+1)\, \alpha^{2k} \beta^{2h-2k+1}.$$

Suppose that F^n is a Douglas space, that is, B^{ij} are $hp(3)$. Separating the above equation in the rational and irrational terms of y^i, we have

$$\begin{aligned} & 2\beta \left\{ \beta^2 (\beta^4 B^2 + \alpha^2 A^2) - (b^2 \alpha^2 - \beta^2)(\beta^2 BK + \alpha^2 AF) \right\} B^{ij} \\ & + 2\alpha^2 \left\{ \beta^3 (\beta BJ + \alpha^2 AB) - (b^2 \alpha^2 - \beta^2)(JK + \alpha^2 \beta BF) \right\} (s^i_0 y^j - s^j_0 y^i) \\ & + \alpha^2 \left\{ \beta r_{00} (\beta^2 BK + \alpha^2 AF) + 2s_0\, \alpha^2 (JK + \alpha^2 \beta BF) \right\} (b^i y^j - b^j y^i) \\ & + \alpha \Big[2\beta \left\{ 2\beta^4 AB - (b^2 \alpha^2 - \beta^2)(\beta^2 BF + AK) \right\} B^{ij} \\ & + 2\alpha^2 \left\{ \beta^2 (AJ + \beta^2 B^2) - (b^2 \alpha^2 - \beta^2)(\beta BK + JF) \right\} (s^i_0 y^j - s^j_0 y^i) \\ & + \alpha^2 \left\{ \beta r_{00} (\beta^2 BF + AK) + 2s_0\, \alpha^2 (\beta BK + JF) \right\} (b^i y^j - b^j y^i) \Big] = 0. \end{aligned} \qquad \text{... (4.21)}$$

Since α is irrational and A, B, C, F, J, K are rational polynomials of y^i, (4.21) is divided into two equations as follows :

$$\begin{aligned} & 2\beta \left\{ \beta^2 (\beta^4 B^2 + \alpha^2 A^2) - (b^2 \alpha^2 - \beta^2)(\beta^2 BK + \alpha^2 AF) \right\} B^{ij} \\ & + 2\alpha^2 \left\{ \beta^3 (\beta BJ + \alpha^2 AB) - (b^2 \alpha^2 - \beta^2)(JK + \alpha^2 \beta BF) \right\} (s^i_0 y^j - s^j_0 y^i) \\ & + \alpha^2 \left\{ \beta r_{00} (\beta^2 BK + \alpha^2 AF) + 2s_0\, \alpha^2 (JK + \alpha^2 \beta BF) \right\} (b^i y^j - b^j y^i) = 0, \end{aligned} \qquad \text{... (4.22)}$$

$$\begin{aligned} & 2\beta \left\{ 4\beta^4 AB - (b^2 \alpha^2 - \beta^2)(\beta^2 BF + AK) \right\} B^{ij} \\ & + 2\alpha^2 \left\{ \beta^3 (JB + \beta^2 B^2) - (b^2 \alpha^2 - \beta^2)(\beta BK + JF) \right\} (s^i_0 y^j - s^j_0 y^i) \\ & + \alpha^2 \left\{ \beta r_{00} (\beta^2 BF + AK) + 2s_0\, \alpha^2 (\beta BK + JF) \right\} (b^i y^j - b^j y^i) = 0. \end{aligned} \qquad \text{... (4.23)}$$

Eliminating B^{ij} from (4.22) and (4.23), we obtain

$$P' (s^i_0 y^j - s^j_0 y^i) + Q' (b^i y^j - b^j y^i) = 0, \qquad \dots (4.24)$$

where
$$P' = 2 \left\{ \beta^3 (\beta BJ + \alpha^2 AB) - (b^2 \alpha^2 - \beta^2) (JK + \alpha^2 \beta BF) \right\}$$
$$\times \left\{ 2 \beta^4 AB - (b^2 \alpha^2 - \beta^2) (\beta^2 BF + AK) \right\}$$
$$- 2 \left\{ \beta^2 (JA + \beta^3 B^2) - (b^2 \alpha^2 - \beta^2) (\beta BK + JF) \right\}$$
$$\times \left\{ \beta^2 (\beta^4 B^2 + \alpha^2 A^2) - (b^2 \alpha^2 - \beta^2) (\beta^2 BK + \alpha^2 AF) \right\}, \qquad \dots (4.25)$$

$$Q' = \left\{ \beta r_{00} (\beta^2 BK + \alpha^2 AF) + 2s_0 \alpha^2 (JK + \alpha^2 \beta BF) \right\}$$
$$\times \left\{ 2 \beta^4 AB - (b^2 \alpha^2 - \beta^2) (\beta^2 BF + AK) \right\}$$
$$- \left\{ \beta r_{00} (\beta^2 BF + AK) + 2s_0 \alpha^2 (\beta BK + JF) \right\}$$
$$\times \left\{ \beta^2 (\beta^4 B^2 + \alpha^2 A^2) - (b^2 \alpha^2 - \beta^2) (\beta^2 BK + \alpha^2 AF) \right\}.$$

Transvection of (4.24) by $b_i y_j$ leads to

$$P' s_0 \alpha^2 + Q' (b^2 \alpha^2 - \beta^2) = 0. \qquad \dots (4.26)$$

The terms of (4.26) which does not contain α^2 are found in $- Q' \beta^2$, and that is expressed as follows:

$$- \beta^7 r_{00} B^2 (AK - \beta^2 FB). \qquad \dots (4.27)$$

Therefore, from (4.24), the term of (4.26) which seemingly does not contain α^2 is $- 8 \beta^{8h+8} r_{00} h^3 (2h - 1)$. Hence, we have $hp (8h + 8) : V_{8h+8}$ such that

$$- 8h^3 (2h - 1) r_{00} \beta^{8h+8} = \alpha^2 V_{8h+8}. \qquad \dots (4.28)$$

We assumed that $\alpha^2 \not\equiv 0$ (mod. β). It will be better to divide our consideration into two cases as follows :

(1′) $V_{8h+8} = 0$ (2′) $V_{8h+8} \neq 0$.

The case of (1′) leads to $r_{00} = 0$ and (4.26) is reduced to

$$s_0 \left\{ P' \alpha^2 + Q'_1 (b^2 \alpha^2 - \beta^2) \right\} = 0, \qquad \dots (4.29)$$

where
$$Q'_1 = 2 (JK + \alpha^2 \beta BF) \left\{ 2 \beta^4 AB - (b^2 \alpha^2 - \beta^2) (\beta^2 BF + AK) \right\}$$
$$- 2 (\beta BK + JF) \left\{ \beta^2 (\beta^4 B^2 + \alpha^2 A^2) - (b^2 \alpha^2 - \beta^2) (\beta^2 BK - \alpha^2 AF) \right\}. \qquad \dots (4.30)$$

If $P' + Q'_1 (b^2 \alpha^2 - \beta^2) = 0$ in (4.29), then the term of this equation which does not contain α^2 is included in $2\beta^8 B^2 (AJ - \beta^3 B^2 - \beta BK + FJ)$. Therefore, the term of (4.29) which does not contain α^2 is $8(4h+1)\cdot\beta^{8h+7}$ Hence there exists $hp(8h+5) : V'_{8h+5}$ such that

$$8(4h+1)\beta^{8h+7} = \alpha^2 V'_{8h+5}.$$

Since $\alpha^2 \not\equiv 0 \pmod{\beta}$, we have $V'_{8h+5} = 0$, which leads a contradiction. Therefore, $P'\alpha^2 + 2Q'_1(b^2\alpha^2 - \beta^2) \neq 0$. Thus $s_0 = 0$ from (4.29). Substituting $s_0 = 0$ and $r_{00} = 0$ in (4.25), we have

$$P'(s'_0 y^j - s^j_0 y^i) = 0$$

by virtue of (4.30). If $P' = 0$, then we have from (4.25)

$$\begin{aligned} &\left\{ \beta^3 (\beta BJ + \alpha^2 BF) - (b^2\alpha^2 - \beta^2)(JK + \alpha^2 \beta BF) \right\} \\ &\times \left\{ 2\beta^4 AB - (b^2\alpha^2 - \beta^2)(\beta^2 BF + AK) \right\} \\ &- \left\{ \beta^3 (KB + \beta^2 B^2) - (b^2\alpha^2 - \beta^2)(\beta BK + JF) \right\} \\ &\times \left\{ \beta^2 (\beta^4 B^2 + \alpha^2 A^2) - (b^2\alpha^2 - \beta^2)(\beta^2 BK + \alpha^2 AF) \right\} = 0. \end{aligned} \qquad \ldots (4.31)$$

The terms of (4.31) which does not contain α^2 are included in the following terms

$$\beta^4 (2\beta^4 AB^2 J + 3\beta^2 ABJK + AK^2 J - 2\beta^5 B^3 K - \beta^7 B^4 - 2\beta^4 B^2 K^2).$$

Therefore, the term of (3.31) which does not contain α^2 is

$$4h(-16h^5 + 8h^4 + 3h^3 - 16h^2 - 7h - 1)\beta^{8h+7}.$$

Hence, there exists $hp(8h+5) : V'_{8h+5}$ such that

$$4h(23h^5 + 8h^4 - 16h^3 + 6h^2 - 3h + 1)\beta^{8h+7} = \alpha^2 V'_{84+5}.$$

Since $23h^5 + 8h^4 - 16h^3 + 6h^2 - 3h + 1 \neq 0$ for positive h and $\alpha^2 \not\equiv 0 \pmod{\beta}$, we have a contradiction. Therefore, we get $P' \neq 0$ and hence $s^i_0 y^j - s^j_0 y^i = 0$. Transvection of this equation by y_j gives $s^i_0 = 0$. Finally, $r_{ij} = s_{ij} = 0$ are concluded, that is, $b_{i;j} = 0$.

In this case of (2'), that is, $V'_{8h+8} \neq 0$ and $\alpha^2 \not\equiv 0$ (mod. β), (4.28) shows that there exists a function $f' = f'(x)$ satisfying

$$r_{00} = f' \alpha^2. \qquad \dots (4.32)$$

Substituting (4.32) into (4.26), we have

$$\begin{aligned} s_0 P' + \Big[& \left\{ f' \beta (\beta^2 BK + \alpha^2 AF) + 2s_0 (JK + \alpha^2 \beta BF) \right\} \\ & \times \left\{ 2\beta^4 AB - (b^2 \alpha^2 - \beta^2)(\beta^2 BF + AK) \right\} \\ & - \left\{ f' \beta (\beta^2 BF + AK) + 2s_0 (\beta BK + JF) \right\} \\ & \times \left\{ \beta^2 (\beta^4 B^2 + \alpha^2 A^2) - (b^2 \alpha^2 - \beta^2)(\beta^2 BK + \alpha^2 AF) \right\} \Big] \\ & \times (b^2 \alpha^2 - \beta^2) = 0. \end{aligned} \qquad \dots (4.33)$$

Therefore, the term of (4.33) which seemingly does not contain α^2 is

$$-\beta^{8h+7} \left\{ 8h^2 (4h^2 + 2h + 1) s_0 + 8h^3 (2h - 1) f' \beta \right\}.$$

Thus there exists $hp\,(8h+6) : V'_{8h+6}$ such that

$$\beta^{8h+7} (c'_1 s_0 + c'_2 f' \beta) = \alpha^2 V'_{8h+6},$$

where $c'_1 = 8h^2 (4h^2 + 2h + 1), c'_2 = 8h^3 (2h - 1)$ and $c'_1 \neq 0, c'_2 \neq 0$ for a positive integer h. Since $\alpha^2 \not\equiv 0$ (mod β), we have $V'_{8h+6} = 0$ and hence

$$c'_1 s_0 + c'_2 f' \beta = 0. \qquad \dots (4.34)$$

Therefore, we get $c'_1 s_i + c'_2 f' b_i = 0$. Transvection of this equation by b^i gives $c'_2 f' b^2 = 0$, which implies $f' b^2 = 0$. If $f' = 0$, then $s_0 = 0$ from (4.34) and $r_{00} = 0$ from (4.32).

On the other hand, we consider $b^2 = 0$. Substituting $b^2 = 0$ in (4.33), we get

$$\begin{aligned} s_0 P'_1 + & \left\{ f' \beta (\beta^2 BK + \alpha^2 AF) + 2s_0 (JK + \alpha^2 \beta BF) \right\} \\ & \times (2\beta^2 AB + \beta^2 BF + AK) \\ & + \left\{ f' \beta (\beta^2 BF + AK) + 2s_0 (\beta BK + JF) \right\} \end{aligned}$$

$$\times(\beta^4 B^2+\alpha^2 A^2+\beta^2 BK+\alpha^2 AF)=0, \qquad \ldots (4.35)$$

where $$P_1'=2(\beta^2 BJ+\alpha^2\beta AB+JK+\alpha^2\beta BF)(2\beta^2 AB+\beta^2 BF+AK)$$

$$-2(JA+\beta^3 B^2+\beta BK+JF)(\beta^4 B^2+\alpha^2 A^2+\beta^2 BK+\alpha^2 AF).$$

The eq. (4.35) is rewritten as the following form

$$2s_0\left\{\beta^2 BJ(2\beta^2 AB+\beta^2 BF+AK)-(JA+\beta^3 B^2)\beta^2(\beta^2 B^2+BK)\right\}$$

$$-f'\beta\left\{\beta^2 BK(2\beta^2 AB+\beta^2 BF+AK)-\beta^2(\beta^2 BK+AK)(\beta^2 B^2+BK)\right\}$$

$$+\alpha^2\Big[2s_0\left\{\beta AB(2\beta^2 AB+\beta^2 BF+AK)-(JA+\beta^3 B^2)(A^2+AF)\right\}$$

$$-f'\beta\left\{AF(2\beta^2 AB+\beta^2 BF+AK)-(\beta^2 BF+AK)(A^2+AF)\right\}\Big]=0.$$

From (4.34), we have $hp(8h): V_{8h}'$ such that

$$\beta^{8h+1}(c_3' s_0+c_4' f'\beta)=\alpha^2 V_{8h}',$$

where $$c_3'=2(256h^7+320)h^6-(96h^5-324h^4+144h^3+32h^3-10)h-1)$$

and $$c_4'=4(8h^5-4)h^4-76h^3+(9h^2-h).$$

Since $\alpha^2\not\equiv 0$ (mod. β), we get

$$c_3' s_0+c_4' f'\beta=0. \qquad \ldots (4.36)$$

From (4.34) and (4.36) we have $f'=0$ by virtue of $c_1' c_4'-c_2' c_3'\neq 0$ for a positive integer h.

Consequently, we obtain $r_{00}=0$ and $s_0=0$. Thus either case of $f'=0$ or $b^2=0$, we obtain $r_{00}=0$ and $s_0=0$. Substituting these equations into (4.24), we have $s_0^i=0$ from $P'\neq 0$. Thus, $r_{ij}=s_{ij}=0$ are concluded, that is, $b_{i;j}=0$.

Conversely, if $b_{i;j}=0$, then $B^{ij}=0$ from (2.5). Hence F^n is a Douglas space.

Summarising up, we have the following

Theorem 4.1 — *An n-dimensional Finsler space F^n $(n>2)$ with the general approximate Matsumoto metric (2.2) is a Douglas space if and only if $b_{i;j}=0$.*

From Theorem 3.1 and Theorem 4.1, we have

Theorem 4.2 — *If an n-dimensional Finsler space F^n $(n>2)$ with the general approximate Matsumoto metric (2.2) is a Douglas space, then it is a Berwald space.*

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