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LRS BIANCHI TYPE II STRING DUST COSMOLOGICAL MODEL IN GENERAL RELATIVITY

RAJ BALI*, R. D. UPADHAYA**, K. N. SINGH***

**Professor in Mathematics, University of Rajasthan, Jaipur* 302 004, *India*
***Department of Mathematics, Government College, Ajmer* 305 001, *India*
****Associate Professor in Mathematics, University of Rajasthan, Jaipur* 302 004, *India*

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An LRS (Locally Rotationally Symmetric) Bianchi Type II Cosmological model filled with string dust as the source for gravitational field, is investigated. For the complete determination of the model, we assume that rest energy density (ε) is equal to the string tension density (λ) (geometrical string). The various physical and geometrical aspects of the models are also discussed.

Key Words : Bianchi Type II String Dust; Cosmology; General Relativity; Locally Rationally Stymmetric (LRS)

1. INTRODUCTION

The large scale distribution of galaxies in our universe shows that the material distribution can be satisfactorily described by a perfect fluid. It is, however, conjectured that universe might have experienced a number of phase transitions after the big bang explosion.[1]. These phase transitions produces vacuum domain walls, strings and monopoles. Among these, cosmic strings play a significant interest as these act as gravitational lenses which gives rise to density perturbation leading to the formation of galaxies.[2] The general relativistic formalism of cosmic strings is due to Letelier[3&4]. Stachel[5] has developed massless strings. Krori *et al.*[6] have studied Bianchi Type II space time filled with string dust and have obtained a particular solution of the field equations. Shri Ram and Singh[7] have investigated Bianchy Type II string dust universes. These models represent expanding, spatially homogeneous and anisotropic universes. Recently, Roy and Banerjee[8] have investigated some LRS Bianchi II string cosmological models in General Relativity. They have obtained cosmological solutions for a cloud of geometrical and massive strings. To get a determinate solutions, they have assumed equation of state ($\varepsilon = \lambda$) for geometrical (Nambu) string and $\frac{\sigma}{\theta}$ = constant for massive string.

In this paper, we have investigated an LRS Bianchi Type II cosmological model filled with string dust as the source for gravitational field. To get a determinate solution, we have assumed an equation of state $\varepsilon = \lambda$. These solutions are general solutions and Shri Ram and Singh's a solution[7] is particular solution of our solution. The physical and geometrical aspects of the models are also discussed.

We consider an LRS Bianchi Type II metric in the form

$$ds^2 = dt^2 - S^2 (dx + z dy)^2 - R^2 (dy^2 + dz^2), \qquad \text{... (1.1)}$$

where R, S are functions of t alone. The energy momentum tensor for string dust is taken as

$$T_i^j = \varepsilon v_i v^j - \lambda x_i x^j \qquad \text{... (1.2)}$$

with $$v^i v_i = -x^i x_i = -1 \qquad \text{... (1.3)}$$

and $$v^i x_i = 0, \qquad \text{... (1.4)}$$

where $\varepsilon = \varepsilon_p + \lambda$ is the rest energy density for a cloud of strings with particles attached to them ε_p, the density of particles, λ is the cloud strings tension density, v^i the four velocity vector and x^i is the direction of strings.

The Einstein field equation

$$R_i^j - \frac{1}{2} R g_i^j = -T_i^j \left[\text{using the unit in which } \frac{8\pi G}{c^4} = 1 \right] \qquad \text{... (1.5)}$$

for the metric (1.1) leads to

$$2\frac{\dot{R}\dot{S}}{RS} + \frac{\dot{R}^2}{R^2} - \frac{1}{4}\frac{S^2}{R^4} = \varepsilon, \qquad \text{... (1.6)}$$

$$2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} - \frac{3}{4}\frac{S^2}{R^4} = \lambda, \qquad \text{... (1.7)}$$

$$\frac{\ddot{S}}{S} + \frac{\ddot{R}}{R} + \frac{\dot{R}\dot{S}}{RS} + \frac{1}{4}\frac{S^2}{R^4} = 0 \qquad \text{... (1.8)}$$

and $$\frac{\dot{\varepsilon}}{\varepsilon} + 2\frac{\dot{R}}{R} = 0. \qquad \text{... (1.9)}$$

Eqs. (1.6) to (1.9) are four equations in four unknowns A, B, ε and λ. To get a determinate solution, we assume

$$\varepsilon = \lambda \qquad \text{... (1.10)}$$

Subtracting (1.7) from (1.6) and using the condition given by (1.10), we have

$$\frac{\dot{R}\dot{S}}{RS} - \frac{\ddot{R}}{R} + \frac{1}{4}\frac{S^2}{R^4} = 0. \qquad \text{... (1.11)}$$

From eqs. (1.11) and (1.8), we have

$$2\frac{\ddot{R}}{R} + \frac{\ddot{S}}{S} = 0 \qquad \text{... (1.12)}$$

Eq. (1.9) on integration lead to

$$\varepsilon = \frac{\text{constant}}{R^2} = \frac{K}{R^2}. \qquad \text{... (1.13)}$$

To obtain exact solutions of (1.11) and (1.12), we make use of the scale transformation

$$dT = S dt \qquad \text{... (1.14)}$$

Using the transformation given by (1.14), eqs. (1.11), (1.12) and (1.9) reduce to

$$\frac{R''}{R} - \frac{1}{4R^4} = 0 \qquad \text{... (1.15)}$$

$$\frac{S''}{S} + \frac{S'^2}{S^2} + \frac{2R''}{R} + 2\frac{R'S'}{RS} = 0 \qquad \text{... (1.16)}$$

and

$$\varepsilon' + 2\varepsilon\frac{R'}{R} = 0, \qquad \text{... (1.17)}$$

where a dash denotes differentiation with respect to T.

Eq. (1.15) lead to

$$\frac{d}{dT}(R'^2) = -\frac{1}{4}\frac{d}{dT}\left(\frac{1}{R^2}\right) \qquad \text{... (1.18)}$$

which on integration leads to

$$R' = \sqrt{a - \frac{1}{4R^2}}, \qquad \text{... (1.19)}$$

where a is the constant of integration.

Eq. (1.19) further leads to

$$\frac{2RdR}{2\sqrt{a}\sqrt{R^2 - \left(\frac{1}{2\sqrt{a}}\right)^2}} = dT. \qquad \text{... (1.20)}$$

Using

$$R^2 - \left(\frac{1}{2\sqrt{a}}\right)^2 = \xi^2, \qquad \text{... (1.21)}$$

eq. (1.20) leads to

$$\frac{\xi\, d\xi}{\sqrt{a}\,\xi} = dT \qquad \text{... (1.22)}$$

which on integration leads to

$$\xi = \sqrt{a}\,T + \eta\ . \qquad \text{.. (1.23)}$$

where η is the constant of integraiton.

Using (1.21), eq. (1.23) gives

$$R^2 = aT^2 + \gamma T + b \qquad \text{... (1.24)}$$

where $$\gamma = 2\sqrt{a}\,\eta \qquad \text{... (1.25)}$$

$$b = \left[\eta^2 + \left(\frac{1}{2\sqrt{a}} \right)^2 \right] \qquad \text{... (1.26)}$$

Putting $r = R^2, s = S^2$ in eqs. (1.15) and (1.16), we have

$$2\,rr'' - r'^2 - 1 = 0 \qquad \text{... (1.27)}$$

and $$\frac{s''}{s} + \frac{r's'}{rs} + \frac{1}{r^2} = 0. \qquad \text{... (1.28)}$$

Using the transformation

$$\gamma = R^2 = aT^2 + \gamma T + b \qquad \text{... (1.29)}$$

in (1.27), we get

$$4ab - \gamma^2 = 1 \qquad \text{... (1.30)}$$

Now using (1.29) in eq. (1.28), we get

$$(aT^2 + \gamma T + b)\, s'' + (2aT + \gamma)\, s' = -\frac{s}{aT^2 + \gamma T + b} \qquad \text{... (1.31)}$$

Using the transformation

$$(aT^2 + \gamma T + b)\,\frac{ds}{dT} = \frac{ds}{d\psi} \qquad \text{... (1.32)}$$

in (1.31), we get

$$\frac{d^2 s}{d\psi^2} = -s \qquad \text{... (1.33)}$$

which leads to

$$\sin^{-1}\frac{s}{l} = \psi - 2\tan^{-1} m, \qquad \text{... (1.34)}$$

where l, m are arbitrary constants and $m > 0$ and

$\psi = 2\tan^{-1}(2aT + \gamma)$ is obtained from the transformation (1.32).

Eq. (1.34) leads to

$$s = S^2 = \frac{8a^2\, l\, m\, T^2 + 4aT\,(l + 2\, rlm - lm^2) + (\gamma - m)\,(1 + \gamma m)\, 2l}{(1 + m^2)\,[4a^2\, T^2 + 4a\, \gamma T + \gamma^2 + 1]} \qquad \text{... (1.35)}$$

where l, m are arbitrary constants.

Hence, the metric (1.1) reduces to the form

$$ds^2 = \frac{(1 + m^2)\,[4a^2\, T^2 + 4a\, \gamma T + \gamma^2 + 1]}{8a^2\, lmT^2 + 4aT\,(l + 2\gamma l\, m - lm^2) + (\gamma - m)\,(1 + \gamma m\,(2l)}\, dT^2$$

$$- \frac{8a^2\, lmT^2 + 4a\,(l + 2\,\gamma lm - lm^2)\, T + (\gamma - m)\,(1 + \gamma m)\, sl}{(1 + m^2)\,(4a^2\, T^2 + 4a\, \gamma T + \gamma^2 + 1)}$$

$$(dx + z\, dy)^2 - (aT^2 + \gamma T + b)\,(dy^2 + dz^2), \qquad \text{... (1.3.5)}$$

when $\gamma = 0$ then we get the metric

$$ds^2 = \frac{(1 + m^2)\,(4a^2\, T^2 + 1)}{8a^2\, lm\, T^2 + 4al\,(1 - m^2)\, T - 2lm}\, dT^2.$$

$$- \frac{8a^2\, lmT^2 + 4al\,(1 - m^2)\, T - 2lm}{(1 + m^2)\,(4a^2\, T^2 + 1)}$$

$$(dx + zdy)^2 - (aT^2 + b)\,(dy^2 + dz^2), \qquad \text{... (1.37)}$$

which is the same metric as obtained by Shri Ram and Singh (1992).

2. Some Physical and Geometrical Features

The energy density (ε), scalar of expansion (θ), shear (σ) and the spatial volume (V) for the metric (1.36) are given by

$$\varepsilon = \frac{2lma}{(1 + m^2)\,(aT^2 + \gamma T + b)} \qquad \text{... (2.1)}$$

$$\theta = \left[\frac{8a^2 lmT^2 + 4a(l + 2\gamma lm - lm^2)\, T + (\gamma - m)\,(1 + \gamma m) 2l}{(1 + m^2)\,(4a^2 T^2 + 4a\gamma T + \gamma^2 + 1)}\right]^{1/2}$$

$$\left[\frac{2aT + \gamma}{aT^2 + \gamma T + b} + \frac{4a^2\, mT + a\,(1 + 2\,\gamma m - m^2)}{4a_m^2\, T^2 + 2aT\,(1 + 2\,\gamma m - m^2) + (\gamma - m)\,(1 + \gamma m)}\right.$$

$$\left. - \frac{4aT^2 + 2a\,\gamma}{4a^2\, T^2 + 4a\, \gamma T + \gamma^2 + 1}\right] \qquad \text{... (2.2)}$$

$$\sigma = \frac{1}{\sqrt{3}}\left[\frac{8a^2lmT^2 + 4aT(l + 2\gamma lm - lm^2) + (\gamma - m)(1 + \gamma m)\,2l}{(1 + m^2)(4a^2T^2 + 4a\gamma T + \gamma^2 + 1}\right.$$

$$\left. + \frac{4a^2 T + 2a\gamma}{4a^2 T + 4a\gamma T^2 + \gamma^2 + 1}\right]^{1/2} \qquad \ldots (2.3)$$

$$V = R^2 S = (aT^2 + \gamma T + b)$$

$$\left[\frac{8a^2\, lm\, T^2 + 4aT(l + 2\gamma lm - lm^2) + (\gamma - m)(1 + \gamma m)\,2l}{(1 + m^2)(4a^2T^2 + 4a\gamma T + \gamma^2 + 1)}\right]^{1/2} \qquad .. (2.4)$$

3. DISCUSSION

The reality condition $\varepsilon > 0$ is satisfied when $l > 0$. There is a big bang in the model at $T = 0$. The scalar of expansion (θ) is monotonically decreasing for $T > 0$ and $\theta \to 0$ when $T \to \infty$. The motion is irrotational but there is shear. Shear is non-zero for $0 \le T < \infty$. Since $\lim_{T \to \infty} \frac{\sigma}{\theta} \neq 0$. hence, the model does not approach isotropy for large-values of T. Then density $\varepsilon \to 0$ when $T \to \infty$. However, if $T = 0$ then $\varepsilon = \frac{2lma}{b(1 + m^2)}$. The spatial volume V is finite at $T = 0$ and it becomes infinite when $T \to \infty$ and $\varepsilon \to 0$ when $T \to \infty$. Thus the model is essentially empty universe at $T \to \infty$ when $\gamma = 0$ then we get the same result as obtained by Shri Ramand Singh[7].

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