Indian J. pure appl. Math., 34(1) : 85-99, January 2003
© Printed in India.

TWIN POSITIVE SOLUTIONS TO SINGULAR BOUNDARY VALUE PROBLEMS OF SECOND ORDER DIFFERENTIAL SYSTEMS*

XIAOJIE XU AND DAQING JIANG

Department of Mathematics, Northeast Normal University, Changchun 130024, *P. R. China*

(*Received* 11 *January* 2002; *accepted* 21 *June* 2002)

In this paper, we establish the existence of two positive solutions to singular boundary value problem

$$\begin{cases} x''(t) + q_1(t) f_1(x(t), y(t)) = 0 \\ y'' + q_2(t) f_2(x(t), y(t)) = 0, \quad 0 < t < 1 \end{cases}$$

with $x(0) = x(1) = y(0) = y(1) = 0$ by using the Lerary-Shauder alternative and the fixed point theorems in cones, where $q_i(t)$ may be singular at $t = 0$ or $t = 1$, nonlinearity f_i may be singular at (0, 0), $i = 1, 2$.

Key Words : Twin Positive Solutions; Fixed Point Theorem; Two-point Boundary Problem; Singular Problem

1. INTRODUCTION

In the paper, we study the existence of two positive solutions to the singular second order boundary value problem

$$\begin{cases} x''(t) + q_1(t) f_1(x(t), y(t)) = 0 \\ y'' + q_2(t) f_2(x(t), y(t)) = 0, \qquad 0 < t < 1 \\ x(0) = x(1) = y(0) = y(1) = 0, \end{cases} \qquad \text{... (1.1)}$$

where $q_i(t)$ may be singular at $t = 0$ or $t = 1$, nonlinearity f_i may be singular at (0, 0), $i = 1, 2$.

Recently, the singular boundary value problems have been studied extensively. For details, see, for instance, papers[1, 4] and the references therein. However, there are only few works on singular boundary value problems for differential systems. As far as the author knows, the works on the existence of twin positive solutions to the singular boundary value problems for differential systems are quite rarely seen.

R. P. Agarwal and D. O'Regan[2] considered the singular boundary value problem

*The work was supported by NNSF of China. E-mail address: daqingjiang@vip.163.com

$$\begin{cases} y''(t)+q(t)\,[g(y(t))+h(y(t))]=0, & 0<t<1, \\ y(0)=y(1)=0 \end{cases}$$

where $q(t)$ may be singular at $t = 0$ or $t = 1$, nonlinarity g may be singular at $y = 0$, h may be superlinear at $y=\infty$. They showed that this problem has twin positive solutions by using a Lerary-Shauder alternative and a fixed point theorem in cones.

The authors[3, 5 & 4] used the Krasnoselski's fixed point theorem in cones to establish the existence of two solutions to singular boundary value problems. However, some strong integrability conditions have to be assumed on g and $g + h$.

Motivated by the results mentioned above, the purpose of this paper is to establish the existence of twin positive solutions of problem (1.1) by applying the method as used in [2].

Throughout this paper, we make the following hypotheses —

(H_1) $q_i \in C((0,1),(0,\infty))$,

and $$\int_0^1 t(1-t)\,q_i(t)\,dt<\infty, \qquad \text{... (1.2)}$$

$$\lim_{t\to 0+} t^2(1-t)\,q_i(t)=0, \text{ if } \int_0^1 (1-t)\,q_i(t)\,dt=\infty,$$

$$\lim_{t\to 1^-} t(1-t)^2\,q_i(t)=0 \text{ if } \int_0^1 tq_i(t)\,dt=\infty,\ i = 1,\ 2. \qquad \text{... (1.3)}$$

(H_2) Let $f_i(x,y)=g_i(x,y)+h_i(x,y)$ on $[0,\infty)^2\ \{O\}$, where $O = (0, 0)$,

with $g_i>0$ continuous and nonincreasing on $[0,\infty)^2\ \{O\}$, ... (1.4)

$h_i \geq 0$ continuous on $[0,\infty)^2$,

and $\frac{h_i}{g_i}$ nondecreasing on $[0,\infty)^2\ \{O\}$. $i = 1,\ 2$. ... (1.5)

(H_3) there exists a constant $r > 0$ such that

$$\begin{cases} \displaystyle\int_0^r \frac{du}{g_1(u,0)} > \left\{1+\frac{h_1(r,r)}{g_1(r,r)}\right\} b_0, \\ \displaystyle\int_0^r \frac{d\upsilon}{g_2(0,\upsilon)} > \left\{1+\frac{h_2(r,r)}{g_2(r,r)}\right\} b_0, \end{cases} \qquad \text{... (1.6)}$$

where
$$b_0 = \max\left\{ 2\int_0^{\frac{1}{2}} t(1-t)\, q_i(t)\, dt;\ 2\int_{\frac{1}{2}}^{1} t(1-t)\, q_i(t)\, dt,\quad i=1,2 \right\} \qquad \dots (1.7)$$

$$(H_4)\ \lim_{x\to\infty} \frac{f_1(x,y)}{x} = \infty,\quad \lim_{y\to\infty} \frac{f_2(x,y)}{y} = \infty. \qquad \dots (1.8)$$

Here and henceforth, we denote the norm of $(x, y) \in R^2$ by $|(x,y)| = \max\ \{|x|, |y|\}$, and write $(x_1, y_1) > (x_2, y_2)$ $((x_1, y_1) \geq (x_2, y_2))$ if $(x_1 - x_2, y_1 - y_2) \in \overline{R}_+^2$

$((x_1 - x_2, y_1 - y_2) \in R_+^2), \overline{R}_+ = (0, \infty)$.

Further, we say that a vector (x, y) is positive (nonnegative) if $(x, y) > (0, 0)$

$((x, y) \geq (0, 0))$.

The hypothesis (H_1) allows $q_i(t)$ to have singularity at $t = 0$ or $t = 1$. For example,

$$q_i(t) = t^{-m}(1-t)^{-n}, 0 < m, n < 2, i = 1, 2$$

satisfy (H_1).

The hypothesis (H_2) allows $f_i(x, y)$ to have singularity at $O = (0, 0)$. For example,

$$f_i(x, y) = (\sqrt{x^2 + y^2})^{-\alpha_i} + \sigma(\sqrt{x^2+y^2})^{\beta_i},\ i = 1, 2$$

satisfy (H_2) and (H_4), where $0 < \alpha_i < +\infty$, $\beta_i > 1, \sigma > 0$.

We have the following main result :

***Theorem* 1** : *Let (H_1)–(H_4) hold. Then the problem (1.1) have twin positive solutions.*

2. PROOF OF THEOREM 1

The following lemmas will be used in our proof. The first result is known nonlinear alternative of Leray-Schauder type [1, Theorem 1.1]. The second result is a more general fixed point theorem in cones [2, Theorem 1.1].

Lemma 2.1 — Assume Ω is a relatively subset of a convex set K in a normed space E. Let $A : \overline{\Omega} \to K$ be a compact map with $0 \in \Omega$. Then either

(A_1) A has a fixed point in $\overline{\Omega}$; or

(A_2) there is a $x \in \partial\Omega$ and a $\lambda < 1$ such that $x = \lambda A(x)$.

Remark 2.1 : By a map being compact we mean it is continuous with relatively compact range.

Lemma 2.2[2] — Let $E=(E, \|\cdot\|)$ be a Banach space and let $K\subset E$ be a cone in E, and let $\|\cdot\|$ be increasing (strictly) with respect to K. Also, r, R are constants with $0 < r < R$. Suppose $A:\Omega_R \cap K \to K$ (here $\Omega_R=\{x\in E, \|x\|<R\}$) is a continuous, compact map and assume the conditions

$$x\neq\lambda A(x), \text{ for } \lambda\in[0,1) \text{ and } x\in\partial_E\Omega_r\cap K \qquad \dots (2.3)$$

and

$$\|Ax\|\geq\|x\| \text{ for } x\in\partial_E\Omega_R\cap K \qquad \dots (2.4)$$

hold. Then A has a fixed point in $K\cap\{x\in E: r\leq\|x\|\leq R\}$.

In this paper, let $\|u\|=\max\limits_{0\leq t\leq 1}|u(t)|$, $u\in C[0,1]$, then $E_1=(C[0,1], \|\cdot\|)$ is a Banach space. Let

$$K_1=\{u\in C[0,1]: u(t)\geq 0 \text{ for } t\in[0,1] \text{ and } u(t) \text{ is concave on } [0,1]\}. \qquad \dots (2.5)$$

Let $E=E_1\times E_1, K=K_1\times K_1$; and $\|z\| = \max\{\|x\|, \|y\|\}$, $\forall z=(x,y)\in E$. Then $(E, \|\cdot\|)$ is a Banach space, and K is a cone in E.

Let $\theta:[0,1]\times[0,1]\to[0,\infty)$, be defined by

$$\theta(t,s)=\begin{cases}\dfrac{t}{s}, & 0\leq t\leq s,\\ \dfrac{1-t}{1-s}, & 0\leq s\leq t.\end{cases} \qquad \dots (2.6)$$

Lemma 2.3[2] — Let $u\in K_1$. Then there exists $t_0\in[0,1]$, with $u(t_0)=\|u\|$ and

$$u(t)\geq\theta(t,s)\|u\|\geq t(1-t)\|u\| \text{ for } t\in[0,1]. \qquad \dots (2.7)$$

Now let's prove Theorem 1. First we consider the problem (1.1) and wish to prove the existence of a positive solution $(x(t), y(t))$ and $\|(x,y)\|<r$.

To show the existence of the solution described in the statement of Theorem 1, we apply Lemma 2.1 at first. We can choose $\varepsilon>0$, and $\varepsilon<r$ such that

$$\begin{cases} \int_{\varepsilon}^{r} \dfrac{du}{g_1(u,0)} > \left\{1 + \dfrac{h_1(r,r)}{g_1(r,r)}\right\} b_0, \\ \int_{\varepsilon}^{r} \dfrac{dv}{g_2(0,v)} > \left\{1 + \dfrac{h_2(r,r)}{g_2(r,r)}\right\} b_0. \end{cases} \qquad \dots (2.8)$$

Let $n_0 \in \{1, 2, \dots\}$ be chosen so that $\frac{1}{n_0} < \varepsilon$. Let $N^+ = \{n_0, n_0 + 1, \dots\}$. We first show that the following boundary value problem

$$\begin{cases} x''(t) + q_1(t) f_1(x(t), y(t)) = 0, \\ y''(t) + q_2(t) f_2(x(t), y(t)) = 0, \quad 0 < t < 1, \\ x(0) = x(1) = y(0) = y(1) = \frac{1}{n}, \quad n \in N^+ \end{cases} \qquad \dots (2.9)^n$$

has a solution $(x_n(t), y_n(t)), n \in N^+, (x_n(t), y_n(t)) > \left(\frac{1}{n}, \frac{1}{n}\right)$ on $(0, 1)$, and $|(x_n(t), y_n(t))| < r$.

To show $(2.9)^n$ has such a solution, $\forall\ n \in N^+$, we will dealt with the modified boundary value problem

$$\begin{cases} x''(t) + q_1(t) F_1(x(t), y(t)) = 0, \\ y''(t) + q_2(t) F_2(x(t), y(t)) = 0, \quad 0 < t < 1, \\ x(0) = x(1) = y(0) = y(1) = \frac{1}{n}, \quad n \in N^+ \end{cases} \qquad \dots (2.10)^n$$

with $F_i(x, y) = g_i^*(x, y) + h_i(x, y)$, where

$$g_1^*(x, y) = \begin{cases} g_1(x, y), & x \geq \frac{1}{n}, \\ g_1\left(\frac{1}{n}, y\right), & x \leq \frac{1}{n}, \end{cases}$$

and

$$g_2^*(x, y) = \begin{cases} g_2(x, y), & y \geq \frac{1}{n}, \\ g_2\left(x, \frac{1}{n}\right), & y \leq \frac{1}{n}. \end{cases}$$

Remark 2.2 : $g_i^*(x, y) \leq g_i(x, y), \ \forall\ (x, y) \in [0, \infty)^2 \ \{O\}$.

Let $\Omega_1 = \Omega_r \times \Omega_r$.

Let $A : \Omega_1 \to E$ be defined by

$$A(x(t), y(t)) = \left(\int_0^1 G(t, s)\, q_1(s)\, F_1(x(s), y(s))\, ds + \frac{1}{n}, \right.$$

$$\left.\int_0^1 G(t,s)\,q_2(s)\,F_2(x(s),y(s))\,ds+\frac{1}{n}\right), \qquad \text{... (2.11)}$$

where
$$G(t,s)=\begin{cases} s(1-t), & 0\le s\le t\le 1,\\ t(1-s), & 0\le t\le s\le 1.\end{cases}$$

Clearly, $t(1-t)\,G(s,s)\le G(t,s)\le s(1-s)$ on $[0,1]\times[0,1]$.

From the definition of A, we know that

$$\begin{cases} (Ax)(t)=\int_0^1 G(t,s)\,q_1(s)\,F_1(x(s),y(s))\,ds+\frac{1}{n},\\ (Ay)(t)=\int_0^1 G(t,s)\,q_2(s)\,F_2(x(s),y(s))\,ds+\frac{1}{n}.\end{cases} \qquad \text{... (2.12)}$$

A standard argument[4] implies $A:\overline{\Omega}_1\to E$ is continuous and completely continuous.

We first show

$$(x,y)\ne\lambda A(x,y) \text{ for } \lambda\in(0,1),\ (x,y)\in\partial\Omega_1. \qquad \text{... (2.13)}$$

Suppose this is false, suppose that there exist a $\lambda\in(0,1)$ and $(x,y)\in\partial\Omega_1$ with $(x,y)=\lambda A(x,y)$. Then we have

$$\begin{cases} x(t)=\lambda(Ax)(t),\\ y(t)=\lambda(Ay)(t),\end{cases} \qquad \text{... (2.14)}$$

that is
$$\begin{cases} -x''(t)=\lambda(Ax)''(t)=\lambda q_1(t)\,F_1(x(t),y(t)),\\ -y''(t)=\lambda(Ay)''(t)=\lambda q_2(t)\,F_2(x(t),y(t)), & 0<t<1,\\ x(0)=x(1)=y(0)=y(1)=\frac{\lambda}{n}, & n\in N^+\end{cases} \qquad \text{... (2.15)}$$

Since $\|x,(y)\|=\max\{\|x\|,\|y\|\}=r$, without loss of generality, we assume that $\|x\|=r$.

Since $x''(t)\le 0$ on $(0,1)$ and $x(t)\ge\frac{\lambda}{n}$ on $[0,1]$, there exists $t_0\in(0,1)$ with $x''(t)\ge 0$ on $(0,t_0)$, $x'(t)\le 0$ on $(t_0,1)$ and $x(t_0)=\|x\|=r$.

Also notice that

$$F_i(x(t),y(t))\le g_i(x(t),y(t))+h_i(x(t),y(t)),\ i=1,2,\ t\in(0,1),$$

then for $z\in(0,1)$, we have

$$-x''(z) \le g_1(x(z), y(z)) \left\{ 1 + \frac{h_1(x(z), y(z))}{g_1(x(z), y(z))} \right\} q_1(z). \quad \text{... (2.16)}$$

Integrate from $t\ (t \le t_0)$ to t_0 to obtain

$$x'(t) \le \left\{ 1 + \frac{h_1(r, r)}{g_1(r, r)} \right\} \int_t^{t_0} g_1(x(z), y(z))\, q_1(z)\, dz, \quad \text{... (2.17)}$$

so we obtain

$$\frac{x'(t)}{g_1(x(t), 0)} + \left\{ 1 = \frac{h_1(r, r)}{g_1(r, r)} \right\} \int_t^{t_0} q_1(z)\, dz, \quad \text{... (2.18)}$$

and then integrate from 0 to t_0 to obtain

$$\int_{\frac{\lambda}{n}}^{r} \frac{du}{g_1(0, 0)} \le \left\{ 1 + \frac{h_1(r, r)}{g_1(r, r)} \right\} \int_0^{t_0} \int_t^{t_0} q_1(z)\, dz. \quad \text{... (2.19)}$$

Consequently, $$\int_{\varepsilon}^{r} \frac{du}{g_1(u, 0)} \le \left\{ 1 + \frac{h_1(r, r)}{g_1(r, r)} \right\} \int_0^{t_0} z q_1(z)\, dz, \quad \text{... (2.20)}$$

and so $$\int_{\varepsilon}^{r} \frac{du}{g_1(u, 0)} \le \left\{ 1 + \frac{h_1(r, r)}{g_1(r, r)} \right\} \frac{1}{1 - t_0} \int_0^{t_0} z(1 - z)\, q_1(z)\, dz. \quad \text{... (2.21)}$$

Similarly, if we integrate (2.16) from $t_0\ (t \ge t_0)$ to t and then from t_0 to 1, we obtain

$$\int_{\varepsilon}^{r} \frac{du}{g_1(u, 0)} \le \left\{ \frac{h_1(r, r)}{g_1(r, r)} \right\} \frac{1}{t_0} \int_{t_0}^{1} z(1 - z)\, q_1(z)\, dz. \quad \text{... (2.22)}$$

Now (2.21), (2.22) imply

$$\int_{\varepsilon}^{r} \frac{du}{g_1(u, 0)} \le \left\{ 1 + \frac{h_1(r, r)}{g_1(r, r)} \right\} b_0. \quad \text{... (2.23)}$$

This contradicts (2.8) and consequently (2.13) is true.

Now Lemma 2.1 imply A has a fixed point $(x_n(t), y_n(t)) \in \overline{\Omega}_1$, i.e., $\frac{1}{n} \le \|(x_n(t), y_n(t))\| \le r$ (note if $\|(x_n(t), y_n(t)\| = r$ then folloing essentially the same argument from (2.16)-(2.23) will yield a contradiction). It follows from the fact $x_n \geq \frac{1}{n}$, we can obtain $(x_n(t), y_n(t))$ is a solution of $(2.9)_n^n$ too.

From (H_2), for $r > 0$, $g_i(x(t), y(t)) \geq g_i(r, r)$, $i = 1, 2$, so we have

$$\begin{cases} x_n(t) \geq \int_0^1 G(t, s)\, q_1(s)\, g_1(r, r)\, ds =: \Psi_1(t), \\ y_n(t) \geq \int_0^1 G(t, s)\, q_2(s)\, g_2(r, r)\, dsa =: \Psi_2(t). \end{cases} \quad \dots (2.24)$$

Notice that $\Psi_i(t) > 0$ on $[0, 1]$ $i = 1, 2$, then $(x_n(t), y_n(t)) > (0, 0)$.

Next we will show that

$$\{x_n\}_{n \in N+}\ \{y_n\}_{n \in N+} \text{ are bounded, equicontinuous families on } [0,1]. \quad \dots (2.25)$$

Returning to (2.16) (with x, y replaced by x_n, y_n), we have

$$\begin{cases} -x_n''(z) \leq g_1(x_n(z), 0)\left\{1 + \dfrac{h_1(r, r)}{g_1(r, r)}\right\} q_1(z), \\ -y_n''(z) \leq g_2(0, y_n(z))\left\{1 + \dfrac{h_2(r, r)}{g_2(r, r)}\right\} q_2(z). \end{cases} \quad \dots (2.26)$$

Since $x_n''(t) \leq 0$ on $(0, 1)$ and $x_n(t) \geq \frac{1}{n}$ on $[0, 1]$, there exists $t_{1n} \in (0, 1)$ with $x_n(t) \geq 0$ on $(0, t_{1n})$, $x_n'(t) \leq 0$ on $(t_{1n}, 1)$ and $x_n(t_{1n}) = \| x_n \| \leq r$.

Also since $y_n''(t) \leq 0$ on $(0, 1)$ and $y_n(t) \geq \frac{1}{n}$ on $[0, 1]$, there exists $t_{2n} \in (0, 1)$ with $y_n'(t) \geq 0$ on $(0, t)$, $y_n'(t) \leq 0$ on $(t_{2n}, 1)$ and $y_n(t_{2n}) = \| y_n \| \leq r$.

Integrate (2.26) of x_n from $t\,(t < t_{1n})$ to t_{1n}, and of y_n from $t\,(t < t_{2n})$ to t_{2n}, we have

$$\frac{x_n'(t)}{g_1(x_n(t), 0)} \leq \left\{1 + \frac{h_1(r, r)}{g_1(r, r)}\right\} \int_t^{t_{1n}} q_1(z)\, dz, \quad \dots (2.27)$$

and

$$\frac{y_n'(t)}{g_2(0, y_n(t))} \leq \left\{1 + \frac{h_2(r, r)}{g_2(r, r)}\right\} \int_t^{t_{2n}} q_2(z)\, dz. \quad \dots (2.28)$$

On the other hand, integrate (2.26) of x_n from $t_{1n}\,(1 > t_{1n})$ to t, and of y_n from $t_{2n}\,(t > t_{2n})$ to t, we obtain

$$\frac{-x_n'(t)}{g_1(x_n(t), 0)} \le \left\{1 + \frac{h_1(r, r)}{g_1(r, r)}\right\} \int_t^{t_{1n}} q_1(z)\,dz, \qquad \text{... (2.29)}$$

and

$$\frac{-y_n'(t)}{g_2(0, y_n(t))} \le \left\{1 + \frac{h_2(r, r)}{g_2(r, r)}\right\} \int_t^{t_{2n}} q_2(z)\,dz. \qquad \text{... (2.30)}$$

Similarly, it follows from the proof of [2], there exist $a_{i0} > 0, a_{i1} < 1, a_{i0} < a_{i1}$ with

$$a_{i0} < \inf\left\{t_{in} : n \in N^+\right\} \le \sup\left\{t_{in} : n \in N^+\right\} < a_{i1}, i = 1, 2.$$

Let $a_0 = \min\{a_{10}, a_{20}\}$, $a_1 = \max\{a_{11}, a_{21}\}$, then we have

$$a_0 < \min\{\inf\{t_{in}\}, i = 1, 2\} \le \max\{\sup\{t_{in}\}, i = 1, 2\} < a_1. \qquad \text{... (2.31)}$$

This implies

$$\frac{|x_n'(t)|}{g_1(x_n(t), 0)} \le \left\{1 + \frac{h_1(r, r)}{g_1(r, r)}\right\} V_1(t), \text{ for } t \in (0, 1) \qquad \text{... (2.32)}$$

and

$$\frac{|y_n'(t)|}{g_2(0, y_n(t))} \le \left\{1 + \frac{h_2(r, r)}{g_2(r, r)}\right\} V_2(t) \text{ for } t \in (0, 1), \qquad \text{... (2.33)}$$

where

$$V_i(t) = \int_{\min(t, a_0)}^{\max(t, a_1)} q_i(z)\,ds$$

Clearly, $V_i \in L^1[0, 1]$[2], $i = 1, 2$.

Let $I_1, I_2 : [0, \infty) \to [0, \infty)$ be defined by

$$I_1(z) = \int_0^z \frac{du}{g_1(u, 0)}, \quad I_2(z) = \int_0^z \frac{d\upsilon}{g_2(0, \upsilon)}.$$

Notice that $I_i : [0, \infty) \to [0, \infty)$ is an increasing map and $I_i(\infty) = \infty$, since $g_i(x, 0) > 0$ and $g_2(0, y) > 0$ are nonincreasing on $(0, \infty)$ with I_i continuous on $[0, B]$ for any $B > 0$, $i = 1, 2$. Notice that

$$\left\{ I_1(x_n) \right\}_{n \in N^+}, \left\{ I_2(y_n) \right\}_{n \in N^+} \text{ are bounded equicontinuous families on } [0, 1].$$

The equicontinuity follows from (here $t, s \in [0, 1]$)

$$|I_1(x_n(t)) - I_1(x_n(s))| = \left| \int_s^t \frac{x_n'(z)}{g_1(x_n(z), 0)} dz \right| \le \left\{ 1 + \frac{h_1(r, r)}{g_1(r, r)} \right\} \left| \int_s^t V_1(z)\, ds \right|.$$

and
$$|I_2(y_n(t)) - I_2(y_n(s))| = \left| \int_s^t \frac{y_n'(z)}{g_2(0, y_n(z),)} dz \right| \le \left\{ 1 + \frac{h_2(r, r)}{g_2(r, r)} \right\} \left| \int_s^t V_2(z)\, ds \right|$$

This inequality, the uniform continuity of $I_i^{-1}, i = 1, 2$ and

$$|x_n(t) - x_n(s)| = |I_1^{-1}(I_1(x_n(t))) - I_1(x_n(s)))|$$

and
$$|y_n(t) - y_n(s)| = |I_2^{-1}(I_2(y_n(t))) - I_2(y_n(s)))|$$

now establish (2.25).

The Arzela-Ascoli theorem guarantees the existence of subsequence $N \in N^+$ $x \in C[0, 1]$ with x_n converging uniformly on $[0, 1]$, $y \in C[0, 1]$ with y_n converging uniformly on $[0, 1]$, $n \in N$. Then (x_n, y_n) is converging uniformly on $[0, 1]$ to (x, y) as $n \to \infty$, for $n \in N$. Also, we have $x(0) = x(1) = y_0 = y(1) = \lim_{n \to \infty} \frac{1}{n} = 0$ and $x(t) \ge \Psi_1(t), y(t) \ge \Psi_2(t)$ on $[0, 1]$. In particular $x(t) > 0$, $y(t) > 0$ on $(0, 1)$.

Fix $t \in (0, 1)$ (without loss of generality assume, $t \ne \frac{1}{2}$). Now $x_n y_n$, $n \in N$, satisfies the integral equations

$$\begin{cases} x_n(z) = x_n\left(\frac{1}{2}\right) + x_n'\left(\frac{1}{2}\right)\left(z - \frac{1}{2}\right) + \int_{\frac{1}{2}}^t (s - z)\, q_1(s) f_1(x_n(s), y_n(s))\, ds, \\ \\ y_n(z) = y_n\left(\frac{1}{2}\right) + y_n'\left(\frac{1}{2}\right)\left(z - \frac{1}{2}\right) + \int_{\frac{1}{2}}^t (s - z)\, q_2(s) f_2(x_n(s), y_n(s)) \cdot ds, \end{cases}$$

for $z \in (0, 1)$. Notice that $\left(\text{take } z = \frac{2}{3}\right) \left\{ x_n'\left(\frac{1}{2}\right) \right\}, \left\{ y_n'\left(\frac{1}{2}\right) \right\}, n \in N$ are bounded sequences since

$\Psi_1(s) \le x_n(s) \le r$, $\Psi_2(s) \le y_n(s) \le r$, $\forall\ s \in [0, 1]$. Thus $\left\{ x_n'\left(\frac{1}{2}\right)\right\}, \left\{ y_n'\left(\frac{1}{2}\right)\right\} n \in N$ denote these subsequences also, and let r_1, r_2 be their limits. Now for the fixed, t, $0 < t < 1$, we have

$$\begin{cases} x_n(z) = x_n\left(\frac{1}{2}\right) + x_n'\left(\frac{1}{2}\right)\left(t - \frac{1}{2}\right) + \int_{\frac{1}{2}}^{t} (s-t)\, q_1(s) f_1(x_n(s), y_n(s))\, ds, \\ y_n(t) = y_n\left(\frac{1}{2}\right) + y_n'\left(\frac{1}{2}\right)\left(t - \frac{1}{2}\right) + \int_{\frac{1}{2}}^{t} (s-t)\, q_2(s) f_2(x_n(s), y_n(s))\, ds, \end{cases}$$

and let $n \to \infty$ through N (cf [2]) to obtain

$$\begin{cases} x(t) = x\left(\frac{1}{2}\right) + r_1\left(t - \frac{1}{2}\right) + \int_{\frac{1}{2}}^{t} (s-t)\, q_1(s) f_1(x(s), y(s))\, ds, \\ y(t) = y\left(\frac{1}{2}\right) + r_2\left(t - \frac{1}{2}\right) + \int_{\frac{1}{2}}^{t} (s-t)\, q_2(s) f_2(x(s), y(s))\, ds. \end{cases}$$

We can do this argument for each $t \in (0, 1)$, and so we have

$$\begin{cases} x''(t) + q_1(t) f_1(x(t), y(t)) = 0, \\ y''(t) + q_2(t) f_2(x(t), y(t)) = 0 \end{cases}$$

for $t \in (0, 1)$. Finally, it is easy to see that $\max\{\|x\|, \|y\|\} < r$ (note if $\|x\| = r$ or $\|y\| = r$, then following essentially the same argument from (2.16)-(2.23) will yield a contradiction again).

Thus we have proved that problem (1.1) has one solution

$$(x(t), y(t)) \in C[0, 1] \times C[0, 1] \cap C^2(0, 1) \times C^2(0, 1) \text{ and } \|(x, y)\| < r.$$

Next we will show problem (1.1) has another positive solution $(x(t), y(t))$, and $\|(x, y)\| > r$.

From $\lim_{x \to \infty} \frac{f_1(x, y)}{x} = \infty$, there exists $R_1 > 0\ (R_1 > r)$ such that $f_1(x, y) \ge x\,\eta_1$, whenever $x \ge R_1$ and η_1 satisfies $a(1-a)\,\eta_1 \int_a^{1-a} G(s, s)\, q_1(s)\, ds \ge 1$, where $a \in (0, 1)$ is fixed.

Also from $\lim\limits_{y \to \infty} \dfrac{f_2(x, y)}{y} = \infty$, there exists $R_2 > 0\ (R_2 > r)$, such that $f_2(x, y) \geq y\, \eta_2$, whenever $y \geq R_2$, and η_2 satisfies $a(1-a)\, \eta_2 \int\limits_a^{1-1a} G(s, s)\, q_2(s)\, ds \geq 1$.

Let $R = \max\{R_1, R_2\}$, then from (1.6) there exists $\varepsilon > 0\ (\varepsilon < r)$, satisfies (2.8). Let $n_0 \in \{1, 2, \ldots\}$ and $\frac{1}{n_0} < \varepsilon, \frac{1}{n_0} < a(1-a)R$. Let $N^+ = \{n_0, n_0 + 1, \ldots\}$.

First we will show that

$$\begin{cases} x''(t) + q_1(t) f_1(x(t), y(t)) = 0, \\ y''(t) + q_2(t) f_2(x(t), y(t)) = 0, \quad 0 < t < 1, \\ x(0) = x(1) = y(0) = y(1) = \frac{1}{n}. \end{cases} \qquad \ldots (2.9)^n$$

has a solution $(x_n(t), y_n(t))$ for each $n \in N^+$ with $x_n(t) > \frac{1}{n}, y_n(t) > \frac{1}{n}$ on (0, 1) and $r \leq \| (x_n, y_n) \| \leq R$. To show $(2.9)^n$ such a solution for each $n \in N^+$, we will dealt with the following problem

$$\begin{cases} x''(t) + q_1(t) F_1(x(t), y(t)) = 0, \\ y''(t) + q_2(t) F_2(x(t), y(t)) = 0, \quad 0 < t < 1, \\ x(0) = x(1) = y(0) = y(1) = \frac{1}{n}. \end{cases} \qquad \ldots (2.10)^n$$

Fix $n \in N^+$. Let $A : K \to E$ be defined by

$$A(x(t), y(t)) = \left(\int_0^1 G(t, s)\, q_1(s)\, F_1(x(s), y(s))\, ds + \frac{1}{n}, \int_0^1 G(t, s)\, q_2(s)\, F_2(x(s), y(s))\, ds + \frac{1}{n} \right) \qquad \ldots (2.34)$$

From [4], it is easy to know that A is completely continuous.

Next we show $A : K \to K$.

If $(x, y) \in K$, then clearly $Ax \in K_1$, $(Ax)(t) \geq 0$, for $t \in [0, 1]$ and

$$(Ax)''(t) = -q_1(t) F_1(x(t), y(t)) \leq 0, \quad t \in (0, 1),$$

$$(Ax)(0) = (Ax)(1) = \frac{1}{n},$$

then $(Ax)(t)$ is concave on [0, 1] and $Ax \in K_1$. Similarly, we can prove $Ay \in K_1$.

Let $\Omega_2 = \Omega_R \times \Omega_R$

We first show $(x, y) \neq \lambda A(x, y)$, $\forall \lambda \in [0, 1)$, $(x, y) \in \partial \Omega_1 \cap K$, ... (2.35)

where $\Omega_1 = \Omega_r \times \Omega_r$ is defined above. Similar to the proof of (2.15)-(2.23), we can show that (2.35) is true.

Next we will show

$$\| A(x, y) \| \geq \| (x, y) \|, \ \forall \ (x, y) \in \partial \Omega_2 \cap K. \qquad \text{... (2.36)}$$

To see this, let $(x, y) \in \partial \Omega_2 \cap K$ such that $\| (x, y) \| = R$. Since $\| (x, y) \| = \max \{\| x \|, \| y \|\}$ $= R$, without loss of generality, we assume that $\| x \| = R$.

Also $x(t)$ is concave on [0, 1], we have from Lemma 2.3 that $x(t) \geq a(1-a)R \geq \frac{1}{n}$, $t \in [a, 1-a]$, then we obtain

$$(Ax)(t) = \int_0^1 G(t, s)\, q_1(s)\, F_1(x(s), y(s))\, ds + \frac{1}{n}$$

$$= \int_0^1 G(t, s)\, q_1 f_1(x(s), y(s))\, ds + \frac{1}{n}$$

$$\geq \eta_1 \int_a^{1-a} G(s, s)\, q_1(s)\, x(s)\, ds$$

$$\geq a(1-a)\, \eta_1 R \int_a^{1-a} G(s, s)\, q_1(s)\, ds$$

$$\geq R,$$

which shows

$$\| A(x, y) \| \geq R = \| (x, y) \|, \text{ for } (x, y) \in \partial \Omega_2 \cap K. \qquad \text{... (2.37)}$$

Now Lemma 2.2 implies A has a fixed point $(x_n(t), y_n(t)) \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, $r <$

$\| (x_n(t), y_n(t) \| \le R$. Clearly, $\| (x_n(t), y_n(t)) \| \neq r$. Consequently $(2.10)^n$ has a solution

$$(x_n(t), y_n(t)) \in C[0,1] \times C[0,1] \cap C^2(0,1) \times C^2(0,1), (x_n(t), y_n(t)) \in K,$$

with
$$(x_n(t), y_n(t)) \ge \left(\frac{1}{n}, \frac{1}{n}\right), t \in [0,1], r < \| (x_n(t), y_n(t) \| \le R, \quad \dots (2.38)$$

which shows that $(2.9)^n$ has a positive solution $(x_n(t), y_n(t))$.

By the same way above, $(x_n(t), y_n(t))$ have subsequences N of N^+, with $(x_n(t), y_n(t))$ converging uniformly on [0, 1] to $(x(t), y(t))$ as $n \to \infty$ through N. Also $(x(0), y(0)) = (x(1), y(1)) = (0, 0)$. It is easy to show that $(x(t), y(t)) \in C[0,1] \times C[0,1] \cap C^2(0,1) \times C^2(0,1)$ is a positive of (1.1) and $r < \| (x, y) \| \le R$.

Thus the proof of Theorem 1 is complete.

3. Example

The singular boundary value problem

$$\begin{cases} x''(t) + \dfrac{1}{\alpha+1}[(\sqrt{x^2+y^2})^{-\alpha} + \sigma(\sqrt{x^2+y^2})^{\beta}] = 0, \\ y''(t) + \dfrac{1}{\alpha+1}[(\sqrt{x^2+y^2})^{-\alpha} + \sigma(\sqrt{x^2+y^2})^{\beta}] = 0, 0 < t < 1, \\ x(0) = x(1) = y(0) = y(1) = 0, \alpha > 0, \beta > 1, \sigma = \left(\dfrac{1}{\sqrt{2}}\right)^{\alpha+\beta} \end{cases}$$

has two solutions $(x_1(t), y_1(t)), (x_2(t), y_2(y)) \in C[0,1] \times C[0,1] \cap C^2(0,1) \times C^2(0,1)$. With

$$(x_1(t), y_1(t)) > (0,0), (x_2(t), y_2(t)) > (0,0) \text{ on } (0, 1) \text{ and } \| (x_1, y_1) \| < 1 < \| (x_2, y_2) \|.$$

To see this we will apply Theorem with $q_i(t) = \frac{1}{\alpha+1}$, $g_i(x, y) = (\sqrt{x^2+y^2})^{-\alpha}$, $h_i(x, y) = \sigma(\sqrt{x^2+y^2})^{\beta}$ $(i = 1, 2)$. Clearly, (H_1), (H_2) and (H_4) hold. Also note,

$$b_0 = \max\left\{\frac{2}{\alpha+1}\int_0^{\frac{1}{2}} t(1-t\,dt, \frac{2}{\alpha+1}\int_{\frac{1}{2}}^{1} t(1-t)\,dt\right\} = \frac{1}{6(\alpha+1)}$$

since there exists $r = 1$ such that

$$\begin{cases} \int_0^r \frac{du}{u^{-\alpha}} > \left\{ 1 + \sigma\sqrt{2r})^{\alpha+\beta} \right\} \frac{1}{6(\alpha+1)}, \\ \int_0^r \frac{d\upsilon}{\upsilon^{-\alpha}} > \left\{ 1 + \sigma(\sqrt{2r})^{\alpha+\beta} \right\} \frac{1}{6(\alpha+1)}, \end{cases}$$

consequently (H_3) holds.

The result now follows from Theorem 1.

REFERENCES

1. R. P. Agarwal and D. O'Regan, *J. Diff. Eq.* **143** (1998) 60-95.
2. R. P. Agarwal, D. O'Regan, *J. math. Anal. Appl.* **240** (1999) 433-45.
3. R. P. Agarwal, D. O'Regan, *Proc. Amer. math. Soc.* **128(7)** (2000) 2085-94.
4. R. P. Agarwal, D. O'Regan, *Existence Theory for Nonlinear Ordinary Differential Equations*, Kluwer, Dordrecht, 1997.
5. Daqing Jiang, *Comput. Math. Appl.* **40** (2000) 249-259.
6. Daqing Jiang, *Ann. Polon. Math.* LXXV (3) (2000) 257-70.