Indian J. pure appl. Math., **34**(2) : 209-218, February 2003

A NEW CLASS OF ANALYTIC FUNCTIONS WITH POSITIVE COEFFICIENTS

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(*Received* 22 *January* 2001; *after revision* 5 *February* 2002; *accepted* 5 *July* 2002)

A systematic investigation of a new class of univalent functions with positive coefficients, which is defined in terms of fractional derivative, is presented. Many interesting and useful properties of this class of functions are given; some of these properties involve coefficient bounds, linear combinations and modified Hadamard products of several functions belonging to the class introduced and studied here.

Key Words : Univalent; Starlike; Convex; Fractional Derivative and Fractional Integral

1. INTRODUCTION

Let S denote the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are analytic and univalent in the unit disk $D = \{z : |z| < 1\}$. A function $f \in S$ is said to be starlike of order $\alpha, 0 \le \alpha < 1$, denoted by $f \in ST(\alpha)$, if $\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha$, for $z \in D$, and is said to be convex of order $\alpha, 0 \le \alpha < 1$, denoted by $CV(\alpha)$, if $\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right) > \alpha$, for $z \in D$.

$ST(0) = ST$ and $CV(0) = CV$ are respectively the classes of starlike and convex functions in S. For $1 < \beta \le 4/3$ and $z \in D$, let $M(\beta) = \left\{f \in S : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < \beta\right\}$ and $L(\beta) = \left\{f \in S : \operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right) < \beta\right\}$. Further, let P be the subclass of S consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \; a_n \ge 0. \qquad \dots (1.1)$$

Let $P^*(\alpha) = ST(\alpha) \cap P, P_k(\alpha) = CV(\alpha) \cap P$ and $P(\beta) = M(\beta) \cap P, U(\beta) = L(\beta) \cap P$. $P^*(0) = P^*$ and $P_k(0) = P_k$ are respectively, the classes of starlike and convex functions in P.

Classes $P(\beta)$ and $U(\beta)$ with positive coefficients was studied recently by Uralegaddi, Ganigi and Sarangi[13]. H. Silverman[7] has studied the univalent functions with negative coefficients.

In this paper we present our studies on a new subclass of univalent functions, using the operator Ω^λ which are motivated from the investigations of Silverman[7], Uralegaddi, Ganigi and Sarangi[13], Srivastava and Owa[10], Owa[4], Kumar, Dixit and Nishimoto[2]. It is worth mentioning that operator Ω^λ was first considered by Owa and Srivastava[5]. A more recent application of this same operator Ω^λ is given by Srivastava, Misra and Das[9]. We now define a new class $P_\lambda(\beta)$.

Definition 1 — Let $P_\lambda(\beta)$ $(0 \le \lambda \le 1; 1 < \beta \le 4/3)$ be the class of functions f in P satisfying the inequality

$$\text{Re}\left\{\frac{z(\Omega^\lambda f(z))'}{\Omega^\lambda f(z)}\right\} < \beta, (z \in D); \qquad \dots (1.2)$$

here linear operator Ω^λ defined by

$$\Omega^\lambda f(z) = \Gamma(2-\lambda)\, z^\lambda D_z^\lambda f(z), \ (f \in P;\ 0 \le \lambda < 1); \qquad \dots (1.3)$$

where $D_z^\lambda f(z)$ denotes the fractional derivative of $f(z)$ of order λ, as defined below, with $D_z^0 f(z) = f(z)$ and $D_z^1 f(z) = f'(z)$. Several essentially equivalent definitions of fractional derivatives and fractional integrals have been given in the literature[10 & 12]. We find it to be convenient to restrict ourselves to the following definitions used recently by Srivastava and Owa[10].

Definition 2 — The fractional integral of order λ is defined, for a function $f(z)$, by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)\, d\zeta}{(z-\zeta)^{1-\lambda}}, \qquad \dots (1.4)$$

where $\lambda > 0, f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 3 — **The fractional** derivative of order λ is defined, for a function $f(z)$, by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)\, d\zeta}{(z-\zeta)^{1-\lambda}}, \qquad \dots (1.5)$$

where $0 \le \lambda < 1, f(z)$ is **analytic function** in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring log $(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Defininition 4 — **Under the** hypothesis of Definition 3, the fractional derivative of order $n+\lambda$ is defined, for a **function** $f(z)$, by

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z), \qquad \dots (1.6)$$

where $0 \le \lambda < 1$ and n is **non-negative** integer. It is easily seen from (1.3) that

$$\Omega^0 f(z) = f(z) \text{ and } \Omega^1 f(z) = z f'(z). \qquad \dots (1.7)$$

It follows from (1.3), (1.7) and Definition 1 that $P_0(\beta) = P(\beta)$ and $P_1(\beta) = U(\beta)$, $(1 < \beta \le 4/3)$. For the class **of functions** belonging to $P_{\lambda}(\beta)$, we prove a number of sharp results including coefficient and **distortion theorems**. In fact by taking linear operator Ω^{λ} we have made a unified study for the classes ***P*(*β*) and** *U*(*β*) studied by Uralegaddi, Ganigi and Sarangi[13].

2. A Theorem on Coefficient Bounds

The following theorems **lays the foundation** of our systematic study of the class $P_{\lambda}(\beta)$ defined in the preceding section.

***Theorem* 1** — *A function f(z) defined by (1.1) is in the class* $P_{\lambda}(\beta)$ *if and only if*

$$\sum_{n=2}^{\infty} \phi(n)(n-\beta)\, a_n \le (\beta - 1), \qquad \dots (2.1)$$

where, for convenience,

$$\phi(n) = \frac{\Gamma(n+1)\,\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}. \qquad \dots (2.2)$$

The result (2.1) is sharp.*t*

PROOF : We assume that the inequality (2.1) holds true and let $|z| = 1$. It suffices to show that

$$\left| \frac{\dfrac{z(\Omega^{\lambda} f(z))'}{\Omega^{\lambda} f(z)} - 1}{\dfrac{z(\Omega^{\lambda} f(z))'}{\Omega^{\lambda} f(z)} - (2\beta - 1)} \right| < 1, \quad z \in D.$$

We have

$$\frac{\left| \dfrac{z\left[z + \sum\limits_{n=2}^{\infty} \phi(n)\, a_n z^n\right]'}{z + \sum\limits_{n=2}^{\infty} \phi(n)\, a_n z^n} - 1 \right|}{\left| \dfrac{z\left[z + \sum\limits_{n=2}^{\infty} \phi(n)\, a_n z^n\right]'}{z + \sum\limits_{n=2}^{\infty} \phi(n)\, a_n z^n} - (2\beta - 1) \right|} \leq \frac{\sum\limits_{n=2}^{\infty} (n-1)\, a_n\, \phi(n)}{2(\beta - 1) - \sum\limits_{n=2}^{\infty} (n - 2\beta + 1)\, \phi(n)\, a_n}.$$

The last expression is bounded above by 1, if

$$\sum_{n=2}^{\infty} \phi(n)(n-1)\, a_n \leq 2(\beta - 1) - \sum_{n=2}^{\infty} (n - 2\beta + 1)\, \phi(n)\, a_n$$

which is equivalent to $\sum\limits_{n=2}^{\infty} \phi(n)(n-\beta)\, a_n \leq (\beta - 1)$, which is true by hypothesis. Hence, we have $f(z) \in P_{\lambda}(\beta)$.

To prove the converse, we assume that $f(z)$ is defined by (1.1) and in the class $P_{\lambda}(\beta)$, so that the condition (1.2) readily yields

$$Re\left\{ \frac{z(\Omega^{\lambda} f(z))'}{\Omega^{\lambda} f(z)} \right\} = Re\left\{ \frac{z + \sum\limits_{n=2}^{\infty} \phi(n)\, n\, a_n z^n}{z + \sum\limits_{n=2}^{\infty} \phi(n)\, a_n z^n} \right\} < \beta, \quad z \in D. \qquad \text{... (2.3)}$$

Choose values of z on the real axis so that $\Gamma(2-\lambda)\, z^{\lambda} D_z^{\lambda} f(z)$ is real. Upon clearing the denominator in (2.3) and letting $z \to 1$ through real values, we have required assertion (2.1).

Finally, we note that the assertion (2.1) of Theorem 1 is sharp, the external function being $f(z) = z + \frac{(\beta - 1)}{\phi(n)\,(n - \beta)} z^n$

Remark 1 : When $\lambda = 0$, Theorem 1 reduces to the corresponding result due to Uralegaddi, Ganigi and Sarangi ([13], p. 226, Theorem 2.3). It follows immediately that $P_0(\beta) = P(\beta)$.

Remark 2 : When $\lambda = 1$, Theorem 1 reduces to the corresponding result due to Uralegaddi, Ganigi and Sarangi ([13], p. 227, Corollary 2.4). It immediately gives $P_1(\beta) = U(\beta), \left(1 < \beta \le \frac{4}{3}\right)$

We record in passing the following interesting consequence of Theorem 1.

Corollary 1 — Let the function $f(z)$ defined by (1.1) belong to the class $P_\lambda(\beta)$. Then

$$a_n \le \frac{(\beta - 1)}{(n - \beta)\,\phi(n)}, \quad (n \ge 2), \qquad \text{... (2.4)}$$

where $\phi(n)$ is given by (2.2).

Remark 3 : The assertion (2.4) of Corollary 1 may be written as

$$a_n \le \frac{(\beta - 1)}{(n - \beta)\,\phi(n)} \le n \quad (n \ge 2) \qquad \text{... (2.5)}$$

for $1 < \beta \le \frac{4}{3}, 0 \le \lambda \le 1$. Thus if A denotes the class of function $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} c_n z^n, \quad (z \in D), \qquad \text{... (2.6)}$$

that are analytic and univalent in D, there do exist functions $f(z) \in P_\lambda(\beta), 1 < \beta \le \frac{4}{3}, 0 \le \lambda < 1$ not necessarily in the class A, for whch the celebrated Bieberbach Conjecture (now DeBranges's Theorem [1]) :

$$|c_n| \le n, \quad (n \ge 2), \text{ holds true.}$$

3. A Distortion Theorem

***Theorem* 2** — *Let the function $f(z)$ defined by (1.1) be in the class $P_\lambda(\beta)$. Then*

$$|f(z)| \ge |z| - \frac{(\beta - 1)(2 - \lambda)}{2(2 - \beta)} |z|^2 \qquad \textbf{... (3.1)}$$

and $$|f(z)| \le |z| + \frac{(\beta-1)(2+\lambda)}{2(2-\beta)}|z|^2. \qquad \text{... (3.2)}$$

Furthermore,

$$|D_z^{\lambda} f(z)| \ge \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)} - \frac{(\beta-1)|z|^{2-\lambda}}{(n-\beta)\,\Gamma(2-\lambda)} \qquad \text{... (3.3)}$$

and $$|D_z^{\lambda} f(z)| \le \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)} + \frac{(\beta-1)|z|^{2-\lambda}}{(n-\beta)\,\Gamma(2-\lambda)}. \qquad \text{... (3.4)}$$

PROOF : Since $f(z) \in P_{\lambda}(\beta)$, in view of Theorem 1, we have

$$\frac{2(2-\beta)}{(2-\lambda)} \sum_{n=2}^{\infty} a_n \le \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\,\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}(n-\beta)\,a_n \le (\beta-1) \qquad \text{... (3.5)}$$

which evidently yields

$$\sum_{n=2}^{\infty} a_n \le \frac{(\beta-1)(2-\lambda)}{2(2-\beta)}, \qquad \text{... (3.6)}$$

and the assertion (3.1) and (3.2) of Theorem 2 follow from (3.6).

Next, by using the second inequality in (3.5), we easily have the assertions (3.3) and (3.4).

Remark 1 : Putting $\lambda = 0$ in Theorem 2, we obtain the corresponding result given by Uralegaddi, Ganigi and Sarangi ([13], p. 227, Theorem 3.1).

Remark 2 : Putting $\lambda = 1$ in Theorem 2, the corresponding result given by Uralegaddi, Ganigi and Sarangi ([13], p. 227, Corollary 3.2) can easily be seen.

The following consequence of Theorem 2 is worthy of note :

Corollary 2 — Under the hypothesis of Theorem 2, $f(z)$ is included in disk with its centre at the origin and radius r given by, $r = 1 + \frac{(\beta-1)(2-\lambda)}{2(2-\beta)}$;

and $D_z^{\lambda} f(z)$ is included in a disk with its centre at the origin and radius R given by

$$R = \frac{1}{\Gamma(2-\lambda)}\left\{1 + \frac{(\beta-1)}{n-\beta}\right\}.$$

4. FURTHER PROPERTIES OF THE CLASS $P_{\lambda}(\beta)$

The following properties are an easy consequences of Theorem 1.

***Theorem* 3** — *Let* $0 \le \lambda \le 1,\ 1 < \beta_1 \le \beta_2 \le \frac{4}{3}$. *Then*

$$P_{\lambda}(\beta_1) \subset P_{\lambda}(\beta_2). \qquad \text{... (4.1)}$$

***Theorem* 4** — *Let* $0 \leq \lambda \leq \mu \leq 1, 1 < \beta \leq \frac{4}{3}$. *Then*

$$P_\lambda (\beta) \supset P_\mu (\beta). \qquad \ldots (4.2)$$

***Theorem* 5** — *Let* $0 \leq \lambda \leq \mu \leq 1, 1 < \beta \leq \frac{4}{3}$. *Then*

$$U (\beta) = P_1 (\beta) \subset P_\lambda (\beta). \qquad \ldots (4.3)$$

5. LINEAR COMBINATIONS OF FUNCTIONS IN THE CLASS $P_\lambda (\beta)$

The proof of the following theorem runs parallel to that of assertion made by Srivastava and Owa[11] and therefore we omit the details involved.

***Theorem* 6** — *Let the functions* $f_1 (z), f_2 (z), \ldots f_m (z)$ *defined by*

$$f_j (z) = z + \sum_{n=1}^{\infty} c_{n,j} z^n \; (c_{n,j} \geq 0), \qquad \ldots (5.1)$$

be in the class $P_\lambda (\beta)$.

Then the function $h(z)$ given by

$$h (z) = \frac{1}{m} \sum_{j=1}^{m} f_j (z), \qquad \ldots (5.2)$$

is also in the class $P_\lambda (\beta)$.

Now we shall determine the extreme points of class $P_\lambda (\beta)$.

***Theorem* 7** — *Let* $f_1 (z) = z$ *and* $f_n (z) = z + \frac{(\beta - 1) z^n}{(n - \beta) \phi (n)}$. *Then* $f \in P_\lambda (\beta)$ *if and only if it can be expressed in the form*

$$f (z) = \sum_{n=1}^{\infty} \lambda_n f_n (z). \qquad \ldots (5.3)$$

PROOF : Proof is similar to that of Lemma (3.2) in [12]. So we omit the proof.

Remark : The extreme points of $P_\lambda (\beta)$ are functions $f_n (z)$, $n = 1, 2, 3, \ldots$ defined in Theorem 7.

6. THEOREMS INVOLVING MODIFIED HADAMARD PRODUCTS

Let $f(z)$ be defined by (1.1), and let

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad (b_n \geq 0). \qquad \text{... (6.1)}$$

The modified Hadamard product of $f(z)$ and $g(z)$ is defined here by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \qquad \text{... (6.2)}$$

The following result describes an interesting property of the modified Hadamard product of several functions :

***Theorem* 8** — *Let the functions $f_1(z), f_2(z), \ldots f_m(z)$ defined by*

$$f_j(z) = z + \sum_{n=2}^{\infty} c_{n,j} z^n, (c_{n,j} \geq 0), \qquad \text{... (6.3)}$$

be *in the classes $P_\lambda(\beta_j)$, $j = 1, 2, \ldots m$ respectively. Also let*

$$\lambda \geq 2\left[\max_{1 \leq j \leq m} \{\beta_j\} - 1\right] \qquad \text{... (6.4)}$$

Then
$$(f_1 * f_2 * \ldots * f_m)(z) \in P_\lambda\left(\prod_{j=1}^{m} \beta_j\right) \qquad \text{... (6.5)}$$

PROOF : The proof of Theorem 8 is much akin to that of Theorem 8 of [11] and therefore we omit the details involved.

For $\beta_j = \beta$, $j = 1, 2, \ldots, m$, Theorem 8 yields.

Corollary 4 — Let each of the functions $f_1(z), f_2(z), \ldots f_m(z)$ defined by (6.3) be in the same class $P_\lambda(\beta)$. Also let $\lambda \geq 2(\beta - 1)$.

Then
$$(f_1 * f_2 * \ldots * f_m)(z) \in P_\lambda(\beta^m). \qquad \text{... (6.6)}$$

Next we prove.

***Theorem* 9** — *Let the functions $f(z)$ defined by (1.1) and $g(z)$ defined by (6.1) be in the classes $P_\lambda(\beta_1)$ and $P_\lambda(\beta_2)$, respectively.*

Then the modified Hadamard product $(f * g)(z)$ belongs to the class $P_\lambda(\beta^2 - 2\beta + 2)$, where

$$\beta = \max\{\beta_1, \beta_2\}. \qquad \text{... (6.7)}$$

PROOF : Since $f(z) \in P_\lambda(\beta_1)$ and $g(z) \in P_\lambda(\beta_2)$, in view of Theorem 1, we have

$$\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\,\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} (n-\beta_1\beta_2)\, a_n b_n$$

$$\leq \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\,\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} (n-\beta_1)\, a_n b_n$$

$$\leq \frac{(\beta_1-1)(\beta_2-1)(2-\lambda)}{2(2-\beta_2)}$$

$$\leq (\beta-1)^2 = \beta^2 - 2\beta + 2 - 1 = (\beta^2 - 2\beta + 2) - 1. \qquad \text{... (6.8)}$$

Moreover, $$1 < \beta^2 - 2\beta + 2 \leq \frac{4}{3} \text{ for } 1 < \beta \leq \frac{4}{3}.$$

Hence, by Theorem 1, the modified Hadamard product $(f*g)(z)$ is in the class $P_\lambda(\beta^2 - 2\beta + 2)$ with β given by (6.7).

Corollary — Under the hypothesis of above theorem, the modified Hadamard product $(f * g)(z)$ belongs to the class $P_\lambda(\beta)$.

PROOF : In view of Theorem 3, we have

$$P_\lambda(\beta) \supset P_\lambda(\beta^2 - 2\beta + 2), \qquad \text{... (6.9)}$$

which in conjunction with Theorem 9 shows that

$$(f*g)(z) \in P_\lambda(\beta), \text{ where } \beta \text{ is given by (6.7).}$$

Finally we give an interesting theorem on the modified Hadamard product (6.2) with extremal functions.

***Theorem* 10** — *Let the functions* $f_j(z)$, $(j = 1, 2)$ *defined by (6.3) be in the class* $P_\lambda(\beta)$.

Then $$(f_1 * f_2)(z) \in P_\lambda(\gamma), \qquad \text{... (6.10)}$$

where $$\gamma(\beta, \lambda) = \frac{2\,[(2-\lambda)(\beta-1)^2 + (2-\beta)^{2}]}{(2-\lambda)(\beta-1)^2 + 2(2-\beta)^2}.$$

The result is sharp, the extremal function being

$$f_j(z) = z + \frac{(\beta-1)(2-\lambda)}{2(2-\beta)} z^2, \; (j = 1, 2).$$

PROOF : The proof of Theorem 10 is much akin to that of Theorem 10 of [11] and therefore we omit its proof.

ACKNOWLEDGEMENT

The authors wish to express their sincere thanks to the referee of this paper for several useful comments and suggestions. Also the first author is thankful to U.G.C., New Delhi for financial support to carry out this work.

REFERENCES

1. De Branges, *Acta Math.*, **154** (1985) 137-52.
2. V. Kumar, K. K. Dixit and Nishimoto, *J. Coll. Eng. Nihon Univ.* **B-25**, (1984) 63-68.
3. K. Nishimoto, *J. College Engng. Nihon Univ. Ser. B*, **17** (1976) 11-19.
4. S. Owa, *Kyungpook Math. J.*, **18** (1978) 53-59.
5. S. Owa and H. M. Srivastava, *Canad. J. Math.* **39** (1987) 1057-77
6. B. Ross, *In: Fractional Calculus and its Applications (Ed. B. Ross)* Springer-Verlag, New York, Heidelberg and Berlin, 1975, 1-36.
7. H. Silverman, *Proc. Amer. math. Soc.* **51** (1975) 109-16.
8. H. M. Srivastava and R. G. Burchman, *Convolution Integral Equations with Special Function Kernels*, John-Wiley and Sons, New York, London, Sydney and Toronto, 1977.
9. H. M. Srivastava, A. K. Misra and M. K. Das, *Comm. Appl. Anal.* **2** (1098) 321-32.
10. H. M. Srivastava and S. Owa, *Math. Japan*, **29** (1984) 383-89.
11. H. M. Srivastava and S. Owa, *Commentarii Mathematici Universitatis Sanct. Pauli*, **35(2)** (1986) 175-88.
12. H. M. Srivastava and S. Owa, *Nagoya Math. J.* **106** (1987) 1-28.
13. B. A. Uralegaddi, M. D. Ganigi and S. M. Sarangi, *Tamkang J. Math.* **25(3)** (1994) 225-30.