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s-INDUCED *L*-FUZZY SUPRA TOPOLOGICAL SPACES

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The aim of this paper is to introduce and study, the concepts of "*s*-induced *L*-fuzzy supra topological space" and "Scott *s*-continuity". Scott *s*-continuity functions turn out to be the natural tool for studying the *s*-induced *L*-fuzzy supra topological space. Finally we discuss the connections between some separation and covering properties of an ordinary topological space and its corresponding *s*-induced *L*-fuzzy supra topological space.

Key Words : Fuzzy Topology; Fuzzy Lattice; Prime Element; Semi Open Set; Scott Continuity; Scott *S*-continuity; Induced Fuzzy Topological Space; *s*-Induced *L*-fuzzy Supra Topological Space

1. INTRODUCTION

M. E. Abd EL-Monsef and A. E. Ramadan[1], introduced the concept of fuzzy supra topology as follows: a family $F \subset I^X$ is called a fuzzy supra topological space on X if 0, $1 \in F$ and it is closed under arbitrary union. The concept of induced fuzzy supra topological space, introduced by Bhaumik and Mukherjee [2], was defined with the notion of *s*-lower semi continuous functions. In section 2, we introduce a new class of functions from a topological space (X, T) to a fuzzy lattice L with its Scott topology, called Scott s-continuous functions as a generalization of the s-lower semi continuous function [2] from (X, T) to the unit closed interval I. Then we study some of their properties and characterizations. We prove that the set $S\ (T)$ of Scott *s*-continuous functions from (X, T) to L is an L-fuzzy supra topological space. Scott *s*-continuous functions turn out to be the natural tool for studying *s*-induced *L*-fuzzy supra topological space (*s-IL-FST* space). In section 3, we discuss the connections between several properties of an ordinary topological space (X, T) and its corresponding *s*-IL-FST spaces $(X, S\ (T))$. For example, (X, T) is semi compact iff the *s*-IL-FST space $(X, S(T))$ is fuzzy supra compact. Throughout this work X, Y will be non-empty ordinary sets and $L = L(\leq, \vee, \wedge, ')$ will denote a fuzzy lattice, i.e., a complete completely distributive lattice with a smallest element 0 and a largest element $1\,(\neq 1)$ and with an order reversing involution $a \to a'\ (a \in L)$ therefore L is a continuous lattice. We will denote $I = [0, 1]$ the unit closed interval and L will denote the lattice of L-fuzzy subsets of X. We will denote by I_A the characteristic function of the ordinary subset A of X. We would like to mention the following definition and results for ready references.

Definition 1.1[4] — An element p of L is called prime iff $p \neq 1$ and whenver $a, b \in L$ with $a \wedge b \leq p$ then $a \leq p$ or $b \leq p$. The set of all prime elements of L will be denoted by $Pr(L)$. In[7], Warner has determined the prime elements of the fuzzy lattice L^X where $Pr(L^X) = \{x_p : x \in X$ and $p \in Pr(L)\}$ for each $x \in X$ and each $p \in Pr(L)$, $x_p : X \to L$ is the fuzzy set defined by

$$\left.\begin{array}{l} x_p(y) = p \text{ if } y = x \\ = 1, \text{ otherwise} \end{array}\right\}.$$

These x_p are called the L-fuzzy points of X and we have x_p is a member of an L-fuzzy set g and we write $x_p \in g$ iff $g(x) \not\leq p$.

Result 1.2[8] — The Scott topology of L is generated by the sets of the form $\{t \in L : t \not\leq p\}$ where $p \in Pr(L)$.

Definition 1.3[3] — Let (X, T) be a topological space and $a \in X$. A function $f : (X, T) \to I$ is called a Scott continuous (or lower semi continuous) at $a \in X$ iff for every $\alpha \in [0, 1)$ with $\alpha < f(a)$ there is a neighbourhood U of 'a' such that $\alpha < f(x)$ for every $x \in U$. f is called Scott continuous (or lower semi continuous) on X iff f is Scott continuous (or lower semi continuous) at every point of X.

Definition 1.4[6] — The set $W(T)$ of Scott continuous function from a topological space (X, T) to L with its Scott topology is an L-fuzzy topology, called the induced L-fuzzy topology (IL-FT).

Definition 1.5[2] — Let (X, T) be a topological space. A function $f(X, T) \to I$ is said to be s-lower semi continuous at $a \in X$ iff for every $\varepsilon > 0$, there is semi open neighbourhood [5] N of 'a' in (X, T) such that $f(x) > f(a) - \varepsilon$ for every $x \in N$. f is called s-lower semi continuous (s-LSC) on X iff f is s-lower semi continuous at every point of X.

This definition can be characterized as follows : $f : (X, T) \to I$ is said to be Scott s-continuous (or s-lower semi continuous) at $a \in X$ iff for every $\alpha \in [0, 1)$ with $f(a) > \alpha$, there is a semi open neighbourhood N of 'a' in (X, T), such that $f(x) > \alpha$ for every $x \in N$.

From this definition it follows that every Scott continuous (or lower semi continuous) function is Scott s-continuous (or s-lower semi continuous).

2. Scott s-Continuous Functions and s-Induced L-Fuzzy Supra Topological Space

Considering a fuzzy lattice with its Scott topology, we introduce the concept of Scott s-continuity as generalization of s-lower semi continuous function[2]. Using this notion we obtained an L-fuzzy supra topology from an ordinary topological space as a generalization of I-fuzzy supra topology[2].

Definition 2.1 — Let (X, T) be a topological space and $a \in X$. A function $f : (X, T) \to L$, where L has its Scott topology, is said to be Scott s-continuous (or s-lower semi continuous) at $a \in X$ iff for every $p \in Pr(L)$ with $f(a) \not\leq p$ there is a semi open neighbourhood U of a in (X, T)

such that $f(x) \nleq p$ for every $x \in U$ i.e. $U \subset f^{-1}$ $\{t \in L : t \nleq p\}$ then f is called Scott s-continuous (or s-lower semi continuous) on X.

When $L = I$, the definition becomes $f : (X, T) \to I$ is Scott s-continuous or s-lower semi continuous at $a \in X$ iff for every $p \in Pr(L) = [0, 1)$ with $f(a) > p$ there is a semi open neighbourhood U of 'a' in (X, T) such that $f(x) > p$ for every $x \in U$.

From this definition it is clear that every Scott continuous function is s-Scott continuous function.

Proposition 2.2 — The characteristic function of every semi open set is Scott s-continuous.

PROOF : Let U be a semi-open set in a topological space (X, T) and $a \in X, p \in Pr(L)$ with $1_U(a) \nleq p$. Then $a \in U$ and U is a semi-open neighbourhood of 'a'. We also have $1_U(x) \nleq p$ for every $x \in U$. Hence 1_U is Scott s-continuous at $a \in X$.

Proposition 2.3 — If $\{f_j : j \in \wedge\}$ is an arbitrary family of Scott s-continuous functions rom a topological space (X, T) to L, then $f = \bigvee_{j \in \wedge} f_j$ is also Scott s-continuous.

PROOF : Let $p \in Pr(L)$ and $a \in x$ with $f(a) = \bigvee_{j \in \wedge} f_j(a) \nleq p$, then there is a $j \in \wedge$ such that $f_j(a) \nleq p$. Since f is Scott s-continuous at 'a', there is a semi open neighbourhood U of 'a' such that $f_j(x) \nleq p$ for all $x \in U$. Henmce, $f(x) = \bigvee_{j \in \wedge} f_j(x) \nleq p$ for all $x \in U$. Thus f is Scott s-continuous at $a \in X$.

Since the intersection of two semi open sets may not be semi open, we have the following.

Proposition 2.4 — Let (X, T) be a topological space. If $g : (X, T) \to L$ are Scott s-continuous functions then $fg : (X, Y) \to L$ is not necessarily Scott s-continuous.

Proposition 2.5 — Let (X, T) be a topological space $f : (X, T) \to L$ is Scott s-continuous if and only if for every $p \in Pr(L), f^{-1}$ $\{t \in L : t \nleq p\}$ can be expressed as a union of semi open sets in (X, T).

PROOF : Let $p \in Pr(L)$ and $x \in f^{-1}$ $\{t \in L : t \nleq p\}$. Then $f(x) \nleq p$. Here f is Scott s-continuous at x, thus there exist a semi-open set N_x, in (X, T) such that $x \in N_x$ and $N_x \subset f^1$ $\{t \in L : t \nleq p\}$. Hence f^1 $\{t \in L : t \nleq p\}$ $= UN_x$, where N_x is semi open.

Conversely, let $a \in X$ and $p \in Pr(L)$ with $f(a) \nleq p$. Then $a \in f^1 \{t \in L : t \nleq p\}$. By the hypothesis, there is a semi open set N in (X, T) such that $a \in N$ and $N \subset f^1\{t \in L : t \nleq p\}$ which implies that f is Scott s-continuous.

***Theorem* 2.6** — *For topological space (X, T), the set* $S(T) = \{f \in L^X . f : (X, T) \to L$ *is Scott s-continuous*$\}$ *is an L-fuzzy supra topology on X.*

PROOF : It follows immediately from Propositions 2.2, 2.3 and 2.4.

Definition 2.7 — The L-fuzzy supra topology $S(T)$, obtained in Theorem 2.6 is called a s-induced L-fuzzy supra topology (s-I-L-FST) and the space $(X, S(T))$ is called s-induced L-fuzzy supra topological space (s-I-L-FST space). The members of $S(T)$ are called fuzzy supra open subsets.

Lemma 2.8 — If A is semi open in a topological space (X, T) then $1_A \in S(T)$.

PROOF : By Proposition 2.2 the lemma follows immediately.

Remark : Since every Scott continuous function from topological space (X, T) to a fuzzy lattice L is Scott s-continuous, we have $W(T) \subset S(T)$ where $W(T)$ is the L-fuzzy topology of Scott continuous functions from (X, T) to L.

In [1], fuzzy supra continuity was defined as follows :

Let (X, F_1) and (Y, F_2) be two fuzzy topological spaces. F_1' and F_2' be two associated fuzzy supra topologies with F_1 and F_2 respectively. A function $f : X \to Y$ is a fuzzy supra continuous if the inverse image of F_2' -supra open subset is F_1'-Supra open. Also we know that a function $f : (X, T_1) \to (Y, T_2)$ is an irresolute if image of semi open subset is semi open.

***Theorem* 2.9** — *Let $S(T_1)$ and $S(T_2)$ be two s-induced L-fuzzy supra tpologies associated with fuzzy topologies F_1 and F_2. Then a function $f : (X, S(T_1)) \to (Y, S(T_2))$ is fuzzy supra continuous iff $f : (X, T_1) \to (Y, T_2)$ is an irresolute function.*

PROOF : Suppose that $f : (X, S(T_1)) \to (Y, S(T_2))$ be fuzzy supra continuous. Let A be a semi open set of (Y, T_2) then $1_A \in S(T_2)$. By fuzzy supra continuity $f^1(1_A) = 1_{f-1(A)} \in S(T_1)$. We shall show that $f^1(A)$ is semi open in (X, T_1), Let $p \in Pr(L)$ and $x \in f^1(A)$. Then $1_{f-1(A)}(x) \leq p$. Since $1_{f-1(A)} \in S(T_1)$, there exists a semi open set N_x in (X, T_1) such that $x \in N_x \subset f^1(A)$. This shows that $f^1(A)$ is semi open in (X, T_1). Consequently $f : (X, T_1) \to (Y, T_2)$ is irresolute.

Conversely, suppose $f: (X, T_1) \to (Y, T_2)$ be an irresolute function. We take $\alpha \in S(T_2)$. We show that $f^1(\alpha) \in S(T_1)$ i.e., $f^1(\alpha): (X, T_1) \to L$ is Scott s-continuous. Let $a \in X$ and $p \in Pr(L)$ with $f^1(\alpha)(a) \not\leq p$. Then $\alpha(f(a)) \not\leq p$. Since $\alpha: (Y, T_2) \to L$ is Scott s-continuous at $f(a) \in Y$, there exists a semi open set N in (Y, T_2) such that $f(a) \in N$ and $\alpha(y) \leq p$ for all $y \in N$. Since N is semi open in $(Y, T_2), f^1(N)$ is also semi open in (X, T_1). Now we have, $a \in f^1(N)$ which implies that there is a semi open set B in (X, T_1) such that $a \in B \subset f^1(N)$. Hence, $f^1(\alpha)(x) = \alpha(f(x)) \not\leq p$ for every $x \in B$. This shows that $f^1(\alpha)$ is Scott s-continuous. Consequently, $f: (X, S(T_1)) \to (Y, S(T_2))$ is fuzzy supra continuous.

3. COVERING AND SEPARATION PROPERTIES OF $(X, S(T))$

Now we study the connections between some separation and covering properties of an ordinary topological space and its corresponding s-induced L-fuzzy supra topological space.

In [5], a topological space (X, T) is called semi compact iff every semi open cover of X admits a finite sub cover of X.

Definition 3.1 — A s-I-L-FST space $(X, S(T))$ is said to be fuzzy supra compact iff for every prime element p of L and every collection $\{f_j : j \in \wedge\}$ of supra open L-fuzzy sets with $(\vee f_j)(x) \not\leq p$ for all $x \in X$, there is a finite subset $\wedge_0$ of $\wedge$ such that $\left(\bigvee_{j \in \wedge_0} f_j\right)(x) \not\leq p$ for all $x \in X$.

***Theorem* 3.2** — *A s-I-LFST space (X, S, (T)) is fuzzy supra compact iff (X, T) is semi compact.*

PROOF : Suppose $(X, S(T))$ is fuzzy supra compact. Let $\{A_j : j \in \wedge\}$ be a semi open covering of X. Then $1_{Aj} \in S(T)$ for every $j \in \wedge$, as A_j is semi open in (X, T). Then $\{1_{Aj} j \in \wedge\}$ is a family of supra open L-fuzzy sets in $(X, S(T))$ with $\left\{\bigvee_{j \in \wedge} 1_{Aj}\right\}(x) \not\leq p$ for all $x \in X$.

From the fuzzy supra compactness of $(X, S(T))$, there exists a finite subset $\wedge_0$ of $\wedge$ such that $\left\{\bigvee_{j \in \wedge} 1_{Aj}\right\}(x) \not\leq p$ for all $x \in X$. Hence $X = \bigcup_{j \in \wedge_0} A_j$. Thus (X, T) is semi compact.

Conversely, let (X, T) be semi compact and $p \in Pr(L)$. Let $B = \{f_j j \in \wedge\}$ be a family of supra open L-fuzzy sets in $(X, S(T))$ with $\left\{ \underset{j \in \wedge}{\vee} f_j \right\}(x) \nleq p$ for all $x \in X$, where $f_j(x) = \alpha_j$ of $x \in A_j$ and $f_j(x) = 0$, otherwise, then A_j is semi open in (X, T) and $\alpha_j \in L$ for every $j \in \wedge$. Then for each $x \in X$ there is an $j \in \wedge$ such that $f_j(x) \nleq p$ i.e. $\alpha_j \nleq p$. Let γ = $\{A_j$: there is an $j \in \wedge$. such that $\alpha_j \nleq p$ and $f_j \in B\}$. Then γ is a family of semi open sets in (X, T) covering X i.e., γ is a semi-open cover of X. From the semi compactness of (X, T), there exists a finite subfamily of γ say γ_0 where $\gamma_0 = \{A_1, A_2, ..., A_n\}$, such that $X = \underset{j\ 1}{\overset{n}{\bigcup}} A_j$.

Hence $\left(\underset{J=1}{\overset{n}{\vee}} f_j \right)(x) \nleq p$ for all $x \in X$ and thus $(X, S(T))$ is fuzzy supra compact.

Definition 3.3 — A topological space (X, T) is called semi completely hausdorff iff for any distinct points x, y of X there are semi open sets A and B such that $x \in A, y \in B$ and $cl(A) \bigcap cl(B) = \phi$.

Definition 3.4 — A s-IL-FST space $(X, S(T))$ is said to be fuzzy supra completely Hausdorff iff for all distinct points x_p, y_q of X and every $p, q \in Pr(L)$, there exists supra open L-fuzzy sets f and g such that $x_p \in f, y_q \in g$ and $\forall z \in X$, $cl(f)(z) = 0$ or $cl(g)(z) = 0$.

***Theorem* 3.5** — *A topological space (X, T) is semi completely hausdorff iff the s-I-LFST space (X, S(T)) is fuzzy supra completely Hausdorff.*

PROOF : Let $x, y \in X(x, y)$ and $p, q \in Pr(L)$. By the semi complete hausdorffnes of (X, T), there exists two semi open sets $\bigcup$ and $\vee$ in (X, T) such that $x \in U$, $y \in V$ and $cl(\bigcup)$ $cl(\vee) = \phi$. Now $1_{\bigcup}, 1_{\vee} \in S(T)$, since $\bigcup$ and $\vee$ are semi open sets in (X, T). Also $1_{\bigcup}(x) \nleq p$ and $1_{\vee}(x) \nleq q$ and $\forall \in X, cl(1_{\bigcup}(x)) = 1_{cl \bigcup}(Z) = 0$ or $cl 1_{\vee}(z) = 1_{cl\vee}(z) = 0$, since $cl(\bigcup) \bigcap cl(\vee) = \phi$. Hence, $(X, S(T))$ is a fuzzy supra completely hausdorff space.

Conversely, let $x, y \in X(x \neq y)$ and $p, q \in Pr(L)$. From the fuzzy supra complete hausdorffness of $(X, S(T))$ there exists L-fuzzy supra open sets f, g which are defined by $f(z) = \alpha$ if

$z \in \cup, f(z) = 0$ otherwise and $g(z) = \beta$ if $z \in \vee$, $g(z) = 0$ otherwise where $\cup$ and $\vee$ are semi open sets in (X, T) and $\alpha, \beta \in L$ such that $x_p \in f, y_q \in g$ and $\forall\ z \in X, cl(f)(z) = 0$ cl $(g(z) = 0$. Thus, we have $x \in \cup$ and $y \in \vee$ where $\cup$ and $\vee$ semi open in (X, T) and cl $(\cup) \cap cl(\vee) = \phi$. Hence (X, T) is semi completely hausdorff.

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