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ANALYTIC SOLUTION FOR THE BLASIUS EQUATION

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Approximate analytical solution is derived for the Blasius equation using δ-perturbation technique. The Pade approximant method is used to accelerate the convergence of the power series. Comparison with the exact result shows the accuracy of the present solution.

Key Words : Analytic Solution; Blasius Equation; δ-perturbation Technique; Pade Approximate Methods

INTRODUCTION

Recent years have witnessed a variety of papers dealing with δ-perturbation expansion method for quantum mechanical (Q-M) problems[1-4]. The method consists of replacing the non-linear terms in the Lagrangian by $(\text{Nonlinear})^{1+\delta}$ and then treating δ as a small parameter. The parameter δ is a measure of nonlinearity of the self interaction term. When $\delta = 0$ the equations are linear and can be solved formally. As δ increases smoothly from zero the nonlinear process gradually turns on. The series in powers of δ convergent because in the neighbourhood of $\delta = 0$ there is usually no abrupt transition, the dependence on δ when $|\delta|$ is small is smooth.

The aim of this paper is to apply the δ-perturbation technique to the Blasius equation which describes the velocity profile of the fluid in the boundary layer which forms when fluid flows along a flat plate[5]. To acceperate the convergence we use the Pade approximants method[5]. Finally, a comparison with some known result has been made.

ANALYSIS

The Blasius equation is one of the basic equations of fluid dynamics and has been the focus of many studies. This nonlinear differential equation describes the velocity profile of the fluid in the boundary layer theory. The equation has the form

$$u'''(x) + u''(x)\,u(x) = 0 \qquad \text{... (1)}$$

Subject to the boundary conditions

$$u(0) = u'(0) = 0, u'(\infty) = 1 \qquad \text{... (2)}$$

where dash (') denotes differentiation with respect to the argument x. We are interested in calculating the value of $\Gamma = u''(0)$ of the velocity profile. This quantity, difficult to compute by any means, plays an important role in many physical analysis. A highly accurate numerical value of $\Gamma = u''(0)$ is given by[5]

$$\Gamma = u''(0) = 0.46960 \qquad \text{... (3)}$$

However, the analytical technique for the calculation of this number Γ is not known so far. In this paper, we wish to compute Γ by using δ-perturbation method.

We do this by replacing $u(x)$ of the Blasius equation by one which contains the parameter δ i.e., we consider the boundary value problem

$$\left.\begin{aligned} & u'''(x) + u''(x)\,(u(x))^{\delta} = 0 \\ & u(0) = u'(0) = 0,\ u'(\infty) = 1 \end{aligned}\right\} \qquad \text{... (4)}$$

Obviously, the Blasius equation is recovered by setting $\delta = 1$. To solve (4) perturbatively we can express the velocity $u(x)$ in the form of a series in powers of δ.

$$u = u_0(x) + \delta u_1(x) + \delta^2 u_2(x) + \ldots \qquad \text{... (5)}$$

Substituting (5) into (4) and comparing different powers of δ, we get a sequence of linear equations with associated boundary conditions. We write the first few equations as

$$\left.\begin{aligned} & u_0'''(x) + u_0''(x) = 0 \\ & u_0(0) = u_0'(0) = 0,\ u_1'(\infty) = 1 \end{aligned}\right\} \qquad \text{... (6)}$$

$$\left.\begin{aligned} & u_1'''(x) + u_1''(x) = -u_0''(x)\log(u_0) \\ & u_1(0) = u_1'(0) = u_1'(\infty) = 0 \end{aligned}\right\} \qquad \text{... (7)}$$

$$u_2'''(x) + u_2''(x) = -u_1''(x)\log(u_0) - u_0''(x)\left\{\frac{u_1}{u_0} + \frac{1}{2}\log^2(u_0)\right\}$$

$$u_2(0) = u_2'(0) = u_2'(\infty) = 0. \qquad \text{... (8)}$$

Indeed all functions $u_n(x)$ satisfy linear differential equations which may be easily obtained. Morever, (6) is a homogeneous linear equation with nonhomogeneous boundary conditions; while (7) and (8) are nonhomogeneous linear equations having homogeneous boundary conditions. This is a typical form of δ-perturbation expansion in general.

Now the appropriate solution of the zero order problem i.e., of (6) being

$$u_0(x) = e^{-x} + x - 1 \qquad \text{... (9)}$$

Consequently the value of Γ is given by

$$\Gamma = u_0''(0) = 1 \qquad \text{... (10)}$$

which is not a bad approximation to the true value $u''(0)$ in (3).

First Order Solution

A close look at eq. (7) reveals a remarkable feature associated with the second derivative of $u_1(x)$ at $x = 0$. For it follows directly that $u_1''(0)$ is given by the integral

$$u_1''(0) = \int_0^\infty e^{-y} \log(e^{-y} + y - 1)\, dy. \qquad \text{... (11)}$$

The exact analytical evaluation of the integral (11) is not known.
However, by numerical integration we find that[6]

$$u_1''(0) = -2.1333275 \qquad \text{... (12)}$$

Hence, to first order in powers of δ our prediction for Γ is

$$\Gamma = 1 - 2.133275\ \delta \qquad \text{... (13)}$$

which is a poor approximation to the true value (4), if we evaluate the δ-series (13) at $\delta = 1$. Of course, we can improve the prediction enormously by using the well known Pade approximate methods[6]. We first convert the δ-series (13) to a (0-1) Pade and then evaluate the result at $\delta = 1$. We obtain the result

$$\Gamma = 0.31914 \qquad \text{... (14)}$$

which is a fairly good approximation to the exact value. We emphasize that the δ-series continues to provide more accurate numerical results as we increase the order of perturbation theory.

Second Order Solution

The second order calculation is straight-forward which requires some more algebra than the first order calculation. The result for the evaluation of $u_2''(0)$ is given by

$$u_2''(0) = \int_0^\infty e^{-y}\, y \log u_{0(y)}\, dy + \frac{1}{2} \int_0^\infty e^{-y} (1-y) \log^2 u_{0(y)}\, dy$$

$$+ \int_0^\infty \frac{e^{-y}}{u_0(y)}\, dy \int_0^y z e^{-z} \log\{u_0(z)\} dz$$

$$+ \frac{1}{2} \{u_1''(0)\}^2 - u_1'(0) + \frac{1}{2} = 5.83101 \qquad \text{... (15)}$$

Therefore, we can compute Γ upto second order in powers of as

$$\Gamma = 1 + \delta u_1''(0) + \delta^2 u_2''(0). \quad \text{... (16)}$$

We compute a Pade approximant from the δ-series (16) and then set $\delta = 1$. The (1 – 1) Pade given $\Gamma = 0.42901$ which is an excellent result to the exact value. We know of no other analytical method to the Blasius equation which is productive.

DISCUSSION

It is worthwhile to remark that although δ-perturbation technique was specifically developed to solve $(Q - M)$ problems but it can be used as a very powerful mathematical tool for obtaining the solutions of nonlinear differential equations. The cardinal idea lies in the fact that as δ-varies, the solution $u(x)$ changes slowly and smoothly as a function of δ. Moreover, the basic advantage of this procedure is that it has a finite non zero radius of convergence[3 & 4] unlike $(Q - M)$ perturbation series which have zero radii of convergence. This is because $(Q - M)$ perturbation series is expanded in powers of the weak coupling constant ε and is called a weak coupling expansion. This technique may be applied to the nonlinear partial differential equations such as Klein-Gordan equation, KdV equation, Burger's equation etc. which will be considered in subsequent papers.

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