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A NECESSARY CONDITION FOR LOCAL ASYMPTOTIC STABILITY OF PERIORIC ORBITS OF NONLINEAR SYSTEMS WITH PARAMETERS

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In this paper we derive a necessary condition for local asymptotic stability of periodic orbits of nonlinear systems with parameters. We illustrate our result with examples from bifurcation theory of periodic orbits.

Key Words : Periodic Orbits; Asymptotic Stability; Nonlinear Systems; Bifurcations; Stability Theory

1. INTRODUCTION

In this paper, we consider a nonlinear system described by

$$\dot{x} = f(x, \mu) \overset{\Delta}{=} f_{\mu}(x) \qquad \ldots (1)$$

where $x \in I\!R^n$ is the *state* and $\mu \in I\!R^k$ the *parameter vector* for the nonlinear system (1). We assume that the vector field f is C^2 in x, and jointly continuous in x and μ. Also, we assume that Γ is an isolated periodic orbit of the dynamics

$$\dot{x} = f_{\mu_0}(x). \qquad \ldots (2)$$

We suppose also that the state x of the system (1) takes values in X, where X is an open neighbourhood of Γ and that the parameter vector μ takes values in an open neighbourhood V of μ_0 in $I\!R^k$.

The Poincaré map for the periodic orbit Γ of the dynamics (2) has the form

$$P_{\mu_0} : \tilde{\chi} = A_{\mu_0} \chi + \phi_{\mu_0}(\chi), \qquad \ldots (3)$$

where χ is defined in an open neighbourhood of the origin of $\mathbb{R}^{n-1}$, A_{μ_0} is an $(n-1)\times(n-1)$ matrix, and ϕ is a C^2 function that vanishes at the origin of $\mathbb{R}^{n-1}$ together with all its partial derivatives.

If the periodic orbit Γ of the dynamics (2) is locally exponentially stable, then it follows from Lyapunov stability theory that the A_{μ_0} is *convergent*; that is, all eigenvalues of A_{μ_0} lie in the open unit disc $|z| < 1$ of the complex plane. Then it follows immediately that the matrix $I - A_{\mu_0}$ is nonsingular. Thus, by Inverse Function Theorem, it follows that for all values of μ near μ_0, the equation

$$(I - P_\mu)(\chi) = \chi - P_\mu(\chi) = y \qquad \ldots (4)$$

is solvable. In particular, taking $y = 0$, this implies that there exists a periodic orbit of the dynamics (1) for all values of μ near μ_0.

We contend that this is the case for locally asymptotically stable periodic orbits as well. That is, if Γ is a locally asymptotically stable periodic orbit of the dynamics (2), then we contend that for all values of μ near μ_0, there exists a periodic orbit for the system (1).

We illustrate our claim with some examples from bifurcation theory of periodic orbits.

Example 1.1 — Consider the planar system

$$\dot{x}_1 = -x_2 + x_1(1 - r^2)[\mu - (r^2 - 1)^2]$$

and

$$\dot{x}_2 = x_1 + x_2(1 - r^2)[\mu - (r^2 - 1)^2] \qquad \ldots (5)$$

where $r^2 = x_1^2 + x_2^2$, and $\mu \in \mathbb{R}$ is a parameter.

In polar coordinates, the plant eq. (5) takes the form

$$\dot{r} = r(1 - r^2)[\mu - (r^2 - 1)^2]$$

and

$$\dot{\theta} = 1 \qquad \ldots (6)$$

When $\mu = 0$, the dynamics (6) reduces to

$$\dot{r} = -r(1 - r^2)^3$$

and

$$\dot{\theta} = 1. \qquad \ldots (7)$$

It is easy to see that the zero-parameter dynamics has a periodic orbit Γ represented by

$$\gamma_0(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}.$$

Using Lyaupnov stability theory for periodic orbits, it can be easily established that Γ is a locally asymptotically stable periodic orbit of the zero-parameter system (7).

Note that for all $\mu \in I\!R$, the system (6) has a one-parameter family of periodic orbits represented by

$$\gamma_0(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}.$$

and for $\mu > 0$, there is another family represented by

$$\overset{\pm}{\gamma}_\mu(t) = \sqrt{1 \pm \sqrt{\mu}} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}.$$

This is an example of a *pitchfork bifurcation* at a nonhyperbolic period orbit[1], (p368-369). □

Example 1.2 — Consider the planar system

$$\dot{x}_1 = -x_2 - x_1[\mu - (r^2 - 1)^2]$$

and $$\dot{x}_2 = x_1 - x_2[\mu - (r^2 - 1)^2], \qquad \text{... (8)}$$

where $r^2 = x_1^2 + x_2^2$ and $\mu \in I\!R$ is a parameter.

In polar coordinates, the plant eq. (8) takes the form

$$\dot{r} = -r[\mu - (r^2 - 1)^2]$$

and $$\dot{\theta} = 1. \qquad \text{... (9)}$$

When $\mu = 0$, the plant dynamics in (9) reduces to

$$\dot{r} = r(r^2 - 1)^2$$

and $$\dot{\theta} = 1. \qquad \text{... (10)}$$

It is easy to see that the zero-parameter plant dynamics (10) has a periodic orbit Γ represented by

$$\gamma_0(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}.$$

As in ([1], p364), it can be easily seen that this periodic orbit Γ is unstable.

Note that for $\mu > 0$, there are two one-parameter families of periodic orbits.

$$\overset{\pm}{\Gamma}_\mu : \overset{\pm}{\gamma}_\mu(t) = \sqrt{1 \pm \sqrt{\mu}} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

but for $\mu < 0$, there is no periodic orbit for the system (9).

This is an example of a *saddle-node bifurcation* at a nonhyperbolic periodic orbit ([1], p364-366). □

2. Persistence of Periodic Orbits for Locally Asymptotically Stable Periodic Orbits of Nonlinear Systems with Parameters

In this section, using degree theory, we derive a necessary condition for Γ to be a locally asymptotically stable periodic orbit of the system

$$\dot{x} = f_{\mu_0}(x).$$

Our main result is similar to the necessary condition obtained by Byrnes and Sundarapandian[2] for local asymptotic stability of equilibria of nonlinear systems with parameters. Our result asserts that any asymptotically stable periodic orbit of a C^2 dynamical system persists as a periodic orbit in a robust way.

***Theorem* 2.1** — *Consider a nonlinear system described by*

$$\dot{x} = f(x, \mu) \overset{\Delta}{=} f_{\mu}(x) \; (x \in \mathbb{R}^n, \mu \in \mathbb{R}^k) \qquad \text{... (11)}$$

where the state x is defined in an open neighbourhood of the periodic orbit Γ of the dynamics

$$\dot{x} = f_{\mu_0}(x), \qquad \text{... (12)}$$

and μ is defined in an open neighbourhood of μ_0 in $\mathbb{R}^k$. We assume that f is C^2 in x, and jointly continuous in x and μ. A necessary condition for Γ to be a locally asymptotically stable periodic orbit of the system (12) is that for all μ near $\mu_0 \in \mathbb{R}^k$, there exists a periodic orbit of the dynamics (11).

PROOF : It is given that Γ is a locally asymptotically stable periodic orbit of the dynamics (12). Hence, by a necessary condition due to Krasnoselski ([3], Theorem 55.2, p366), it follows that

$$\kappa_{\mu_0}(\Gamma) = \text{ index } (I - P_{\mu_0}, 0) = 1, \qquad \text{... (13)}$$

where P_{μ_0} is the Poincaré map for the dynamics (12), and $\kappa_{\mu_0}(\Gamma)$ is the *index* of the periodic orbit Γ for the dynamics (12).

Since the index operator is robust with respect to small variations in the parameter, it follows that for all values of μ near μ_0, we have

$$\text{index } (I - P_{\mu}, 0) \neq 0.$$

Now, we can apply the degree theqry [4] to conclude that the map $I - P_\mu$ is locally onto, *i.e.* the equation

$$\chi - P_\mu(\chi) = y \qquad \text{... (14)}$$

is locally solvable. In particular, taking $y = 0$ in (14), we conclude that for all values of μ near μ_0, there exists a periodic orbit for the dynamics (11). This completes the proof. □

The following example shows that the converse of Theorem 2.1 is not true.

Example 2.2 — Consider the planar system

$$\dot{x}_1 = -x_2 - x_1(1 - r^2)(1 + \mu - r^2)$$

and
$$\dot{x}_2 = x_1 - x_2(1 - r^2)(1 + \mu - r^2) \qquad \text{... (15)}$$

where $r^2 = x_1^2 + x_2^2$ and $\mu \in I\!R$ is a parameter.

In polar coordinates, the plant dynamics in (15) takes the form

$$\dot{r} = -r(1 - r^2)(1 + \mu - r^2)$$

and
$$\dot{\theta} = 1. \qquad \text{... (16)}$$

When $\mu = 0$, the plant dynamics in (16) reduces to

$$\dot{r} = -r(1 - r^2)^2$$

and
$$\dot{\theta} = 1, \qquad \text{... (17)}$$

which has the periodic orbit

$$\Gamma : \gamma_0(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

It can be easily seen that Γ is an unstable periodic orbit for the dynamics (17).

Note, however, that for all $\mu \in I\!R$, the system (16) has a one-parameter family of periodic orbits represented by

$$\gamma_0(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

and for $\mu > -1$, there is another one-parameter family of periodic orbits represented by

$$\gamma_\mu(t) = \sqrt{1 + \mu}\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}.$$

This is an example of a *transcritical bifurcation* at a nonhyperbolic periodic orbit ([1], p 366-367), and demonstrates that the converse of Theorem 2.1 is not true. □

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