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SOME PROPERTIES OF SMOOTH IDEALS

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We introduce the concept of smooth ideals and smooth ideal bases. We investigate the images and preimages of smooth ideals. Furthermore, we define the product smooth ideals.

Key Words : Smooth Ideal; Smooth Ideal Base

1. INTRODUCTION

The concept of fuzzy sets was introduced by Zadeh in classical paper[10] and thereafter, the paper of Chang[3] paved the way for the subsequent development of numerous fuzzy topological concepts. One of the recent directions is the study of *L*-fuzzy filter[4] and generalised filters[1 & 2] and its applications. Ramadan *et al.*[9] introduced the smooth ideal which is the dual of a smooth filter.

In this paper, we investigate the images and preimages of smooth ideals and smooth ideal bases. From the existence of preimages of smooth ideals, we can define the product smooth ideals.

2. PRELIMINARIES

For the sake of fixing notation, we recall some basic definitions. We shall let X be a nonempty set and I be the closed unit interval. A fuzzy set in X is an element of the set I^X of all functions from the set X into I. The fuzzy set which assigns to each element in X the value $\alpha, 0 \le \alpha \le 1$, is denoted by $\tilde{\alpha}$. We denote the characteristic function of a subset A of X by 1_A.

Definition 1.1 — If X is a set, then an ideal on X is a nonempty $D^* \subset 2^X$ satisfying the following conditions :

(1) $X \notin D^*$,

(2) if $A, B \in D^*$, then $A \cup B \in D^*$,

(3) if $B \in D^*$ and $A \subset B$, then $A \in D^*$, that is, D^* is a lower set.

Definition 1.2 — If X is a set, then a preideal on X is a nonempty $\mathcal{D} \subset I^X$ satisfying the following conditions :

(P1) $\tilde{1} \notin \mathcal{D}$,

(P2) if $\lambda, \mu \in \mathcal{D}$, then $\lambda \vee \mu \in \mathcal{D}$,

(P3) if $\mu \in \mathcal{D}$ and $\lambda \leq \mu$, then $\lambda \in \mathcal{D}$.

Definition 1.3 — A nonempty subset $\mathcal{D}$ is called a *preideal base* on X if it satisfies the following conditions :

(PB1) $\tilde{1} \notin \mathcal{D}$

(PB2) if $\mu_1, \mu_2 \in \mathcal{D}$, there exists $\mu_3 \in \mathcal{D}$ such that $\mu_1 \vee \mu_2 \leq \mu_3$.

2. The Preimages of Smooth Ideals

Definition 2.1[9] — A nonzero function $I : I^X \to I$ is called a *smooth ideal on* X if it satisfies the following conditions :

(S1) $I(\tilde{1}) = 0$.

(S2) $I(\lambda \vee \mu) \geq I(\lambda) \wedge I(\mu)$, for $\lambda, \mu \in I^X$

(S3) If $\lambda \geq \mu$, $I(\lambda) \leq I(\mu)$.

If I_1 and I_2 are smooth ideals on X, we say I_1 is *finer* than I_2 (or I_2 is *coarser* than I_1), denoted by $I_2 \leq I_1$, iff $I_2(\lambda) \leq I_1(\lambda)$ for all $\lambda \in I^X$.

Notation 2.2 — Let $\mathcal{B} : I^X \to I$ be a function and $\lambda \in X$. We denote

$$\langle \mathcal{B} \rangle (\lambda) = \sup_{\lambda \leq \mu} \mathcal{B}(\mu).$$

Definition 2.3 — A nonzero function $\mathcal{B} : I^X \to I$ is called a *smooth ideal base* on X if it satisfies the following conditions :

(SB1) $\mathcal{B}(\tilde{1}) = 0$.

(SB2) $\langle \mathcal{B}(\lambda \vee \mu) \rangle \geq \mathcal{B}(\lambda) \wedge \mathcal{B}(\mu)$, for $\lambda, \mu \in I^X$.

Remark 2.4 : (1) A smooth ideal is a smooth ideal base.

(2) If a function $I: I^X \to I$ is a smooth ideal (base), for $r \in [0, 1)$, $I^r = \left\{ \mu \in I^X \mid I(\mu) > r \right\}$ is a preideal (base).

***Theorem* 2.5** — *If a function $\mathcal{B}: I^X \to I$ is a smooth ideal base, then $\langle \mathcal{B} \rangle$ is the coarsest smooth ideal satisfying $\mathcal{B}(\lambda) \leq \langle \mathcal{B} \rangle (\lambda)$ for each $\lambda \in I^X$.*

PROOF : The conditions (S1) and (S3) are easily checked. Suppose there exist $\lambda, \mu \in I^X$ and $t \in (0, 1)$ such that

$$\langle \mathcal{B} \rangle (\lambda) \wedge \langle \mathcal{B} \rangle (\mu) > t > \langle \mathcal{B} \rangle (\lambda \vee \mu). \quad \text{... } (A)$$

Since $\langle \mathcal{B} \rangle (\lambda) > t$ and $\langle \mathcal{B} \rangle (\mu) > t$, there exist $\lambda_1, \mu_1 \in I^X$ with $\lambda \leq \lambda_1, \mu \leq \mu_1$ such that

$$\langle \mathcal{B} \rangle (\lambda) \wedge \langle \mathcal{B} \rangle (\mu) \geq \mathcal{B}(\lambda_1) \wedge \mathcal{B}(\mu_1) > t.$$

Since $\mathcal{B}$ is a smooth ideal base,

$$\langle \mathcal{B} \rangle (\lambda_1 \vee \mu_1) \geq \mathcal{B}(\lambda_1) \wedge \mathcal{B}(\mu_1) > t.$$

Since $\lambda \vee \mu \leq \lambda_1 \vee \mu_2$

$$\langle \mathcal{B} \rangle (\lambda \vee \mu) \geq \langle \mathcal{B} \rangle (\lambda_1 \vee \mu_1) > t.$$

It is a contradiction for the eq. (A). Thus, for every $\lambda, \mu \in I^X$,

$$\langle \mathcal{B} \rangle (\lambda) \wedge \langle \mathcal{B} \rangle (\mu) \leq \langle \mathcal{B} \rangle (\lambda \vee \mu).$$

Hence, $\langle \mathcal{B} \rangle$ is a smooth ideal.

If I is smooth ideal satisfying $\mathcal{B}(\lambda) \leq I(\lambda)$ for each $\lambda \in I^X$, we will show that $\langle \mathcal{B} \rangle \leq I$. Suppose there exist $\mu \in I^X$ and $r \in (0, 1)$ such that

$$\langle \mathcal{B} \rangle (\mu) > r > I(\mu). \quad \text{... } (B)$$

Since $\langle \mathcal{B} \rangle (\mu) > r$, there exists μ_1 with $\mu \leq \mu_1$ such that

$$\langle \mathcal{B} \rangle (\mu) \geq \mathcal{B}(\mu_1) > r.$$

On the other hand, since $\mathcal{B}(\mu_1) \leq I(\mu_1)$, we have

$$I(\mu) \geq I(\mu_1) \geq \mathcal{B}(\mu_1) > r.$$

It is a contradiction for the eq. (B). Hence $\langle \mathcal{B} \rangle \leq I$.

Definition 2.6 — Let I and I^* be two smooth ideals on X and Y, respectively, and $f: X \to Y$ be a function.

(1) f is said to be a *smooth ideal map* (for short, I-map) iff $I^*(\mu) \leq I(f^{-1}(\mu))$ for each $\mu \in I^Y$.

(2) f is said to be a *smooth ideal preserving map* (for short, I-preserving map) iff $I(\lambda) \leq I^*(f(\lambda))$ for each $\lambda \in I^X$.

Naturally, the composition of I-maps (resp. I-preserving maps) is an I-map (resp. I-preserving map).

***Theorem* 2.7** — *Let $\mathcal{B}$ and $\mathcal{B}^*$ be two smooth ideal bases on X and Y, respectively, and $f: X \to Y$ be a function.*

(1) $f: (X, \langle \mathcal{B} \rangle) \to (Y, \langle \mathcal{B}^* \rangle)$ is an I-map iff $\mathcal{B}^*(\mu) \leq \langle \mathcal{B} \rangle (f^{-1}(\mu))$ for each $\mu \in I^Y$.

(2) $f(X, \langle \mathcal{B} \rangle) \to (Y, \langle \mathcal{B}^* \rangle)$ is an I-preserving map iff $\mathcal{B}(\lambda) \leq \langle \mathcal{B}^* \rangle (f(\lambda))$ for each $\lambda \in I^X$

(3) If $\mathcal{B}^*(\mu) \leq \mathcal{B}(f^{-1}(\mu))$ for each $\mu \in I^Y$, then $f: (X, \langle \mathcal{B} \rangle) \to (Y, \langle \mathcal{B}^* \rangle)$ is a I-map.

(4) If $\mathcal{B}(\lambda) \leq \mathcal{B}^*(f(\lambda))$ for each $\lambda \in I^X$, then $f(X, \langle \mathcal{B} \rangle) \to (Y, \langle \mathcal{B}^* \rangle)$ is an I-preserving map.

PROOF : (1) ($\Rightarrow$) Since $\mathcal{B}^*(\mu) \leq \langle \mathcal{B}^* \rangle (\mu)$ for each $\mu \in I^Y$, it is trivial.

($\Leftarrow$) Suppose there exist $\mu \in I^Y$ and $r \in (0, 1)$ such that

$$\langle \mathcal{B}^* \rangle (\mu) > r > \langle \mathcal{B} \rangle (f^{-1}(\mu)). \qquad \ldots (C)$$

Since $\langle \mathcal{B}^* \rangle (\mu) > r$, there exists μ_1 with $\mu \leq \mu_1$ such that

$$\langle \mathcal{B}^* \rangle (\mu) \geq \mathcal{B}^*(\mu_1) > r.$$

On the other hand, since $\mathcal{B}^*(\mu_1) \leq \langle \mathcal{B} \rangle (f^{-1}(\mu_1))$ we have

$$\langle \mathcal{B} \rangle (f^{-1}(\mu)) \geq \langle \mathcal{B} \rangle (f^{-1}(\mu_1)) \geq \mathcal{B}^*(\mu_1) > r.$$

It is a contradiction for the eq. (C). Thus, $\langle \mathcal{B}^* \rangle (\mu) \leq \langle \mathcal{B} \rangle (f^{-1}(\mu))$, for each $\mu \in I^Y$.
(2), (3) and (4) are similarly proved.

Example 2.8 — Let $X = \{a, b, c, d\}$ be a set. We define two functions $\mathcal{B}_1, \mathcal{B}_2 : I^X \to I$ as follows :

$$\mathcal{B}_1(\lambda) = \begin{cases} 0.5, & \text{if, } \lambda = \tilde{0}, \\ 0.4, & \text{if, } \lambda \in \left\{ 1_{(a)}, 1_{(b)}, 1_{(a, b, c)} \right\}, \\ 0, & \textit{otherwise}. \end{cases}$$

$$\mathcal{B}_2(\mu)=\begin{cases}0.4, & \text{if, } \mu=\tilde{0},\\ 0.3, & \text{if, } \mu\in\left\{1_{(a)}, 1_{(b)}, 1_{(a,b)}\right\},\\ 0, & \textit{otherwise.}\end{cases}$$

Since $\langle \mathcal{B}\rangle (1_{(a)} \vee 1_{(b)}) = \mathcal{B}_1(1_{(a,b,c)}) = 0.4$, $\mathcal{B}_1$ is a smooth base. Similarly, $\mathcal{B}_2$ is a smooth base. We obtain

$$\langle \mathcal{B}_1\rangle(\lambda)=\begin{cases}0.5, & \text{if, } \lambda=\tilde{0},\\ 0.4, & \text{if, } \tilde{0}<\lambda\le 1_{(a,b,c)},\\ 0, & \textit{otherwise.}\end{cases}$$

$$\langle \mathcal{B}_2\rangle(\mu)=\begin{cases}0.4, & \text{if, } \mu=\tilde{0},\\ 0.3, & \text{if, } \tilde{0}<\mu\le 1_{(a,b)},\\ 0, & \textit{otherwise.}\end{cases}$$

Then $id_X : (X, \langle \mathcal{B}_1\rangle) \to (X, \langle \mathcal{B}_2\rangle)$ be an I-map and $id_X : (X, \langle \mathcal{B}_2\rangle) \to (X, \langle \mathcal{B}_1\rangle)$ is an I-preserving map. But $0 = \mathcal{B}_1(1_{(a,b)}) < \mathcal{B}_2(1_{(a,b)}) = 0.3$.

Hence, the converse of Theorem 2.7 (3-4) need not be true.

Notation 2.9 — Let I be a smooth ideal on X. We denote

$$I^0=\left\{\lambda\in I^X \mid I(\lambda)>0\right\}.$$

Theorem 2.10 — *Let $g_i : X \to X_i$ be a function, for each $i\in\Gamma$. Let $\left\{\mathcal{B}_i\right\}_{i\in\Gamma}$ a family of smooth ideal bases on X_i satisfying the following condition*

(*C*) If $\nu_i \in (\mathcal{B}_i)^0$ for all $i\in\Gamma$, then we have $\vee_{i\in K}\, g_i^{-1}(\nu_i) \neq \tilde{1}$ for every finite subset K of Γ.

We define a function $\vee_{i\in\Gamma}\, g_i^{-1}(\mathcal{B}_i) : I^X \to I$ as

$$\bigvee_{i\in\Gamma} g_i^{-1}(\mathcal{B}_i)(\lambda)=\begin{cases}\sup\left\{\wedge_{i\in K}\,\mathcal{B}_i(\nu_i)\right\} & \text{if } \lambda=\vee_{i\in K}\, g_i^{-1}(\nu_i),\ \nu_i\in(\mathcal{B}_i)^0,\\ \tilde{0} & \text{if } \textit{otherwise.}\end{cases}$$

where the supremum is taken for every finite subset K of Γ such that $\lambda = \vee_{i\in K}\, g_i^{-1}(\mu_i)$. Let $\mathcal{B} = \vee_{i\in\Gamma}\, g_i^{-1}(\mathcal{B}_i)$ be given. Then :

(1) $\mathcal{B}$ is a smooth ideal base on X

(2) $\langle \mathcal{B} \rangle$ is the coarsest smooth ideal on X for wich each function $g_i : (X, \langle \mathcal{B} \rangle \to (X_i, \langle \mathcal{B}_i \rangle)$ is an I-map.

(3) A map $g : (Y, I^*) \to (X, \langle \mathcal{B} \rangle)$ is an I-map iff for each $i \in \Gamma$, $g_i \circ g : (Y, I^*) \to (X_i, \langle I_i \rangle)$ is an I-map.

(4) $\langle \vee_{i \in \Gamma} g_i^{-1}(\mathcal{B}_i) \rangle = \langle \vee_{i \in \Gamma} g_i^{-1}(\langle \mathcal{B} \rangle_i) \rangle$.

PROOF : (1) Let $\mathcal{B} = \vee_{i \in \Gamma} g_i^{-1}(\mathcal{B}_i)$. We show that $\mathcal{B}$ is a smooth ideal base.

Since $\mathcal{B}_i$ is nonzero function, there exists $\mu_i \in (\mathcal{B}_i)^0$ such that

$$\mathcal{B}(g_i^{-1}(\mu_i)) \geq \mathcal{B}_i(\mu_i) > 0.$$

Thus, $\mathcal{B}$ is nonzero function.

(SB1) It is trivial that $\mathcal{B}(\tilde{1}) = 0$.

(SB2) Suppose there exist $\lambda_1, \lambda_2 \in I^X$ and $r \in (0, 1)$ such that

$$\langle \mathcal{B}(\lambda_1 \vee \lambda_2) \rangle < r < \mathcal{B}(\lambda_1) \wedge \mathcal{B}(\lambda_2). \qquad \text{... } (D)$$

Since $\mathcal{B}(\lambda_1) > r$ and $\mathcal{B}(\lambda_2) > r$, by definition of $\mathcal{B}$, there exist two finite subsets K and J of Γ with $\lambda_1 = \vee_{k \in K} g_k^{-1}(\nu_k)$ and $\lambda_2 = \vee_{j \in J} g_j^{-1}(\mu_j)$ such that

$$\mathcal{B}(\lambda_1) \geq \wedge_{k \in K} \mathcal{B}_k(\nu_k) > r,$$

$$\mathcal{B}(\lambda_2) \geq \wedge_{j \in J} \mathcal{B}_j(\mu_j) > r.$$

Put $m \in K \bigcup J$ such that

$$\rho_m = \begin{cases} \nu_m & \text{if } m \in K - (K \wedge J), \\ \mu_m & \text{if } m \in J - (K \wedge J), \\ \nu_m \vee \mu_m & \text{if } m \in K \wedge J. \end{cases}$$

For each $m \in K \wedge J$, since $\mathcal{B}_m(\nu_m) > r$ and $\mathcal{B}_m(\mu_m) > r$, we have

$$\langle \mathcal{B}_m \rangle (\nu_m \vee \mu_m) \geq \mathcal{B}_m(\nu_m) \wedge \mathcal{B}_m(\mu_m) > r.$$

From the definition of $\langle \mathcal{B}_m \rangle$, there exists $\omega_m \in I^{X_m}$ with $\omega_m \geq \nu_m \vee \mu_m$ such that

$$\langle \mathcal{B}_m \rangle (\nu_m \vee \mu_m) \geq \mathcal{B}_m(\omega_m) > r. \qquad \text{... } (E)$$

Since

$$\lambda_1 \vee \lambda_2 = (\vee_{k \in K}\, g_k^{-1}(\nu_k)) \vee (\vee_{j \in J}\, g_j^{-1}(\mu_j))$$

$$= \vee_{m \in (K \cup J)}\, g_m^{-1}(\rho_m)$$

$$\leq (\vee_{m \in (K \cup J) - (K \wedge J)}\, g_m^{-1}(\rho_m)) \vee (\vee_{m \in (K \cap J)}\, g_m^{-1}(\omega_m)),$$

there exists a finite index $K \cup J$ such that

$$\langle \mathcal{B} \rangle (\lambda_1 \vee \lambda_2) \geq (\wedge_{m \in (K \cup J) - (K \cap J)}\, \mathcal{B}_m(\rho_m)) \wedge (\wedge_{m \in (K \cap J)}\, \mathcal{B}_m(\omega_m))$$

$$\geq (\wedge_{m \in (K \cup J) - (K \cap J)}\, \mathcal{B}_m(\rho_m)) \wedge r \text{ (by (E))}$$

$$> r.$$

It is a contradiction for the eq. (D). Hence, $\langle \mathcal{B} \rangle (\lambda_1 \vee \lambda_2) \geq \mathcal{B}(\lambda_1) \wedge \mathcal{B}(\lambda_2)$, for all $\lambda_1, \lambda_2 \in I^X$.

(2) From Theorem 2.7(3), we only show that $\mathcal{B}(g_i^{-1}(\lambda_i)) \geq \mathcal{B}_i(\lambda_i)$ for each $i \in \Gamma$ from the following :

If $\mathcal{B}_i(\lambda_i) = 0$, it is trivial.

If $\mathcal{B}_i(\lambda_i) > 0$, for a one family $\{\lambda_i \in \mathcal{B}_i^0\}$, we have $\mathcal{B}(g_i^{-1}(\lambda_i)) \geq \mathcal{B}_i(\lambda_i)$.

Let $I(g_i^{-1}(\lambda_i)) \geq \langle \mathcal{B}_i \rangle (\lambda_i)$ for each $i \in \Gamma$. Suppose there exist $\lambda \in I^X$ and $r \in (0, 1)$ such that

$$\langle \mathcal{B} \rangle (\lambda) > r > I(\lambda). \qquad \ldots (F)$$

Since $\langle \mathcal{B} \rangle (\lambda) > r$, by definition of $\langle \mathcal{B} \rangle$, there exists a finite subset K of Γ with $\lambda \leq \vee_{k \in K}\, g_k^{-1}(\nu_k)$ such that

$$\langle \mathcal{B} \rangle (\lambda) \geq \wedge_{k \in K}\, \mathcal{B}_k(\nu_k) > r.$$

On the other hand, since $I(g_k^{-1}(\nu_k)) \geq \langle \mathcal{B}_k \rangle (\nu_k)$ for all $k \in K$, we have

$$I(\lambda) \geq I \vee_{k \in K}\, \mathcal{B}_k(\nu_k)) \text{ (by (S3))}$$

$$\geq \wedge_{k \in K}\, I(g_k^{-1}(\nu_k))$$

$$\geq \wedge_{k \in K}\, \langle \mathcal{B}_k \rangle (\nu_k)$$

$$\geq \wedge_{k \in K} \mathcal{B}_k (\nu_k)$$

$$> r.$$

It is a contradiction for the eq. (F). Hence $I \geq \langle \mathcal{B} \rangle$.

(3) Necessity of the composition condition is clear since the composition of I-maps is an I-map.

Conversely, suppose $g : (Y, I^*) \to (X, \langle \mathcal{B} \rangle)$ is not an I-map. There exist $\mu \in I^X$ and $r \in (0, \cdot 1)$ such that

$$\langle \mathcal{B} \rangle (\mu) \geq \wedge_{k \in K} \mathcal{B}_k (\nu_k) > r$$

$$r \in (0, 1) \text{ such that} \qquad \dots (G)$$

$$\langle \mathcal{B} \rangle (\mu) > r > I^{*k} (g^{-1} \mu)).$$

Since the other hand, since for each $i \in \Gamma$, $g_i \circ g : (Y, I^*) \to (X_i, \langle \mathcal{B}_i \rangle)$ is a I-map,

$$\langle \mathcal{B} \rangle (\mu) \geq \wedge_{k \in K} \mathcal{B}_k (\nu_k) > r.$$

On the other hand, since for each $i \in \Gamma$, $g_i \circ g : (Y, I^*) \to (X_i, \langle \mathcal{B}_i \rangle)$ is a I-map.

$$\langle \mathcal{B}_i \rangle (\nu_i) \leq I^* (g^{-1} (g_i^{-1} (\nu_i))).$$

It follows $I^* (g^{-1} (g_k^{-1} (\nu_k))) \geq \langle \mathcal{B}_k \rangle (\nu_k)$ for all $k \in K$. Since $g^{-1} (\mu) \leq \vee_{k \in K} g^{-1} (g_k^{-1} (\nu_k))$, we have

$$I^* (g^{-1} (\mu)) \geq \wedge_{k \in K} I^* (g^{-1} (g_k^{-1} (\nu_k))) \text{ (by (S3))}$$

$$\geq \wedge_{k \in K} \langle \mathcal{B}_k \rangle (\nu_k)$$

$$\geq \wedge_{k \in K} \mathcal{B}_k (\nu_k)$$

$$> r.$$

It is a contradiction for the eq. (G). Hence g is an I-map.

(4) Let $I = \langle \vee_{i \in \Gamma} g_i^{-1} (\langle \mathcal{B}_i \rangle) \rangle$. Since $\mathcal{B}_i (\lambda_i) \leq \mathcal{B} (g_i^{-1} (\lambda_i))$, by Theorem 2.7 (3), $g_i : (X, \langle \mathcal{B} \rangle) \to (X_i, \langle \mathcal{B}_i \rangle)$ is an I-map. From (3), the identity map $id_X : (X, \langle \mathcal{B} \rangle) \to (X, I)$ is an I-map. Thus,

$$I \leq \langle \mathcal{B} \rangle.$$

From the definition of $I, \langle \mathcal{B}_i \rangle (\lambda_i) \leq I(g_i^{-1}(\lambda_i))$, that is, $g_i : (X, I) \to (X_i, \langle \mathcal{B}_i \rangle)$ is an I-map. From (2),

$$\langle \mathcal{B} \rangle \leq I.$$

***Theorem* 2.11** — *Let* $\{ I_i \}_{i \in \Gamma}$ *a family of smooth ideals on X satisfying the following condition :*

(C) If $\lambda_i \in (I_i)^0$ *for all* $i \in \Gamma$, *then we have* $\vee_{i \in K} \lambda_i \neq \tilde{1}$ *for every finite subset K of* Γ.

We define a function $\vee_{i \in \Gamma} I_i : I^X \to I$ *as*

$$\bigvee_{i \in \Gamma} I_i(\mu) = \begin{cases} \sup \left\{ \wedge_{i \in K} I_i(\mu_i) \right\} & \text{if } \mu = \vee_{i \in K} \mu_i, \mu_i \in (I_i)^0, \\ \tilde{0} & \text{if } otherwise. \end{cases}$$

where the supremum is taken for every finite subset K of Γ *such that* $\mu = \vee_{i \in K} \mu_i$. *Then I is the coarset smooth ideal finer than* I_i *for each* $i \in \Gamma$.

PROOF : From Theorem 2.10, put $(X_i, \mathcal{B}_i) = (X, I_i)$ and $g_i = id_X$ where id_X is an identity map for each $i \in \Gamma$. Let $I \doteq \vee_{i \in \Gamma} I_i$ be given. We only show that $I = \langle I \rangle$. It is trivially show

$$I \leq \langle I \rangle.$$

Suppose that

$$I \ngeq \langle I \rangle. \qquad \text{... } (H)$$

There exist $\nu \in I^X$ and $r \in (0, 1)$ such that

$$I(\nu) < r < \langle I \rangle (\nu).$$

Since $\langle I \rangle (\nu) > r$, there exists $\mu \in I^X$ with $\nu \leq \mu$ such that

$$\langle I \rangle (\nu) \geq I(\mu).$$

By Definition of I, there exists a finite index set K with $\mu = \vee_{k \in K} \mu_k$ such that

$$I(\mu) \geq \wedge_{k \in K} I_i(\mu_k) > r.$$

On the other hand, since $\nu = \nu \wedge \mu = \vee_{k \in K} (\nu \wedge \mu_k)$, we have

$$\begin{aligned} I(\nu) &\geq \wedge_{k \in K} I_k(\nu \wedge \mu_k) \\ &\geq \wedge_{k \in K} I_k(\mu_k) \\ &> r. \end{aligned}$$

It is a contradiction for the eq. (H). Hence $I \geq \langle I \rangle$.

Example 2.12 — Let $X = \{a, b, c, d\}$ be a set. We define a function $\mathcal{B}: I^X \to I$ as follows:

$$\mathcal{B}(\lambda) = \begin{cases} 0.5, \text{ if, } \lambda = \tilde{0}, \\ 0.4, \text{ if } \lambda \in \left\{ 1_{(a)}, 1_{(b)}, 1_{(a,b,c)} \right\}, \\ 0, \quad \textit{otherwise.} \end{cases}$$

(1) Let $A = \{a, b, c\}$ be set and $i_A : A \to X$ an inclusion map. Since $\overset{-1}{i_A}(1_{(a,b,c)}) = 1_{(a,b,c)}$, we can not define $\overset{-1}{i}(\mathcal{B})$ from the condition (*C*) of Theorem 2.10.

(2) Let $B = \{a, b, d\}$ be set and $i_B : B \to X$ an inclusion map. We can obtain $\overset{-1}{i_B}(\mathcal{B}) : I^B \to I$ as follows :

$$\overset{-1}{i_B}\mathcal{B}(\mu) = \begin{cases} 0.5, \text{ if, } \mu = \tilde{0}, \\ 0.4, \text{ if } \mu \in \left\{ 1_{(a)}, 1_{(b)}, 1_{(a,b)} \right\}, \\ 0, \quad \textit{otherwise.} \end{cases}$$

From Theorem 2.10, we have $\langle \overset{-1}{i_B}(\mathcal{B}) \rangle = \langle \overset{-1}{i_B}(\langle \mathcal{B} \rangle) \rangle$ as follows :

$$\langle \overset{-1}{i_B}(\mathcal{B}) \rangle (\mu) = \begin{cases} 0.5, \text{ if, } \mu = \tilde{0}, \\ 0.4, \text{ if } , \tilde{0} < \mu \le 1_{(a,b)}, \\ 0, \quad \textit{otherwise.} \end{cases}$$

Example 2.13 — Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$ be sets. We define a function $I : 2^Y \to I$ as follows :

$$I(v) = \begin{cases} 1, \text{ if } v = \tilde{0}, \\ \frac{1}{2}, \text{ if } v = x_t, \ \forall t \in (0, 1] \\ 0, \quad \textit{otherwise.} \end{cases}$$

Then I is a smooth ideal on X. Let $g : X \to Y$ be a function as follows :

$$g(a) = g(b) = x, \; g(c) = y.$$

$$g^{-1}(I)(\lambda) = \begin{cases} 1, \text{ if } \lambda = \tilde{0}, \\ \frac{1}{2}, \text{ if } \lambda = t\, 1_{(a,b)}, \ t \in (0, 1] \\ 0, \quad \textit{otherwise.} \end{cases}$$

Then $g^{-1}(I)$ is a smooth base on X. Since

$$\frac{1}{2} = g^{-1}(I)(1_{(a,b)}) \not\leq g^{-1}(I)(1_{(a)}) = g^{-1}(I)(1_{(b)}) = 0,$$

$g^{-1}(I)$ is not a smooth ideal on X.

$$\langle g^{-1}(I) \rangle (\lambda) = \begin{cases} 1, & \text{if } \lambda = \tilde{0}, \\ \frac{1}{2}, & \text{if } \tilde{0} < \lambda \leq 1_{(a,b)}, \\ 0, & \textit{otherwise}. \end{cases}$$

Then $\langle g^{-1}(I) \rangle \neq g^{-1}(I)$.

***Theorem* 2.14** — *Let $\{\mathcal{B}_i\}_{i \in \Gamma}$ a family of smooth ideal bases on X_i. Let $X = \prod_{i \in \Gamma} X_i$ be a product set and $\pi_i : X \to X_i$ a projection map, for each $i \in \Gamma$. We define a function $\vee_{i \in \Gamma} \pi_i^{-1}(\mathcal{B}_i) : I^X \to I$ as*

$$\bigvee_{i \in \Gamma} \pi_i^{-1}(\mathcal{B}_i)(\mu) = \begin{cases} \sup\left\{ \wedge_{i \in K} \mathcal{B}_i(\mu_i) \right\} & \text{if } \mu = \wedge_{i \in K} \pi_i^{-1}(\mu_i), \mu_i \in (\mathcal{B}_i)^0, \\ \tilde{0} & \text{if } \textit{otherwise} \end{cases}$$

where the supremum is taken for every finite subset set K of Γ such that $\mu = \vee_{i \in K} \pi_i^{-1}(\mu_i)$. Let $\mathcal{B} = \vee_{i \in \Gamma} \pi_i^{-1}(\mathcal{B}_i)$ be given

Then :

(1) $\langle \mathcal{B} \rangle$ is the coarsest smooth ideal on X for which each projection map $\pi_i : (X, \langle \mathcal{B} \rangle) \to (X_i, \langle \mathcal{B}_i \rangle$ is an I-map.

(2) A map $f : (Y, I) \to (X, \langle \mathcal{B} \rangle)$ is a I-map iff for each $i \in \Gamma$, $\pi_i \circ f : (Y, I) \to (X_i, \langle \mathcal{B}_i \rangle)$ is a I-map.

PROOF : From Theorem 2.10, we only show that a family $\{\mathcal{B}_i\}_{i \in \Gamma}$ satisfies the condition (C).

Let $\nu_i \in (\mathcal{B}_i)^0$ for all $i \in K \subset \Gamma$ and every finite subset set $K \subset \Gamma$. Since $\mathcal{B}_i(\nu_i) > 0$ for all $i \in K$, we have $\nu_i \neq \tilde{1}$. For each $i \in K$, there exists $x_i \in X_i$ with $\nu_i(x_i) < 1$. Pick up $x \in X$ such that $\pi_i(x) = x_i$ for all $i \in K$. Then $\vee_{i \in K} \pi_i^{-1}(\mu_i)(x) = \vee_{i \in K} \mu_i(\pi_i(x)) < 1$. Thus $\vee_{i \in K} \pi_i^{-1}(\mu_i) \neq \tilde{1}$.

From Theorem 2.14, we can define the product smooth ideal space.

Definition 2.15 — Let $\{ I_i \}_{i \in \Gamma}$ a family of smooth ideals on X_i, $X = \prod_{i \in \Gamma} X_i$ a product set, and the function $\pi_i : X \to X_i$ a projection map, for each $i \in \Gamma$. The structure $\langle \vee_{i \in \Gamma} \pi_i^{-1}(I_i) \rangle$ is called a *product smooth ideal* on X.

Example 2.16 — Let $X = \{a, b\}$ and $Y = \{x, y\}$ be sets. We define smooth ideals $I_1 : I^X \to I$ and $I_2 : I^X \to I$ as follows :

$$I_i(\nu) = \begin{cases} 0.6, & \text{if } \nu = \tilde{0}, \\ 0.4, & \text{if } \nu = a_t, t \in (0, 1] \\ 0, & \textit{otherwise.} \end{cases}$$

$$I_2(\nu) = \begin{cases} 0.7, & \text{if } \nu = \tilde{0}, \\ 0.3, & \text{if } \nu = x_s, t \in (0, 1] \\ 0, & \textit{otherwise.} \end{cases}$$

Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be projection maps.

We can obtain the product smooth ideal $I = \pi_1^{-1}(I_1) \vee \pi_2^{-1}(I_2)$ as follows :

$$I(\lambda) = \begin{cases} 0.7, & \text{if } \lambda = \tilde{0}, \\ 0.4, & \text{if } \tilde{0} < \lambda \le 1_{(a, x), (a, y)}, \\ 0.3, & \text{if } 1_{(a, x), (a, y)} \not\geq \lambda \le 1_{(a, x), (a, y), (b, x)}, \\ 0, & \textit{otherwise.} \end{cases}$$

3. The Images of Smooth Ideals

***Theorem* 3.1** — *Let $f_i : X_i \to X$ be a function, for each $i \in \Gamma$. Let $\{ I_i \}_{i \in \Gamma}$ a family of smooth ideals on X satisfying the following condition :*

(C) If $\mu_i \in (I_i)^0$ for all $i \in \Gamma$, then we have $\vee_{i \in K} f_i(\mu_i) \neq 1$ for every finite subset K of Γ.

We define a function $\vee_{i \in \Gamma} f_i(I_i) : I^X \to I$ as

$$\bigvee_{i \in \Gamma} f_i(I_i(\nu)) = \begin{cases} \sup\left\{\wedge_{i \in K} I_i(\mu_i)\right\} & \text{if } \nu = \vee_{i \in K} f_i(\mu_i), \mu_i \in (I_i)^0, \\ \tilde{0} & \text{if } otherwise. \end{cases}$$

where the supremum is taken for every finite subset K *of* Γ *such that* $\nu = \vee_{i \in K} f_i(\mu_i)$. *Put* $I = \vee_{i \in \Gamma} f_i(I_i)$. *Then* :

(1) I *is the coarsest smooth ideal on* X *for which each function* $f_i : X_i \to X$ *is an* I*-preserving map.*

(2) A map $f : (X, I) \to (Y, \mathcal{H})$ *is a* I*-preserving map iff for each* $i \in \Gamma$, $f \circ f_i : (X_i, I_i) \to (Y, \mathcal{H})$ *is a* I*-preserving map.*

PROOF : First, we show that I is a smooth ideal. Since I_i is nonzero function, there exists $\mu_i \in (I_i)^0$ such that $I(f_i(\mu_i)) \geq I_i(\mu_i) > 0$.

Thus, I is nonzero function.

(S1) It is trivial that $I(\tilde{1}) = 0$.

(S2) Suppose there exist $\nu_1, \nu_2 \in I^Y$ and $r \in (0, 1)$ such that

$$I(\nu_1 \vee \nu_2) < r < I(\nu_1) \wedge I(\nu_2). \qquad \ldots (L)$$

Since $I(\nu_1) > r$ and $I(\nu_2) > r$, by definition of I, there exist two finite subset K and J of Γ with $\nu_1 = \vee_{k \in K} f_k(\lambda_k)$ and $\nu_2 = \vee_{j \in J} f_j(\mu_j)$ such that

$$I(\nu_1) \geq \wedge_{k \in K} I_k(\lambda_k) > r,$$

$$I(\nu_2) \geq \wedge_{j \in J} I_j(\mu_j) > r.$$

Put $m \in K \bigcup J$ such that

$$\rho_m = \begin{cases} \lambda_m & \text{if } m \in K - (K \bigcap J) \\ \mu_m & \text{if } m \in J - (K \bigcap J) \\ \lambda_m \vee \mu_m & \text{if } m \in K \bigcap J. \end{cases}$$

Since $\nu_1 \vee \nu_2 = (\vee_{k \in K} f_k(\lambda_k)) \vee (\vee_j f_j(\mu_j)) = \vee_{m \in (K \bigcup J)} f_m(\rho_m)$, there exists a finite subset $K \bigcup J$ of Γ such that

$$I(\nu_1 \vee \nu_2) \geq (\wedge_{m \in (K \cup J)} I_m(\rho_m))$$

$$\geq (\wedge_{k \in K} I_k(\lambda_k)) \wedge (\wedge_{j \in J} I_j(\mu_j))$$

$$> r.$$

It is a contradiction for the eq. (L). Hence, $I(\nu_1 \vee \nu_2) \geq I(\nu_1) \wedge I(\nu_2)$, for all $\nu_1, \nu_2 \in I^Y$.

Suppose there exist $\lambda, \mu \in I^X$ with $\lambda \leq \mu$ and $r \in (0, 1)$ such that

$$I(\lambda) < r < I(\mu). \qquad \text{... } (M)$$

Since $I(\mu) > r$, by definition of I, there exists a finite subset K of Γ with $\mu = \vee_{k \in K} \mu_k$ such that

$$I(\mu) \geq \wedge_{k \in K} I(\mu_k) > r.$$

On the other hand, since $\lambda = \lambda \wedge \mu = \vee_{k \in K}(\lambda \wedge \mu_k)$, we have

$$I(\lambda) \geq \wedge_{k \in K} I_k(\lambda \wedge \mu_k)$$

$$\geq \wedge_{k \in K} I_k(\mu_k)$$

$$> r.$$

It is a contradiction for the eq. (M). Hence, if $\lambda \leq \mu$, then $I(\lambda) \geq I(\mu)$.

We will show that $I(f_i(\lambda_i)) \geq I_i(\lambda_i)$ for each $i \in \Gamma$ from the following :

If $I_i(\lambda_i) = 0$, it is trivial.

If $I_i(\lambda_i) > 0$, for a one family $\{\lambda_i \in I_i^0\}$, we have $I(f_i(\lambda_i)) \geq I_i(\lambda_i)$.

If $f_i : (X_i, I_i) \to (X, \mathcal{K})$ is an I-preserving map, that is, $\mathcal{K}(f_i(\lambda_i)) \geq I_i(\lambda_i)$ for each $i \in \Gamma$, we will show that $\mathcal{K} \geq I$.

Suppose there exist $\nu \in I^X$ and $r \in (0, 1)$ such that

$$I(\nu) > r > \mathcal{K}(\nu). \qquad \text{... } (N)$$

Since $I(\lambda) > r$, by definition of I, there exists a finite index set K with $\nu = \vee_{k \in K} f_k(\lambda_k)$ such that

$$I(\nu) \geq \wedge_{k \in K} I_k(\lambda_k) > r.$$

On the other hand, since $\mathcal{K}(f_k(\lambda_k)) \geq I_k(\lambda_k)$ for all $k \in K$, we have

$$\mathcal{K}(\nu) \geq \wedge_{k \in K} \mathcal{K}(f_k(\lambda_k)) \text{ (by (S2))}$$

$$\geq \wedge_{k \in K} I_k(\lambda_k) > r.$$

It is a contradiction for the equation (N).

(3) Necessity of the composition condition is clear since the composition of I-preserving maps is an I-preserving map.

Conversely, suppose $f : (X, I) \to (Y, \mathcal{H})$ is not an I-preserving map.

Suppose there exist $\mu \in I^X$ and $r \in (0, 1)$ such that

$$I(\mu) > r > \mathcal{H}(f(\mu)). \qquad \ldots (O)$$

Since $I(\mu) > r$, by definition of I, there exists a finite subset K of Γ with $\mu = \vee_{k \in K} f_k(\lambda_k)$ such that

$$I(\mu) \geq \wedge_{k \in K} I_k(\lambda_k) > r.$$

On the other hand, since for each $i \in \Gamma$, $f \circ f_i : (X_i, I_i) \to (Y, \mathcal{H})$ is a I-preserving map,

$$I_i(\lambda_i) \leq \mathcal{H}(f(f_i(\lambda_i))).$$

So, $\mathcal{H}(f(f_k(\lambda_k))) \geq I_k(\lambda_k)$ for all $k \in K$. Since $f(\mu) = \vee_{k \in K} f(f_k(\lambda_k))$, we have

$$\mathcal{H}(f(\mu)) \geq \wedge_{k \in K} \mathcal{H}(f(f_k(\lambda_k)))$$

$$\geq \wedge_{k \in K} I_k(\lambda_k)$$

$$> r.$$

It is a contradiction the eq. (O). Hence, f is an I-preserving map.

Corollary 3.2 — Let $f_i : X_i \to X$ be a function, for each $i \in \Gamma$. Let $\left\{ \mathcal{B}_i \right\}_{i \in \Gamma}$ a family of smooth ideal bases on X satisfying the following condition :

(C) If $\mu_i \in (\mathcal{B}_i)^0$ for all $i \in \Gamma$, then we have $\vee_{i \in K} f_i(\mu_i) \neq \tilde{1}$ for every finite subset K of Γ.

We define a function $\vee_{i \in \Gamma} f_i(\mathcal{B}_i) : I^X \to I$ as

$$\bigvee_{i\in\Gamma} f_i(\mathcal{B}_i)(\nu) = \begin{cases} \sup\left\{\wedge_{i\in K}\mathcal{B}_i(\mu_i)\right\} & \text{if } \nu=\vee_{i\in K} f_i(\mu_i),\ \tilde{0}\neq\mu_i\in(\mathcal{B}_0)^0, \\ \tilde{0} & \text{if } otherwise. \end{cases}$$

where the supremum is taken for every finite subset K of Γ such that $\nu=\vee_{i\in K} f_i(\mu_i)$. Put $\mathcal{B}=\vee_{i\in\Gamma} f_i(\mathcal{B}_i)$.

Then :

(1) $\vee_{i\in\Gamma} f_i(\mathcal{B}_i)$ is the smooth ideal base on X for which each function $\mathcal{B}_i(\lambda_i)\le\vee_{i\in\Gamma} f_i(\mathcal{B}_i)(f_i(\lambda_i))$ for each $\lambda_i\in I^{X_i}$.

(2) $\langle\mathcal{B}\rangle=\bigvee_{i\in\Gamma} f_i(\langle\mathcal{B}_i\rangle)$.

(3) A map $f:(X,\langle\mathcal{B}\rangle)\to(Y,\mathcal{H})$ is an I-preserving map iff for each $i\in\Gamma, f\circ f_i:(X_i,\vee\mathcal{B}_i\rangle)\to(Y,\mathcal{H})$ is an I-preserving map.

PROOF : (1) and (3) are similarly proved from Theorem 3.1,

(2) Let $I=\vee_{i\in\Gamma} f_i(\langle\mathcal{B}_i\rangle)$. Since $\mathcal{B}_i(\lambda_i)\le\mathcal{B}(f_i(\lambda_i))$, by Theorem 2.7 (4), $f_i:(X_i,\langle\mathcal{B}_i\rangle)\to(X,\langle\mathcal{B}\rangle)$ is an I-preserving map. Since $id_X\circ f_i:(X_i,\langle\mathcal{B}_i\rangle)\to(X,\langle\mathcal{B}\rangle)$ is an I-preseving map, by Theorem 3.1 (2), the identity map $id_X:(X,I)\to(X,\langle\mathcal{B}\rangle)$ is an I-preserving map. Thus,

$$I\le\langle\mathcal{B}\rangle.$$

We have

$$\left\{\mu_i\in I^{X_i}\mid\mu_i\in\mathcal{B}_i^0\right\}\subset\left\{\lambda_i\in I^{X_i}\mid\lambda_i\in\langle\mathcal{B}_i\rangle^{0)}\right\}.$$

Thus, $I\ge\mathcal{B}$. From Theorem 2.5, it implies

$$I\ge\langle\mathcal{B}\rangle.$$

Example 3.4 — Let $X=\{a,b,c\}$, and $Y=\{x,y\}$ be sets. We define smooth ideal $I:I^X\to I$ as follows :

$$I(\lambda)=\begin{cases} 0.6, & \text{if } \nu=\tilde{0} \\ 0.4, & \text{if } \tilde{0}\neq\lambda\le\mu \\ 0, & otherwise \end{cases}$$

where $\mu(a) = 0.5, \mu(b) = 1, \mu(c) = 1$. Let $f: X \to Y$ be defined by $f(a) = x$, $f(b) = x$, $f(c) = y$. Since $f(\mu)(x) = 1, f(\mu)(y) = 1$, that is, $f(\mu) = \tilde{1}$, we cannot define a smooth ideal $f(I)$ from the condition (C) of Theorem 3.1.

Example 3.5 — Let $X = \{a, b, c, d\}$ be a set. We defne smooth ideal bases $\mathcal{B}_i : I^X \to I$ as follows :

$$\mathcal{B}_1(\lambda) = \begin{cases} 0.5, & \text{if, } \lambda = \tilde{0}, \\ 0.4, & \text{if, } \lambda \in \left\{ 1_{(a)}, 1_{(b)}, 1_{(a,b,c)} \right\}, \\ 0, & \textit{otherwise.} \end{cases}$$

$$\mathcal{B}_2(\mu) = \begin{cases} 0.6, & \text{if, } \mu = \tilde{0}, \\ 0.3, & \text{if, } \mu \in \left\{ 1_{(c)}, 1_{(b)}, 1_{(a,b,c)} \right\}, \\ 0, & \textit{otherwise.} \end{cases}$$

$$\mathcal{B}_3(\mu) = \begin{cases} 0.6, & \text{if, } \mu = \tilde{0}, \\ 0.3, & \text{if, } \mu \in \left\{ 1_{(c)}, 1_{(b)}, 1_{(a,b,d)} \right\}, \\ 0, & \textit{otherwise.} \end{cases}$$

We obtain,

$$\langle \mathcal{B}_1 \rangle (\lambda) = \begin{cases} 0.5, & \text{if, } \lambda = \tilde{0}, \\ 0.4, & \text{if, } \tilde{0} < \lambda \leq 1_{(a,b,c)}, \\ 0, & \textit{otherwise.} \end{cases}$$

$$\langle \mathcal{B}_2 \rangle (\mu) = \begin{cases} 0.6, & \text{if, } \lambda = \tilde{0}, \\ 0.3, & \text{if, } \tilde{0} < \mu \leq 1_{(a,b,c)}, \\ 0, & \textit{otherwise.} \end{cases}$$

Let $id_X : X \to X$ be an identity function. We cannot define $id_X(\mathcal{B})_1 \vee id_X(\mathcal{B})_2$ because

$$1_{(a,b,d)} \vee = 1_X.$$

We have :

$$id_X(\mathcal{B})_1 \vee id_X(\mathcal{B})_2(\mu) = \begin{cases} 0.6, & \text{if, } \mu = \tilde{0}, \\ 0.4, & \text{if, } \mu \in \left\{ 1_{(a)}, 1_{(b)}, 1_{(a,b,c)} \right\} \\ 0.3, & \text{if, } \mu \in \left\{ 1_{(c)}, 1_{(b,c)}, 1_{(a,c)}, 1_{(a,b)} \right\}, \\ 0, & \textit{otherwise.} \end{cases}$$

$$\langle id_X(\mathcal{B}_1) \vee id_X(\mathcal{B}_2) \rangle (\mu) = \begin{cases} 0.6, & \text{if, } \mu = \tilde{0}, \\ 0.4, & \text{if, } \tilde{0} < \mu \leq 1_{(a,b,c)}, \\ 0, & \textit{otherwise.} \end{cases}$$

Thus $$\langle id_X(\mathcal{B}_1) \vee id_X(\mathcal{B}_2) \rangle = id_X(\langle \mathcal{B}_1 \rangle) \vee id_X(\langle \mathcal{B}_2 \rangle).$$

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