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ON BLOW-UP OF SOLUTIONS FOR A CLASS OF NONLINEAR PARABOLIC EQUATIONS WITH NEUMANN BOUNDARY CONDITIONS*

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The Hopf's maximum principles are utilized to deal with the problem on blow-up of the solutions for a class of nonlinear parabolic equations $u_{,t} = \nabla(a(u)\nabla u) + g(x,t)f(u)$, subject to Neumann boundary conditions. Some nonexistence theorems of global solutions and the bounds of "blow-up time" are obtained.

Key Words : Nonlinear Parabolic Equations; Blow-up Solutions; Blow-up Time; Maximum Principles

1. INTRODUCTION

Blow-up solutions are studied by many authors. Paper [1 & 2] discussed the following problems :

$$\begin{cases} u_{,t} = \Delta u & \text{in } D \times (0, T), \\ \dfrac{\partial u}{\partial n} = b(u) & \text{on } \partial D \times (0, T), \\ u(x, 0) = u_0(x) > 0 & \text{in } \overline{D}, \end{cases}$$

where n is outer normal vector and $\overline{D}$ is closure of D. Paper [3 & 4] researched the following problems :

$$\begin{cases} u_{,t} = \nabla(a(u)\nabla u)u & \text{in } D \times (0, T), \\ \dfrac{\partial u}{\partial n} = g(x, t) & \text{on } \partial D \times (0, T), \\ u(x, 0) = u_0(x) > 0 & \text{in } \overline{D}, \end{cases}$$

where ∇ is gradient symbol. Paper [5] deals with the following problems :

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$$\begin{cases} u_{,t} = \nabla(a(u)\nabla u) & \text{in } D\times(0,T), \\ \dfrac{\partial u}{\partial n} = b(u) & \text{on } \partial D\times(0,T), \\ u(x,0) = u_0(x) > 0 & \text{in } \overline{D}, \end{cases}$$

The above three problems can be regarded as heat conduction problems in which there are no heat source in D, and the heat supply is provided through the boundary ∂D. The blow-up and global solutions of these problems reflect the balance between the diffusion and heat supply. However, in physics and engineering there are a variety of heat conduction problems in which there is heat source in D and ∂D is insulated against heat. In order to resolve the balance problems between the diffusion and the heat supply, one needs to discuss the blow-up and global solutions of the following problem :

$$u_{,t} = \nabla(a(u)\nabla u) + g(x,t)f(u) \quad \text{in } D\times(0,T), \qquad \text{... (1.1)}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial D\times(0,T), \qquad \text{... (1.2)}$$

$$u(x,0) = u_0(x) \geq 0 \quad \text{in } \overline{D}, \qquad \text{... (1.3)}$$

where $D\subset R^N$ is a smooth bounded domain, $N\geq 2, 0<T<+\infty$.

In this paper, we shall consider the problems (1.1)-(1.3). The function a is assumed to be a positive C^2 function, the function f a nonnegative C^2 function, and the function g a nonnegative C^1 function. We obtain some nonexistence theorems of global solution u and bounds for the "blow-up time" T^*. Our results in this paper extend and supplement those obtained in [1-5]. Our approach depends heavily upon Hopf's maximum principles.

The content of this paper is organized as follows. In section 2, we shall give the main result and its proof. In section 3, we shall give a few examples to which the theorems obtained in this paper may be applied.

2. The Main Result and its Proof

Our main result is the following theorem :

***Theorem* 1** — *Let u be a $C^3(D\times(0,T))\cap C^2(\overline{D}\times[0,T))$ solution of (1.1)-(1.3). Suppose that*

(i) For $s\in R$,

$$a(s)>0, a'(s)\geq 0, f(s)\geq 0, \left(\frac{f'(s)}{a(s)}\right)\geq 0. \qquad \text{... (2.1)}$$

(ii) For $x\in D$ and $t\in R^+$,

$$g(x, t) \geq 0, \; g_t(x, t) \geq 0. \qquad \ldots (2.2)$$

(*iii*) At the point $x \in \overline{D}$ where $f(u_0(x)) = 0$,

$$a(u_0)\,\Delta u_0 + a'(u_0)\,|\nabla u_0|^2 \geq 0. \qquad \ldots (2.3)$$

(*iv*) The constant

$$\beta = \min_{D_1} \left\{ \frac{a(u_0)}{f(u_0)} \left[a(u_0)\,\Delta u_0 + a'(u_0)\,|\nabla u_0|^2 + g(x, 0) f(u_0) \right] \right\} > 0, \qquad \ldots (2.4)$$

where $\quad D_1 = \left\{ x \mid x \in \overline{D}, f(u_0(x)) \neq 0 \right\} \neq \phi.$

(*v*) The integration

$$\int_{M_0}^{+\infty} \frac{a(s)}{f(s)}\, ds < +\infty, \qquad \ldots (2.5)$$

where $\quad M_0 = u_0(x_0) = \max_{\overline{D}} u_0(x).$

Then $u(x, t)$ must blow-up in finite time T^* and

$$T^* \leq \frac{1}{\beta} \int_{M_0}^{+\infty} \frac{a(s)}{f(s)}\, ds. \qquad \ldots (2.6)$$

PROOF : It follows from (1.1) that

$$a\,\Delta u - u_{,t} = -a'\,|\nabla u|^2 - gf \leq 0. \qquad \ldots (2.7)$$

From (2.7), (1.2) and (1.3), it is easy to see that the solution $u(x, t)$ is nonnegative. Consider the function

$$h = -a(u)\,u_{,t} + \beta f(u), \qquad \ldots (2.8)$$

from which we find

$$\nabla h = -a'\,u_{,t}\,\nabla u - a\,\nabla u_{,t} + \beta f'\,\nabla u, \qquad \ldots (2.9)$$

$$\Delta h = -a''\,u_{,t}\,|\nabla u|^2 - 2a'\,\nabla u \cdot \nabla u_{,t} - a'\,u_{,t}\,\Delta u - a\,\Delta u_{,t} + \beta f''\,|\nabla u|^2 + \beta f'\,\Delta u \qquad \ldots (2.10)$$

and

$$h_{,t} = -a'\,(u_{,t})^2 - a u_{,tt} + \beta f'\,u_{,t}$$

$$= -a'\,(u_{,t})^2 - a\,(a\,\Delta u + a'\,|\nabla u|^2 + gf)_{,t} + \beta f'\,u_{,t}$$

$$= -a'(u_{,t})^2 - aa'u_{,t}\Delta u - a^2\Delta u_{,t} - aa''u_{,t}|\nabla u|^2 - 2aa'\nabla u\cdot\nabla_{u,t}$$

$$- ag_t'f - agf'u_{,t} + \beta f'u_{,t} \qquad \text{... (2.11).}$$

By (2.10) and (2.11), it follows that

$$a\Delta h - h_{,t} = \beta af''|\nabla u|^2 + \beta af'\Delta u + a'(u_{,t})^2 + ag'_t f + agf'u_{,t} - \beta f'u_{,t}. \qquad \text{... (2.12)}$$

With (1.1), we have

$$\Delta u = \frac{1}{a}(u_{,t} - a'|\nabla u|^2 - gf). \qquad \text{... (2.13)}$$

Combining (2.12) and (2.13), it follows that

$$a\Delta h - h_{,t} = \beta af''|\nabla u|^2 - \beta a'f'|\nabla u|^2 - \beta gff' + a'(u_{,t})^2 + ag'_t f + agf'u_{,t}. \qquad \text{... (2.14)}$$

By (2.8), we have

$$u_{,t} = \frac{1}{a}(-h + \beta f). \qquad \text{... (2.15)}$$

By means of (2.14) and (2.15) we find

$$a\Delta h + gf'h - h_{,t} = a^2\left(\frac{f'}{a}\right)'\beta|\nabla u|^2 + a'(u_{,t})^2 + ag'_t f. \qquad \text{... (2.16)}$$

The assumptions (2.1) and (2.2) guarantee that right side in the equality (2.16) is nonnegative, i.e.,

$$a\Delta h + gf'h - h_{,t} \geq 0. \qquad \text{... (2.17)}$$

It follows from (2.3) and (2.4) that

$$\max_{\overline{D}} h(x,0)$$

$$= \max_{\overline{D}} \left\{-a(u_0)[a(u_0)\Delta u_0 + a'(u_0)|\nabla u_0|^2 + g(x,0)f(u_0)] + \beta f(u_0)\right\}$$

$$= 0. \qquad \text{... (2.18)}$$

On $\partial D\times(0,T)$ we have $\dfrac{\partial u_{,t}}{\partial n} = 0$

and therefore $\dfrac{\partial h}{\partial n} = -a'u_{,t}\dfrac{\partial u}{\partial n} - a\dfrac{\partial u_{,t}}{\partial n} + \beta f'\dfrac{\partial u}{\partial n} = 0.$ ... (2.19)

By Hopf's maximum principles [6], we find that in $\overline{D}\times[0,T)$, the maximum of h is 0. Hence we have $\overline{D}\times[0,T)$, $h\leq 0$ and

$$u_{,t}\geq\frac{\beta f(u)}{a(u)}.$$

At the point x_0 where $u_0(x_0)=M_0$, we get by integration

$$\frac{1}{\beta}\int_{M_0}^{u(x_0,t)}\frac{a(s)}{f(s)}\,ds\geq t.$$

By using the assumption (2.5), it follows that $u(x,t)$ must blow-up for a finite value $t=T^*$. Further the following inequality must hold :

$$T^*\leq\frac{1}{\beta}\int_{M_0}^{+\infty}\frac{a(s)}{f(s)}\,ds.$$

The proof of Theorem 1 is complete.

In Theorem 1, if $a(u)\equiv 1$, then the following conclusion holds :

Corollary 1 — Let u be a $C^3(D\times(0,T))\cap C^2(\overline{D}\times[0,T))$ solution of the problem

$$\begin{cases} u_{,t}=\Delta u+g(x,t)f(u) & \text{in } D\times(0,T), & \quad \ldots (2.20)\\ \dfrac{\partial u}{\partial n}=0 & \text{on } \partial D\times(0,T), & \quad \ldots (2.21)\\ u(x,0)=u_0(x)\geq 0 & \text{in } \overline{D}, & \quad \ldots (2.22)\end{cases}$$

where $D\subset R^N$, $N\geq 2$, $0<T<+\infty$. Suppose that

(*i*) For $s\in R$, $f(s)\geq 0$, $f''(s)\geq 0$. ... (2.23)

(*ii*) For $x\in D$ and $t\in R^+$,

$$g(x,t)\geq 0,\ g_t'(x,t)\geq 0. \qquad \ldots (2.24)$$

(*iii*) At the point $x\in\overline{D}$ where $f(u_0(x))=0$,

$$\Delta u_0\geq 0. \qquad \ldots (2.25)$$

(*iv*) The constant

$$\beta=\min_{D_1}\left\{g(x,0)+\frac{\Delta u_0}{f(u_0)}\right\}>0, \qquad \ldots (2.26)$$

where $D_1 = \left\{ x \mid x \in \overline{D}, f(u_0(x)) \neq 0 \right\} \neq \phi.$

(*v*) The integration

$$\int_{M_0}^{+\infty} \frac{1}{f(s)} ds < +\infty, \qquad \text{... (2.27)}$$

where $M_0 = u_0(x_0) = \max_{\overline{D}} u_0(x).$

Then $u(x, t)$ must blow-up in finite time T^* and

$$T^* \leq \frac{1}{\beta} \int_{M_0}^{+\infty} \frac{1}{f(s)} ds. \qquad \text{... (2.28)}$$

In Theorem 1, if $u_0(x) \equiv c$ (constant) ≥ 0, then the following conclusion holds :

Corollary 2 — Let u be a $C^3(D \times (0, T)) \cap C^2(\overline{D} \times [0, T))$ solution of the problem

$$\begin{cases} u_{,t} = \nabla(a(u)\nabla u) + g(x,t) f(u) & \text{in } D \times (0, T), & \text{... (2.29)} \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial D \times (0, T), & \text{... (2.30)} \\ u(x, 0) = u_0(x) \equiv c \geq 0 & \text{in } \overline{D}, & \text{... (2.31)} \end{cases}$$

where $D \subset R^N, N \geq 2, 0 < T < +\infty$. Suppose that

(*i*) For $s \in R$,

$$a(s) > 0, a'(s) \geq 0, f(s) > 0, \left(\frac{f'(s)}{a(s)} \right)' \geq 0. \qquad \text{... (2.32)}$$

(*ii*) For $x \in D$ and $t \in R^+$,

$$g(x, t) \geq 0, g_t'(x, t) \geq 0. \qquad \text{... (2.33)}$$

(*iii*) The constant

$$\beta = a(c) \min_{\overline{D}} g(x, 0) > 0. \qquad \text{... (2.34)}$$

(*iv*) The integration

$$\int_{c}^{+\infty} \frac{a(s)}{f(s)} ds < +\infty. \qquad \text{... (2.35)}$$

Then $u(x, t)$ must blow-up in finite time T^* and

$$T^* \leq \frac{1}{\beta} \int_c^{+\infty} \frac{a(s)}{f(s)} ds. \qquad \dots (2.36)$$

In Theorem 1, if $g(x, t) \equiv 1$, then the following conclusion holds :

Corollary 3 — Let u be a $C^3(D \times (0, T)) \cap C^2(\overline{D} \times [0, T))$ solution of the problem

$$\begin{cases} u_{,t} = \nabla(a(u) \nabla u) + f(u) & \text{in } D \times (0, T), \qquad \dots (2.37) \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial D \times (0, T), \qquad \dots (2.38) \\ u(x, 0) = u_0(x) \geq 0 & \text{in } \overline{D}, \qquad \dots (2.39) \end{cases}$$

where $D \subset R^N, N \geq 2, 0 < T < +\infty$. Suppose that

(*i*) For $s \in R$,

$$a(s) > 0, a'(s) \geq 0, f(s) \geq 0, \left(\frac{f'(s)}{a(s)}\right) \geq 0. \qquad \dots (2.40)$$

(*ii*) At the point $x \in \overline{D}$ where $f(u_0(x)) = 0$,

$$a(u_0) \Delta u_0 + a'(u_0) | \nabla u_0 |^2 \geq 0. \qquad \dots (2.41)$$

(iii) The constant

$$\beta = \min_{D_1} \left\{ \frac{a(u_0)}{f(u_0)} [a(u_0) \Delta u_0 + a'(u_0) | \nabla u_0 |^2 + f(u_0)] \right\} > 0. \qquad \dots (2.42)$$

where $$D_1 = \left\{ x \mid x \in \overline{D}, f(u(x)) \neq 0 \right\} \neq \phi.$$

(*iv*) The integration

$$\int_{M_0}^{+\infty} \frac{a(s)}{f(s)} ds < +\infty, \qquad \dots (2.43)$$

where $$M_0 = u_0(x_0) = \max_{\overline{D}} u_0(x).$$

Then $u(x, t)$ must blow-up in finite time T^* and

$$T^* \leq \frac{1}{\beta} \int_{M_0}^{+\infty} \frac{a(s)}{f(s)} ds. \qquad \dots (2.44)$$

3. Concluding Remarks and Applications

One may give extensions of the blow-up problem for a uniformly parabolic equation $u_{,t} = \nabla(a(u)\nabla u) + f(x, u, t)$ under suitable assumptions, as shown in [7].

In the following, we present a few examples to which Theorem 1 and its Corollary 1-3 may be applied.

Example 1 — Let u be a $C^3(D\times(0,T))\cap C^2(\overline{D}\times[0,T))$ solution of the problem

$$\begin{cases} u_{,t} = \Delta u + \left(1 + \sum_{i=1}^{N} x_i^2 + t\right) u^2 & \text{in } D\times(0,T), \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial D\times(0,T), \\ u(x,0) = u_0(x) = 1 & \text{in } \overline{D}, \end{cases}$$

where

$$D = \left\{ x \;\middle|\; \sum_{i=1}^{N} x_i^2 < 1 \right\},\ N \ge 2,\ 0 < T < +\infty .$$

We now have

$$f(u) = u^2,\ g(x,t) = 1 + \sum_{i=1}^{N} x_i^2 + t,\ u_0(x) = 1.$$

It is easy to check that (2.23), (2.24) and (2.25) hold. By (2.26), we find $\beta = 1$. It follows from Corollary 1 that $u(x, t)$ must blow-up in finite time T^* and

$$T^* \le \frac{1}{\beta} \int_{M_0}^{+\infty} \frac{1}{f(s)}\, ds = \int_{1}^{+\infty} \frac{1}{s^2}\, ds = 1.$$

Example 2 — Let u be a $C^3(D\times(0,T))\cap C^2(\overline{D}\times[0,T))$ solution of the problem

$$\begin{cases} u_{,t} = \nabla(e^u\, \nabla u) + \left(1 + \sum_{i=1}^{N} x_i^2 + t\right) e^{2u} & \text{in } D\times(0,T), \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial D\times(0,T), \\ u(x,0) = u_0(x) = 1 & \text{in } \overline{D}, \end{cases}$$

where

$$D = \left\{ x \;\middle|\; \sum_{i=1}^{N} x_i^2 < 1 \right\}, N \geq 2, 0 < T < +\infty .$$

We now have

$$a(u) = e^u, f(u) = e^{2u}, g(x, t) = 1 + \sum_{i=1}^{N} x_i^2 + t, u_0(x) = 1.$$

It is easy to check that (2.32) and (2.33) hold. By (2.35), we find $\beta = e$. It follows from Corollary 2 that $u(x, t)$ must blow-up in finite time T^* and

$$T^* \leq \frac{1}{\beta} \int_1^{+\infty} \frac{a(s)}{f(s)} ds = \frac{1}{e} \int_1^{+\infty} e^{-s} ds = \frac{1}{e^2}.$$

Example 3 — Let u be a $C^3(D \times (0, T)) \cap C^2(\overline{D} \times [0, T))$ solution of the problem

$$\begin{cases} u_{,t} = \nabla(e^u \nabla u) + e^{2u} & \text{in } D \times (0, T), \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial D \times (0, T), \\ u(x, 0) = u_0(x) = 1 & \text{in } \overline{D}, \end{cases}$$

where

$$D \subset R^N, N \geq 2, 0 < T < +\infty.$$

We now have

$$a(u) = e^u, f(u) = e^{2u}, u_0(x) = 1.$$

It is easy to check that (2.40) holds. By (2.42), we find $\beta = e$. It follows from Corollary 3 that $u(x, t)$ must blow-up in finite time T^* and

$$T^* \leq \frac{1}{\beta} \int_1^{+\infty} \frac{a(s)}{f(s)} ds = \frac{1}{e} \int_1^{+\infty} e^{-s} ds = \frac{1}{e^2}.$$

Example 4 — Let u be a $C^3(D\times(0,T))\cap C^2(\overline{D}\times[0,T))$ solution of the problem

$$\begin{cases} u_{,t}=\nabla(e^u\nabla u)+\left(t+\sum_{i=1}^{2}x_i^2\right)16\,e^{2u} & \text{in } D\times(0,T),\\ \dfrac{\partial u}{\partial n}=0 & \text{on } \partial D\times(0,T),\\ u(x,0)=u_0(x)=2\sum_{i=1}^{2}x_i^2-\left(\sum_{i=1}^{2}x_i^2\right)^2 & \text{in } \overline{D}, \end{cases}$$

where

$$D=\left\{x\;\middle|\;\sum_{i=1}^{N}x_i^2<1\right\},\ 0<T<+\infty.$$

We now have

$$a(u)=e^u, f(u)=16\,e^{2u}, g(x,t)=t+\sum_{i=1}^{2}x_i^2, u_0(x)=2\sum_{i=1}^{2}x_i^2-\left(\sum_{i=1}^{2}x_i^2\right)^2.$$

It is easy to check that (2.1) and (2.2) hold. In order to determine β, suppose

$$s=\sum_{i=1}^{2}x_i^2,$$

then $0\le s\le 1$ and

$$\beta=\min_{\overline{D}}\left\{\frac{a(u_0)}{f(u_0)}\left[a(u_0)\Delta u_0+a'(u_0)\,|\nabla u_0|^2+g(x,0)f(u_0)\right]\right\}$$

$$=\min_{\overline{D}}\left\{\frac{1}{16}e^{-u_0}\left[e^{u_0}\Delta u_0+e^{u_0}|\nabla u_0|^2+16\,e^{2u_0}\sum_{i=1}^{2}x_i^2\right]\right\}$$

$$=\min_{\overline{D}}\left\{\frac{1}{16}\Delta u_0+\frac{1}{16}|\nabla u_0|^2+e^{u_0}\sum_{i=1}^{2}x_i^2\right\}$$

$$=\min_{\overline{D}}\left\{\frac{1}{2}\left(1-2\sum_{i=1}^{2}x_i^2\right)+\sum_{i=1}^{2}x_i^2\left(1-\sum_{i=1}^{2}x_i^2\right)^2+e^{u_0}\sum_{i=1}^{2}x_i^2\right\}$$

$$= \min_{0 \le s \le 1} \left\{ \frac{1}{2}(1-2s) + s(1-s)^2 + se^{2s-s^2} \right\} = \frac{1}{2}.$$

It follows from Theorem 1 that $u(x, t)$ must blow-up in finite time T^* and

$$T^* \le \frac{1}{\beta} \int_{M_0}^{+\infty} \frac{a(s)}{f(s)} ds = 2 \int_{1}^{+\infty} e^{-s} ds = \frac{2}{e}.$$

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