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CHAOS IN ATTITUDE MOTION OF A SATELLITE UNDER A THIRD BODY TORQUE IN AN ELLIPTIC ORBIT (I)

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Attitude motion of a satellite in an elliptic orbit under the influence of a third body torque is being studied in Papers (I) and (II). Paper (I) deals with the Melnikov's method and Paper (II) deals with non-resonance case, the resonance case and the chaotic nature of the given dynamical system. In paper (I), by using the Melnikov's metod we have shown that the equations of motion are non-integrable. We have also shown this non-integrability by graphical representation of Melnikov's function in the Earth-Moon-Artificial Satellite (1958 B2 Vanguard 1) system.

Key Words : Chaos; Celestial Mechanics, Solar System

1. INTRODUCTION

Planar oscillations of a satellite in an elliptic orbit have been studied by Beletskii[1], Cherousko[2], Zlatanstov *et al.*[5], Singh[3 & 4]. None of them have taken the third body effect. Bhatnagar and Bhardwaj[6] have taken the effect of third body torque, but the satellite is assumed to move in an almost circular orbit. We have modified the problem by taking the orbit of the satellite as elliptic. We have determined hyperbolic equilibrium solution and double asymptotic solutions corresponding to unperturbated hamiltonian H_0. The non-integrability of the system has been shown through Melnikov's integral.

Poincare[7], has discovered that transversal crossing of asymptotic surfaces of unstable periodic solution leads to complex structure of phase curves. Melnikov's result has allowed us to formulate theorems about non-integrability of systems with transversal homoclinic (Hetroclinic) orbits.

Let us consider 2π-periodic Hamiltonian system with one degree of freedom. Hamiltonian function is assumed to be analytic with respect to its arguments and depends on small parameter :

$$H = (H(x, t, e) = H_0(x) + e H_1(x, t) + 0(e^2), x = (q, p) \quad \text{... (1.1)}$$

We assume that unperturbed system ($e = 0$) possess hyperbolic equilibrium $x_0 = 0$ and let $\bar{x}(t)$ be double asymptotic solution to x_0 i.e., lim $x(t) = x_0$ as $t \to \pm\infty$. In the extended phase space (x, t), we have two asymptotic surfaces $\overset{0}{\omega_u}, \overset{0}{\omega_s}$ formed by solutions tending asymptotically to x_0 as $t \to -\infty$ and $t \to +\infty$ respectively. In the unperturbed system, they are doubled (coincide). For sufficiently small e, there exists hyperbolic 2π-perodic solution $x_0(t)$. In general, its asymptotic surfaces ω_u^0, ω_s^0 do not coincide and cross transversely. Points belonging to both of the surfaces are called homoclinic.

Condition for transversal crossing of the separatrices can be expressed in terms Melnikov's integral. Namely, if function :

$$M(t_0) = \int_{-\infty}^{\infty} (H_0, H_1)(\bar{x}(t-t_0), t)\, dt \qquad \ldots (1.2)$$

has simple zero then perturbed asymptotic surfaces cross transversely and the Hamiltonian system is non-integrable. In such situation, for the Poincare's map the following theorems are valid.

Theorem 1 — *If Σ is a phase space and σ is a shift map, then σ has*

(a) Countably infinite periodic orbits (including orbits with arbitrarily long period).

(b) Uncountably infinite non-periodic orbits (c) A dense orbit.

Theorem 2 — *Suppose $f: R^n \to R^n$ is a c^r $(r \geq 2)$, diffeo-morphism having a hyperbolic periodic point p. Furthermore, suppose that two asymptotic surfaces $w^s(p)$ and $w^u(p)$ have a point $n \geq 1$ such that f^n has an invariant cantor set Δ. Moreover, there exists a homeomorphism $\phi: \Delta \to \Sigma$ such that $\phi \cdot f^n = \sigma \cdot \phi$.*

When the condition $\phi \cdot f^n = \sigma \cdot \phi$ holds, with $\phi: \Delta \to \Sigma$, the dynamical system $f^n: \Delta \to \Delta$ and $\sigma: \Sigma \to \Sigma$ are said to be topologically conjugate.

For the proofs and details of the above theorems see Smale[8] or Wiggins[9 & 10].

As a consequence of these theorems, the same method gives us more information about the chaotic nature of the dynamical system which we propose to study in Paper II. With the help of the theory developed in these two papers we have also made a study of the Earth-Moon-Artificial Satellite (1958 B2 Vanguard 1) system.

2. Equation of Motion

Let us consider a rigid satellite moving in an elliptic orbit (semi-major axis a, eccentricity e) under the influence of a central body of mass M and its moon of mass m whose orbit is assumed circular and coplanar with the orbit of the satellite (Fig. 1). The satellite is assumed to be a triaxial ellipsoid with principal moments of inertia $A < B < C$, and C is the moment of inertia about the spin axis which is perpendicular to the orbital plane. We approximate the influence of the moon by resolving potential of the torque with respect to the r/R ratio, where r is the radius of the satellite and R the radius of moon's orbit.

Let the true anomaly be v, the orientation of the satellite's long axis be θ. Then $\theta - v = \delta/2$ measures the orientation of the satellite's long axis relative to the satellite's radius vector.

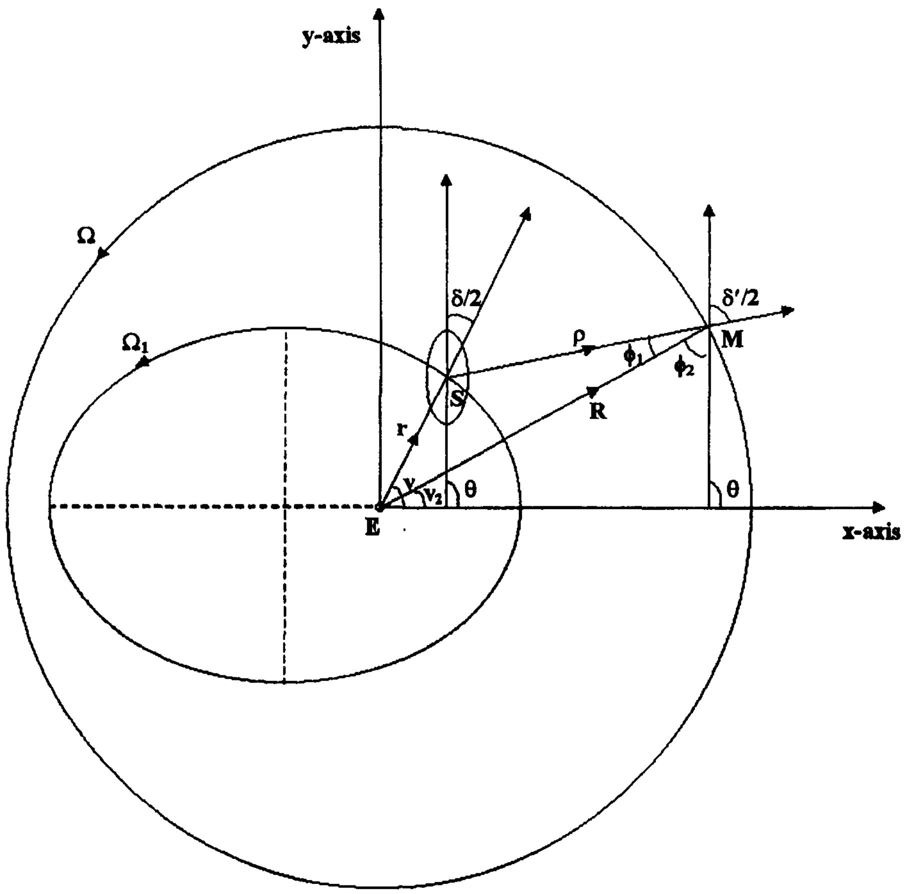


FIG. 1. Satellite's motion in elliptic orbit with third body perturbation

The **equation of motion**, which we have finally obtained is

$$[1 + e \cos v] \frac{d^2 q}{dv^2} - 2e \sin v \frac{dq}{dv} - 4e \sin v + n^2 \sin q$$

$$- n^2 \varepsilon (1 - 3e \cos v) [\sin (cv - q) + de \sin v \cos (cv - q)] = 0, \qquad \text{... (2.1)}$$

where, ε is the parameter due to the third body torque,

$$n^2 = 3 \frac{B - A}{C},$$

$$c = 2\left[1 - \Omega \sqrt{\frac{l^3}{\mu}} \right],$$

$$d = 4 \, \Omega \sqrt{\frac{l^3}{\mu}},$$

$$\varepsilon = \frac{m}{M} \Omega^2 \frac{l^3}{\mu},$$

$q = \delta,$

e — eccentricity of satellite's elliptical orbit,

v — true anomaly,

m — mass of the moon $\approx 7.348 \times 10^{22}$ kg,

M — mass of the Earth $\approx 5.9742 \times 10^{24}$ kg,

Ω — angular velocty of the moon $= \dfrac{2\pi}{30 \times 24 \times 60 \times 60}$ rad/s

l — length of semilatus rectum of the satellite's orbit

and μ — gravitational constant of the earth

$\cong GM \approx 3.986005 \times 10^5\, km^3/s^2.$

2.1 HAMILTON'S EQUATION

Eq. (2.1) is equivalent to Hamilton's equations

$$\frac{dq}{dv} = \frac{\partial H}{\partial p}, \frac{dp}{dv} = -\frac{\partial H}{\partial q}, \qquad \text{... (2.1.1)}$$

where p is the generalized momenta. Takeing e of the order of ε (parameter de to the third body torque),

$$e = e_1 \in (0 < e_1 << 1, 0 < \varepsilon << 1).$$

The Hamiltonian function H can be written as

$$H = H_0 + \varepsilon H_1 + 0\,(\varepsilon^2).$$

In our problem

$$H = -2p + \frac{p^2}{2} - n^2 \cos q + \varepsilon\,[-p^2 e_1 \cos v - n^2 \cos (cv - q) - n^2 e_1 \cos v \cos q)],$$

where $$H_0 = -2p + \frac{p^2}{2} - n^2 \cos q$$

$$H_1 = -p^2 e_1 \cos v - n^2 \cos (cv - q) + n^2 e_1 \cos v \cos q.$$

Since ε being small therefore $0\,(\varepsilon^2)$ and higher are neglected.

2.2 Equilibrium and Double Asymptotic Solution

The equilibrium solutions corresponding to H_0 are given by

$$\frac{dq}{dv} = 0, \frac{dp}{dv} = 0.$$

Corresponding to H_0 we get from eq. (2.1.1)

$$\frac{dq}{dv} = p - 2, \frac{dp}{dv} = -n^2 \sin q$$

which gives us $p = 2$ and $q = 0, \pi$.

Thus in the phase space $[q, p]$; $[0, 2]$ and $[\pi, 2]$ are the equilibrium points.

We have further calculated that $[0, 2]$ is stable where as $[\pi, 2]$ is unstable. Also

$$\frac{dp}{dq} = \pm \frac{n^2 \sin q}{p - 2},$$

which on integration gives

$$\frac{p^2}{2} - 2p = n^2 \cos q + n^2 - 2$$

But $$p = 2 + \frac{dq}{dv}.$$

Hence, the unperturbed double asymptotic solutions at $[\pi, 2]$ are obtained as

$$p^{\pm}(v) = 2 \pm \frac{2n}{\cosh(nv)},$$

$$\sin[q^{\pm}(v)] = \pm 2 \frac{\sinh(nv)}{\cosh^2(nv)}$$

and $$\cos[q^{\pm}(v)] = \frac{2}{\cosh^2(nv)} - 1.$$

2.3 Melnikov's Function

Melnikov's function is obtained as

$$M^{\pm}(v_0) = \pm 2\pi e_1 \sin v_0 \left[4\, sech \frac{\pi}{2n} + 3\, cosech \frac{\pi}{2n} \right] \pm 2\pi c^2 \sin cv_0 \left[sech \frac{\pi c}{2n} + cosech \frac{\pi c}{2n} \right] \quad \ldots (2.3.1)$$

It is easy to observe that for any value of the mass parameter $n > 0$ and the third body torque parameter e_1 $(0 < e_1 << 1)$ the above function has a simple zero.

Thus the eqns. (2.1.1) are non-integrable.

2.4 Graphical Representation of Melnikov's Function

We have studied the graphical representation of Melnikov's function in the Earth-Moon-Artificial Satellite (1958 B2 Vanguard 1) system.

For the Artificial Satellite (1958 B2 Vanguard 1) the data is eccentricity $e = 0.190$

Semi major axes $a = 8676$ km

In our problem

$$c = 2\left[1 - \Omega\sqrt{\frac{l^3}{\mu}}\right].$$

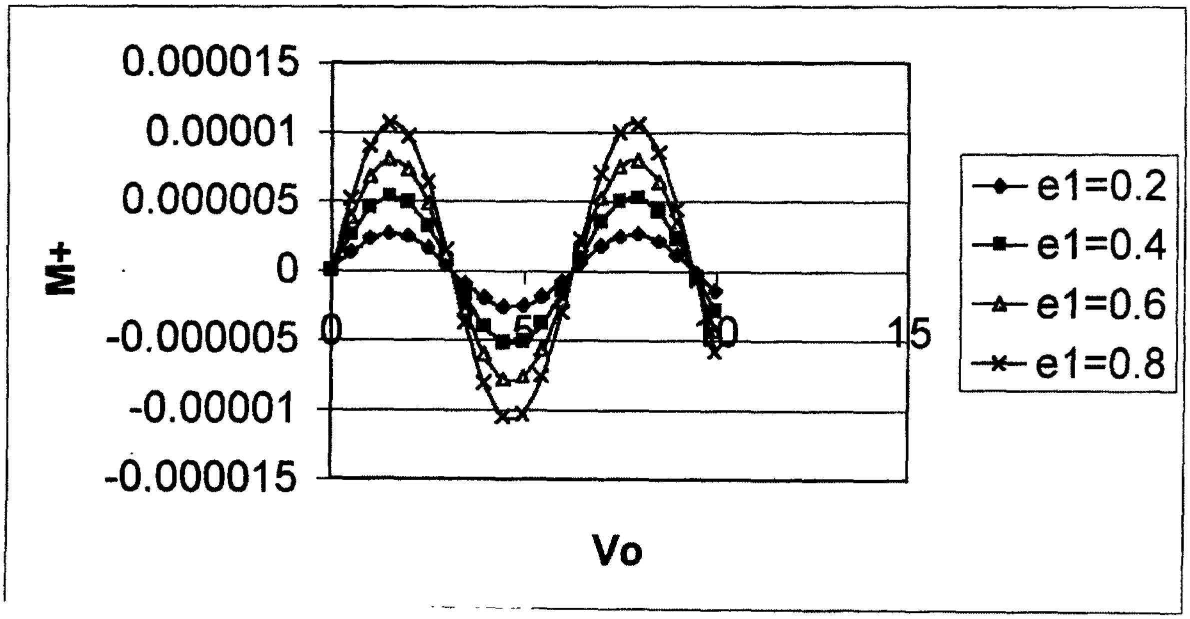
For the Earth-Moon-Artificial Satellite (1958 B2 Vanguard 1) system we have evaluated $c = 1.9999$

For this fixed value of c we have studied the graphs of Melnikov's function.

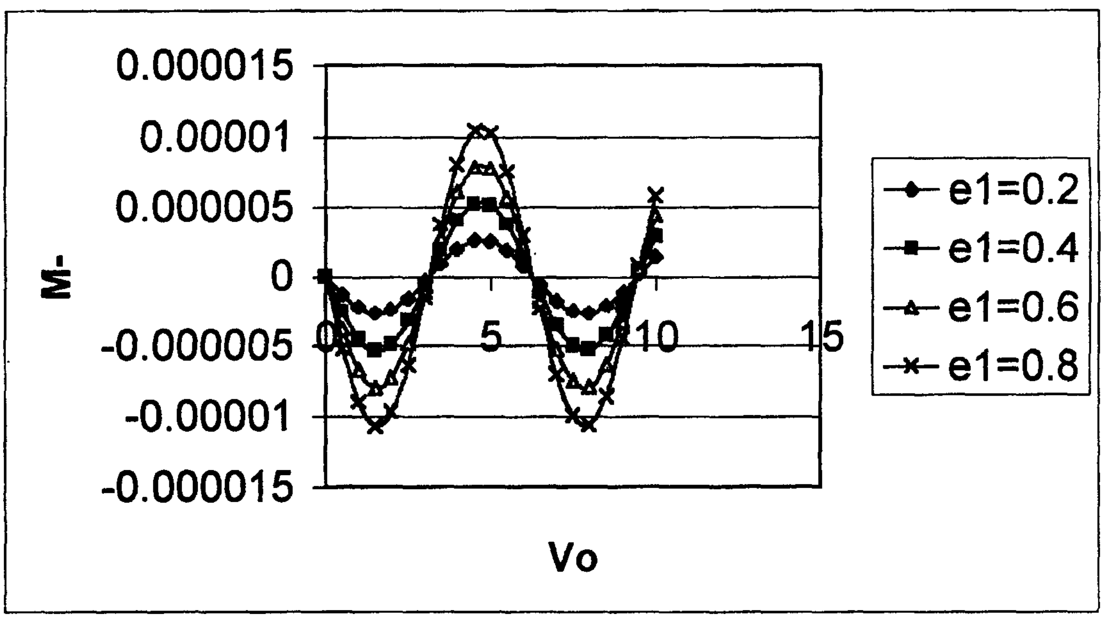
1. Figures 2, 3, 4 and 5 illustrate the graphs of $M^+(v_0, e_1, n)$ and $M^-(v_0, e_1, n)$ for $0 \le v_0 \le 10$

Figure Number	n	e_1	$M^+(v_0, n, e_1)$		$M^-(v_0, n, e_1)$	
			Max. Value	Min.Value	Max. Value	Min. Value
2	0.1	0.2	2.66E-06	–2.61E-06	2.61E-06	–2.66E-06
	01	0.4	5.32E-06	–5.22E-06	5.22E-06	–5.32E-06
	0.1	0.6	7.99E-06	–7.83E-06	7.83E-06	–7.99E-06
	0.1	0.8	1.06E-05	–1.04E-05	1.04E-05	–1.06E-05
3	0.3	0.2	0.094	–0.091	0.091	–0.094
	0.3	0.4	0.187	–0.182	0.182	–0.187
	0.3	0.6	0.281	–0.274	0.274	–0.281
	0.3	0.8	0.375	–0.366	0.366	–0.375
4	0.5	0.2	0.836	–0.832	0.832	–0.836
	0.5	0.4	1.550	–1.562	1.562	–1.550
	0.5	0.6	2.303	–2.291	2.291	–2.303
	0.5	0.8	3.062	-3.021	3.021	-3.062
	0.7	0.2	2.598	-2.447	2.447	-2.598
5	0.7	0.4	4.234	–4.190	4.190	–4.234
	0.7	0.6	5.982	–5.977	5.977	–5.982
	0.7	0.8	7.730	–7.765	7.765	–7.730

We observe from the figures that as v_0 changes from 0 to 10 the Melnikov's functions $M^+(v_0, n, e_1)$ and $M^-(v_0, n, e_1)$ behave almost like the Sine functions. They have simple zeroes. Also we observe that in each graph when $3 \le v_0 \le 3.5$, $M^+(v_0, n, e_1)$ changes sign from positive to negative



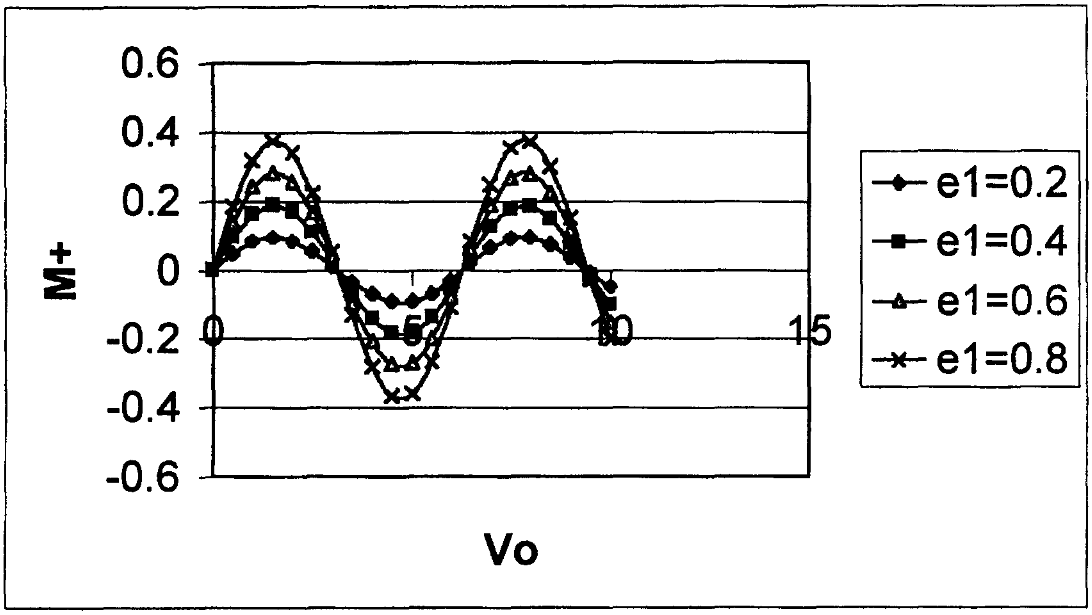
Melnikov's function $M^{+}(v_o, n, e_1)$ for $c = 1.9999$, $0 \le v_o, \le 10, n = 0.1, e_1 = 0.2, 0.4, 0.6, 0.8$



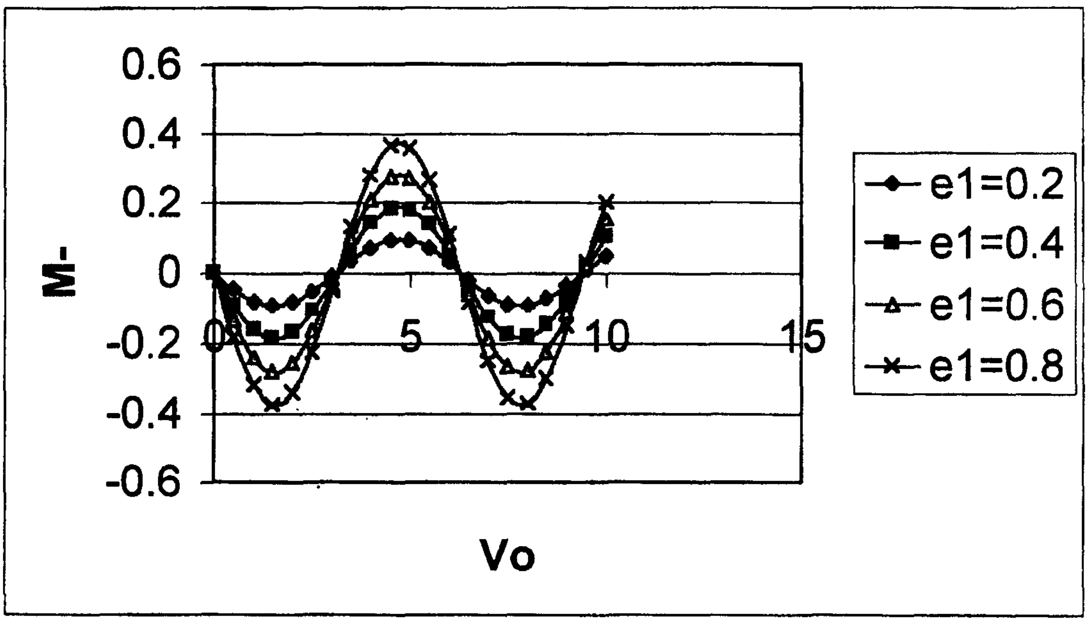
Melnikov's function $M^{-}(v_o, n, e_1)$ for $c = 1.9999$, $0 \le v_o, \le 10, n = 0.1, e_1 = 0.2, 0.4, 0.6, 0.8$

FIG. 2.

and when $6 \le v_0 \le 6.5$ $M^{+}(v_0, n, e_1)$ changes sign from negative to positive. Thus in all these graphs abscissae remain almost the same.



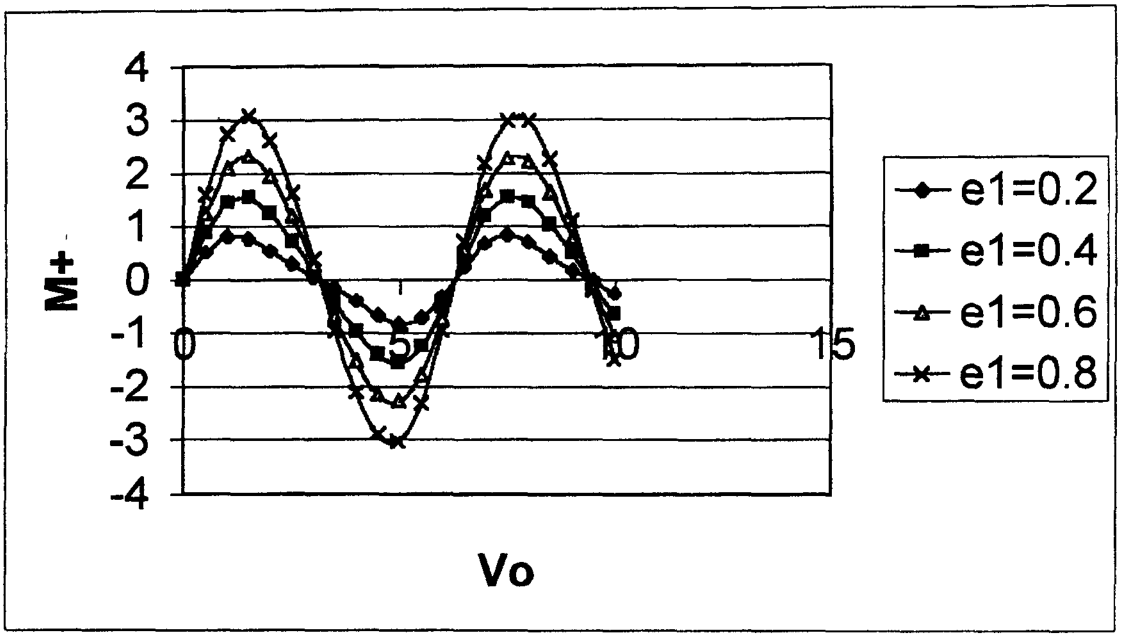
Melnikov's function $M^{+}(v_o, n, e_1)$ for $c = 1.9999,\ 0 \le v_o, \le 10, n = 0.3, e_1 = 0.2, 0.4, 0.6, 0.8$



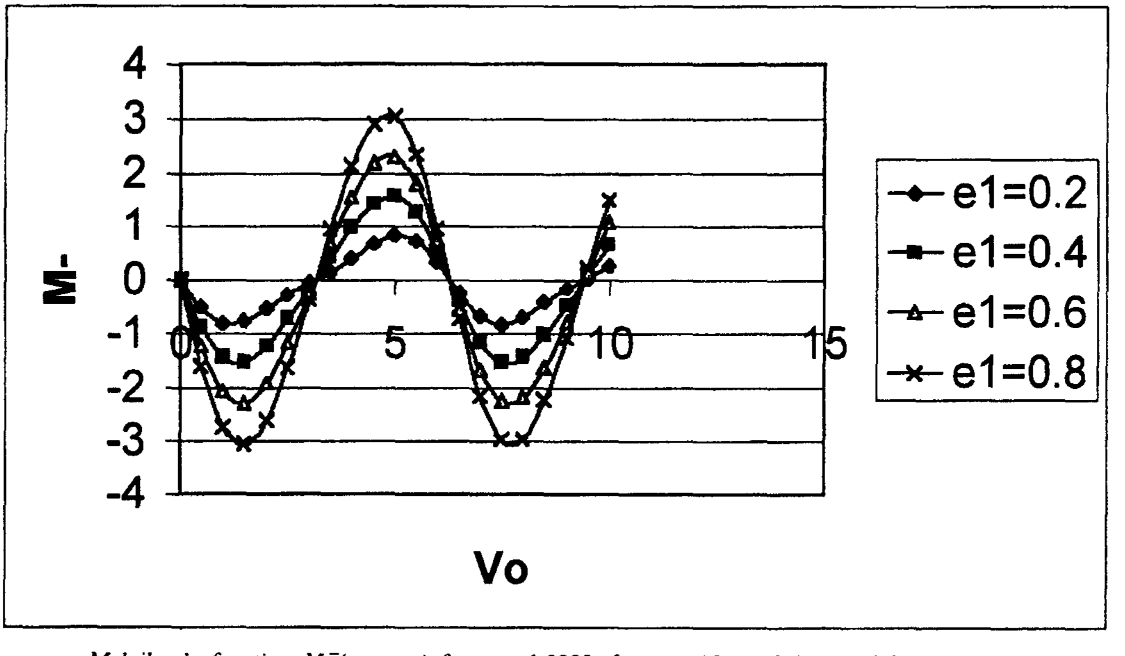
Melnikov's function $M^{-}(v_o, n, e_1)$ for $c = 1.9999,\ 0 \le v_o, \le 10, n = 0.3, e_1 = 0.2, 0.4, 0.6, 0.8$

FIG. 3.

We also observe from the table that as e_1 and n both vary $e_1 = 0.2, 0.4, 0.6, 0.8$, $n = 0.1, 0.3, 0.5, 0.7$ $M^{+}(v_0, n, e_1)$ and $M^{-}(v_0, n, e_1)$ elongate along the ordinate, abscissa remaining almost the same.



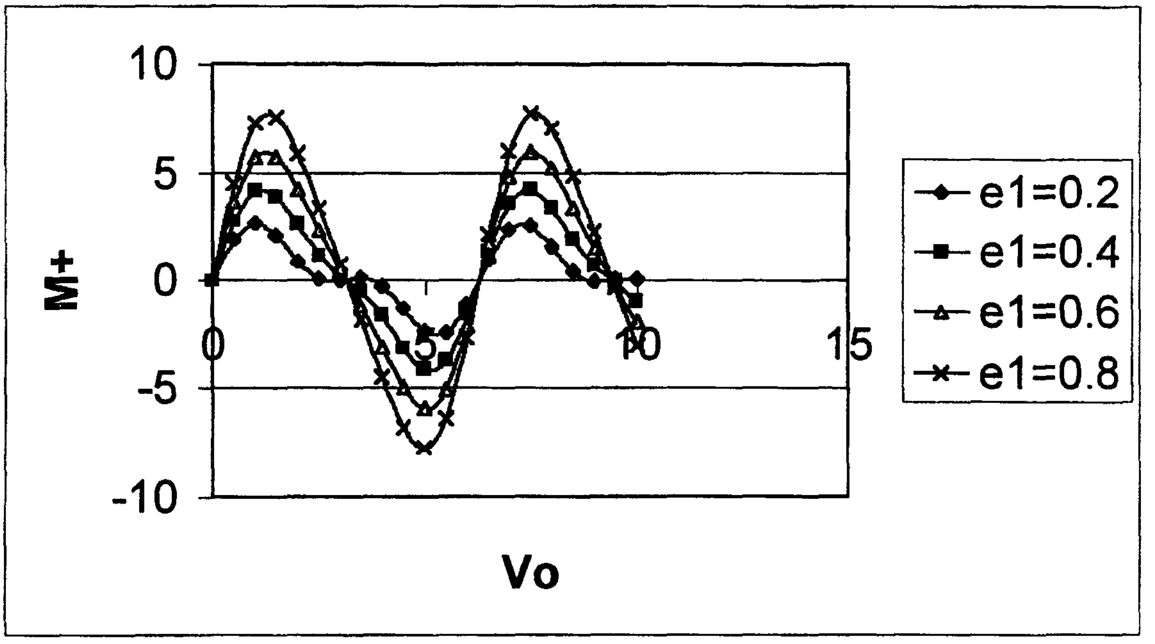
Melnikov's function $M^{+}(v_o, n, e_1)$ for c = 1.9999, $0 \leq v_o, \leq 10, n = 0.5, e_1$ = 0.2, 0.4, 0.6, 0.8



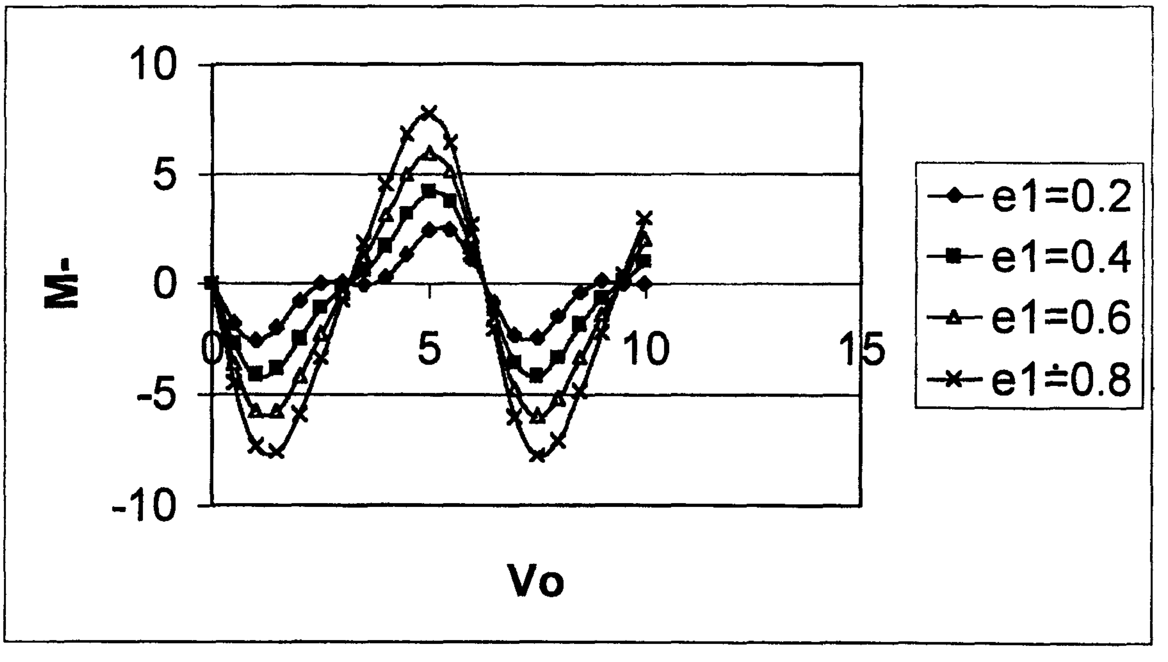
Melnikov's function $M^{-}(v_o, n, e_1)$ for c = 1.9999, $0 \leq v_o, \leq 10, n = 0.5, e_1$ = 0.2, 0.4, 0.6, 0.8

FIG. 4.

2. Figure 6 illustrates the graphs of $M^{\pm}(v_0, e_1, n)$ for $0 \leq e_1 \leq 0.9, n = 0.1$, and v_0 = 0.2, 0.4, 0.6, 0.8.

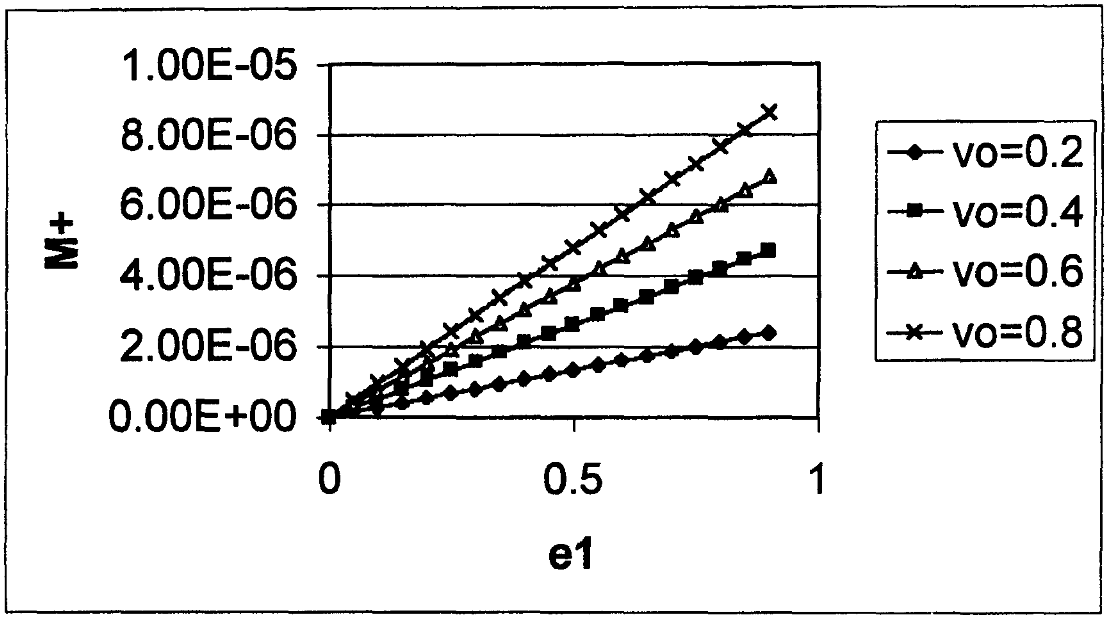


Melnikov's function $M^{+}(v_o, n, e_1)$ for $c = 1.9999$, $0 \leq v_o, \leq 10, n = 0.7, e_1 = 0.2, 0.4, 0.6, 0.8$

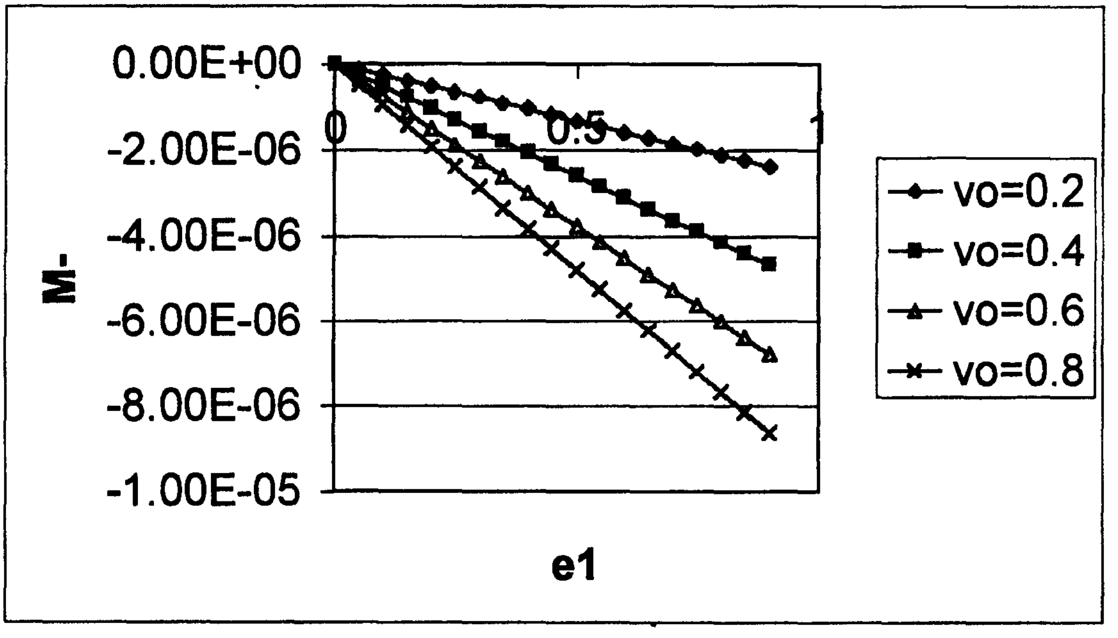


Melnikov's function $M^{-}(v_o, n, e_1)$ for $c = 1.9999$, $0 \leq v_o, \leq 10, n = 0.7, e_1 = 0.2, 0.4, 0.6, 0.8$

FIG. 5.

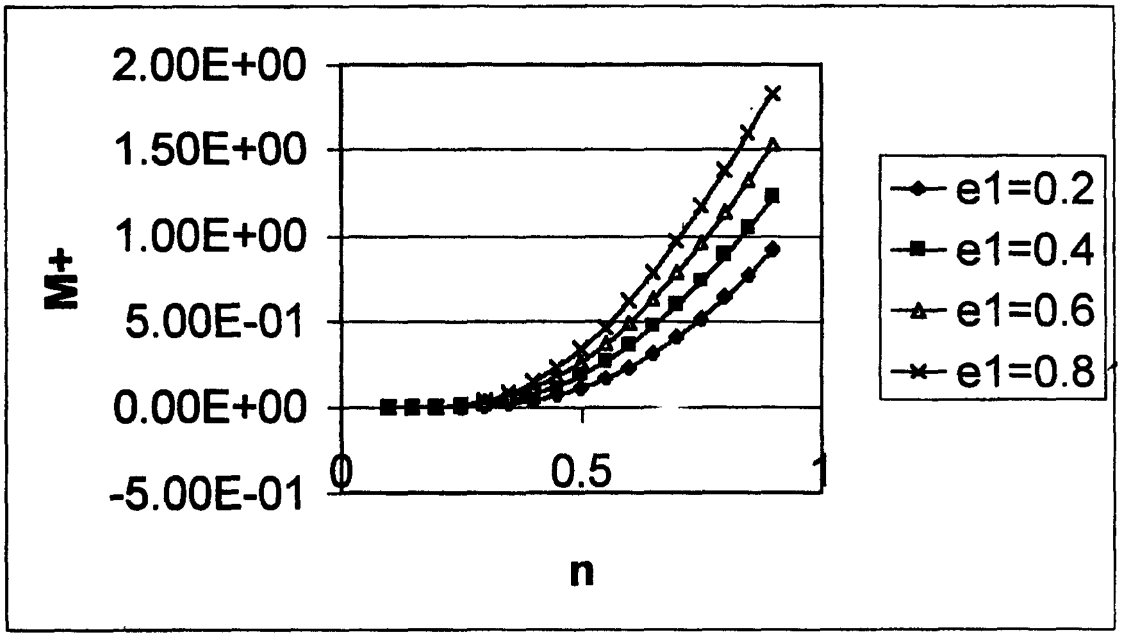


Melnikov's function $M^{+}(v_o, n, e_1)$ for $c = 1.9999,\ 0 \leq e_1, \leq 0.9, n = 0.1, v_o = 0.2,\ 0.4,\ 0.6,\ 0.8$

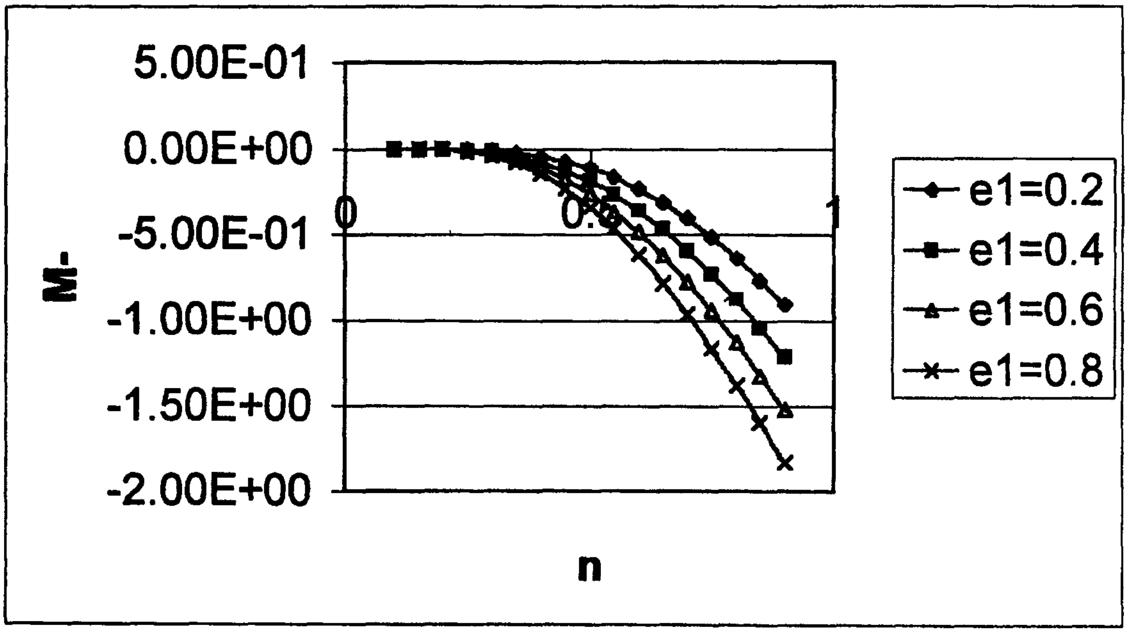


Melnikov's function $M^{-}(v_o, n, e_1)$ for $c = 1.9999,\ 0 \leq e_1, \leq 0.9, n = 0.1, v_o = 0.2,\ 0.4,\ 0.6,\ 0.8$

FIG. 6.



Melnikov's function $M^{+}(v_o, n, e_1)$ for $c = 1.9999$, $0.1 \le n \le 0.9$, $v_o = 0.1$, $e_1 = 0.2, 0.4, 0.6, 0.8$



Melnikov's function $M^{-}(v_o, n, e_1)$ for $c = 1.9999$, $0.1 \le n \le = 0.9$, $v_o = 0.1$, $e_1 = 0.2, 0.4, 0.6, 0.8$

FIG. 7.

It has been observed that as the parameter due to the third body effect 'e_1' changes from 0 to 0.9 ($0 \le e_1 \le 0.9$) the value of the Melnikov's function $M^+(v_0, e_1, n)$ increases monotonically for each value of v_0 and that of $M^-(v_0, e_1, n)$ decreases monotonically for each value of v_0.

Also we vary both v_0 and e_1, v_0 = 0.2, 0.4, 0.6 and 0.8 and $0 \le e_1 \le 0.9$ the graphs elongate along the ordinate.

3. Figure 7 illustrates the graphs of $M^\pm(v_0, e_1, n)$ for $0.1 \le n \le 0.9$, $v_0 = 0.1$, and e_1 = 0.2, 0.4, 0.6, 0.8.

We have observed that as the mass distribution parameter 'n' of the satellite changes from 0.1 to 0.9 ($0.1 \le n \le 0.9$) the value of $M^+(v_0, e_1, n)$ for each e_1 initially increases very slowly and then exponentially increases to $+\infty$. Also the value of $M^-(v_0, e_1, n)$ for each e_1 initially decreases very slowly and then exponentially decreases to $-\infty$.

Also as we vary both e_1 and n, e_1 = 0.2, 0.4, 0.6, 0.8 and $0.1 \le n \le 0.9$, the graphs elongate along the ordinate.

CONCLUSION

We thus conclude that the non-linear rotational equations of motion of the planar oscillation of a satellite in an elliptic orbit under the influence of third body torque are non-integrable.

Also we have observed graphically that in the Earth-Moon-Artificial Satellite (1958 B2 Vanguard 1) system the Melnikov's function has a simple zero and hence the equations of motion are non integrable.

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