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ON LIE IDEALS AND JORDAN GENERALIZED DERIVATIONS OF PRIME RINGS

MOHAMMAD ASHRAF*, NADEEM-UR-REHMAN** AND SHAKIR ALI**

**Department of Mathematics, Faculty of Science, King Abdul Aziz University, P.O. Box* 80203, *Jeddah* 21589, *Saudi-Arabia*
(E-mail: mashraf80@hotmail.com)
***Department of Mathematics, Aligarh Muslim University, Aligarh* 202002, *India*
(E-mail : rehman100@postmark.net; shakir50@rediffmail.com)

Let R be a ring and S a nonempty subset of R. An additive mapping $F: R \to R$ is called a generalized derivation (resp. Jordan generalized derivation) on S if there exists a derivation $d: R \to R$ such that $F(xy) = F(x)y + xd(y)$ (resp. $F(x^2) = F(x)x + xd(x)$) holds for all $x, y \in S$. Suppose that R is a 2-torsion free prime ring and U a nonzero Lie ideal of R such that $u^2 \in U$ for all $u \in U$. In the present paper it is shown that if F is a Jordan generalized derivation on U, then F is a generalized derivation on U.

Key Words : Lie Ideals; Prime Rings; Jordan Generalized Derivations; Generalized Derivations; Derivations; Torsion Free Rings

1. INTRODUCTION

Throughout the present paper R will denote an associative ring with centre $Z(R)$. Recall that R is prime if $aRb = (0)$ implies that $a = 0$ or $b = 0$. As usual $[x, y]$ will denote the commutator $xy - yx$. An additive subgroup U of R is said to be Lie ideal of R if $[u, r] \in U$ for all $u \in U, r \in R$. An additive mapping $d: R \to R$ is called a derivation (resp. Jordan derivation) if $d(xy) = d(x)y + xd(y)$, (resp. $d(x^2) = d(x)x + xd(x)$), holds for all $x, y \in R$. Obviously, every derivation is a Jordan derivation. The converse need not be true in general. A famous result due to Herstein[10] states that every Jordan derivation on a 2-torsion free prime ring is a derivation. A brief proof of this result is presented in [7]. Further, Awtar[4] generalized this result on Lie ideals.

Following Havla[11], an additive mapping $F: R \to R$ is called a generalized derivation (resp. Jordan generalized derivation) if there exists a derivation $d: R \to R$ such that $F(xy) = F(x)y + xd(y)$ (resp. $F(x^2) = F(x)x + xd(x)$) holds for all $x, y \in R$. Clearly, every generalized derivation on a ring is a Jordan generalized derivation. But the converse statement does not hold in general. It is shown in [1] that if R is a ring with a commutator which is not a divisor of zero, then every Jordan generalized derivation on R is a generalized derivation. The aim of the present paper is to establish another set of conditions under which every Jordan generalized derivation on a ring is a generalized derivation. This lead to the discovery of a new result which can be regarded as a contribution to the theory of Jordan derivations in rings.

2. Preliminary Results

We begin with the following result which is essentially proved in [5].

Lemma 2.1 — If $U \not\subseteq Z$ is a Lie ideal of a 2-torsion free prime ring R and $a, b \in R$ such that $aUb = (0)$, then $a = 0$ or $b = 0$.

To facilitate our discussion, we define a mapping $\delta : R^2 \rightarrow R$ such that $\delta(x, y) = F(xy) - F(x)y - xd(y)$. It is easy to see that $\delta(x, y+z) = \delta(x, y) + \delta(x, z)$ and $\delta(x+y, z) = \delta(x, z) + \delta(y, z)$, for all $x, y, z \in R$. Moreover, if δ is zero then F is a generalized derivation on R. We shall make use of commutator identities; $[x, yz] = [x, y]z + y[x, z]$ and $[xy, z] = [x, z]y + x[y, z]$.

Lemma 2.2 — Let R be a 2-torsion free ring and U be a nonzero Lie ideal of R such that $u^2 \in U$, for all $u \in U$. If $F : R \rightarrow R$ is an additive mapping satisfying $F(u^2) = F(u)u + ud(u)$, for all $u \in U$, then —

(*i*) $F(uv + vu) = F(u)v + F(v)u + ud(v) + vd(u)$, for all $u, v \in U$.

(*ii*) $F(uvu) = F(u)vu + ud(v)u + uvd(u)$, for all $u, v \in U$.

(*iii*) $F(uvw + wvu) = F(u)vw + F(w)vu + ud(v)w + uvd(w)$

$+ wd(v)u + wvd(u)$, for all $u, v, w \in U$.

Proof : (*i*), (*ii*), (*iii*) are easily obtained in the way similar to that in [1].

Using similar techniques as used to prove Theorem 3 in [8], one can prove the following.

Lemma 2.3 — Let R be a 2-torsion free ring and U be a nonzero Lie ideal of R such that $u^2 \in U$, for all $u \in U$. If $F : R \rightarrow R$ is an additive mapping satisfying $F(u^2) = F(u)u + ud(u)$, for all $u \in U$, then $\delta(u, v)w[u, v] = 0$, for all $u, v, w \in U$.

3. Main Result

The main result of the present paper states as follows :

Theorem — *Let R be a 2-torsion free prime ring and U a nonzero Lie ideal of R such that $u^2 \in U$ for all $u \in U$. If F is an additive mapping of R into itself satisfying $F(u^2) = F(u)u + ud(u)$ for all $u \in U$, then $F(uv) = F(u)v + ud(v)$, for all* u, $v \in U$.

PROOF : If U is a commutative Lie ideal of R, i.e., $[u, v] = 0$, for all $u, v \in U$, then use the same arguments as used in the proof of Lemma 1.3 of [10], $U \subseteq Z$. Now, by Lemma 2.2 (*iii*), we have

$$F(uvw + wvu) = F(u)vw + F(w)vu + ud(v)w + uvd(w) + wd(v)u + wvd(u). \qquad \text{... (3.1)}$$

Since $u^2 \in U$ for all $u \in U$, we find that $uv + vu \in U$ for all $u, v \in U$. This yields that $2uv \in U$ for all $u, v \in U$. As the ideal U is commutative, in view of Lemma 2.2 (*i*) we have

$$\begin{aligned} 2F(uvw + wvu) &= F((2uv)w + w(2uv)) \\ &= F(2uv)w + 2uvd(w) + 2F(w)uv + wd(2uv) \\ &= 2\{F(uv)w + uvd(w) + F(w)uv + wd(u)v + wud(v)\} \end{aligned}$$

This shows that for all $u, v \in U$

$$F(uvw + wvu) = F(uv)w + uvd(w) + F(w)uv + wd(u)v + wud(v). \qquad \text{... (3.2)}$$

Combining (3.1) and (3.2) and using the fact that $uv = vu$, we obtain

$$\delta(u, v)w = 0, \text{ for all } u, v, w \in U. \qquad \text{... (3.3)}$$

Now, replacing w by $[w, r]$ in (3.3) and using (3.3), we get $\delta(u, v)rw = 0$, for all $u, v, w \in U$ and $r \in R$ and hence $\delta(u, v)RU = (0)$, for all $u, v \in U$. Since $U \neq (0)$ and R is prime the above expression yields that $\delta(u, v) = 0$, for all $u, v \in U$. Hence, we get the required result.

Hence, onward we shall assume that U is a non-commutative Lie ideal of R i.e. $U \not\subseteq Z(R)$. By Lemma 2.3, we have $\delta(u, v)w[u, v] = 0$, for all $u, v, w \in U$ i.e. $\delta(u, v)U[u, v] = (0)$, for all $u, v \in U$. Thus in view of Lemma 2.1, we find that for each pair $u, v \in U$ either $\delta(u, v) = 0$ or $[u, v] = 0$. For each $u \in U$, let $U_1 = \{v \in U \mid \delta(u, v) = 0\}$ and $U_2 = \{v \in U \mid [u, v] = 0\}$. Hence, U_1 and U_2 are additive subgroups of U whose union is U. By Brauer's trick, we have either $U = U_1$ or $U = U_2$. Again by using the same method we find that either $U = \{u \in U \mid U = U_1\}$ or $U = \{u \in U \mid U = U_2\}$. Since U is non-commutative, we find that $\delta(u, v) = 0$, for all $u, v \in U$ i.e. F is a generalized derivation on U.

Corollary — Let R be a 2-torsion free prime ring and $F : R \to R$ be a Jordan generalized derivation. Then F is a generalized derivation on R.

The following example shows that the primeness is necessary in the hypotheses of the above theorem.

Example — Let S be a ring such that the square of each element in S is zero, but the product of some elements in S is nonzero. Next, let $R = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \mid x, y \in S \right\}$

Define a map $F : R \to R$ such that $F\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ Then with $d = 0$ and $U = R$, it can be easily seen that $F(r^2) = F(r)\, r = F(r)\, s = 0$ for all $r, s \in R$ but $F(rs) \neq 0$ for some $r, s \in R$.

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