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ON DIMENSION $\tau_i \tau_j$ *dm X*

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A new property of $\tau_i \tau_j$ dm$X$ is proved for the class of normal space (X, τ_j) of bitopological space X for $i \neq j$ and $i, j \in \{1, 2\}$.

Key Words : *p*-Open Cover; $\tau_i \tau_j$ *dmX*; Nerve of the Cover; Normal Space

Let (X, τ_1, τ_2) be a bitopological space (briefly bispace X). Fletcher, Hoyle and Patty in [5] called a cover $\mathcal{U}$ of X p-open if $\mathcal{U} \subset \tau_1 \cup \tau_2$ and if furthermore $\mathcal{U}$ contains a non-empty member of τ_1 and a non-empty member of τ_2. Let $\mathcal{V} = \left\{ V_i \mid i = 1, ..., k \right\}$ be a finite τ_2-open cover of X. A nerve of the cover $\mathcal{V}$ is a partially ordered set. Duschnik and Miller in [4] defined the dimension *ds* of a partially ordered set as the smallest number of linear orderings whose intersection is the original partial ordering. Using that definition in [6] we defined $\tau_i \tau_j$ dm$X$ as follows:

Definition 1 — Let X be a bispace. $\tau_i \tau_j$ dm$X$ = - 1, if $X = \varnothing$; $\tau_i \tau_j$ dm X = 0 if for every finite τ_i-open cover $\mathcal{U}$ of X there exists a finite τ_j-open refinement $\mathcal{V}$ of X whose nerve $\mathcal{N}(\mathcal{V})$ is totally unordered (x without compare y, for all $x, y \in \mathcal{N}(\mathcal{V})$). $\tau_i \tau_j$ dm $X \leq n, n \in N$, if each finite τ_i-open cover $\mathcal{U}$ of X has a finite τ_j-open refinement $\mathcal{V}$ of $\mathcal{U}$ such that *ds* $\mathcal{N}(\mathcal{V}) \leq n + 1$; $\tau_i \tau_j$ dm$X$ = n means $\tau_i \tau_j$ dm$X \leq n$ holds but $\tau_i \tau_j$ dm $X \leq n - 1$ fails; $\tau_i \tau_j$ dm $X = \infty$ if $\tau_i \tau_j$ dm $X > n$ for each $n = -1, 0, 1, ..., i \neq j$ and $i, j \in \{1, 2\}$.

If $\tau_i = \tau_j$ then $\tau_i \tau_j$ dm X = dm$X$ introduced by Adnadevic in [1].

Some properties of $\tau_i \tau_j$ dm X will be proved now.

Proposition 2 — Let F be a τ_j-closed set of a bispace X with $\tau_i\,\tau_j$ dm$F$ = 0. Let (X, τ_j) be a normal space and let τ_j-dm$K \le m\ (m > 0)$ for every τ_j-closed subset K of X such that $K \cap F = \phi$, $i \neq j$ and $i, j \in \{1, 2\}$. Then $\tau_i\,\tau_j\, dm\, X \le m + 2$.

PROOF : Let $\mathcal{U} = \{U_1, U_2, \ldots, U_s\}$ be a τ_i-open cover of X. Then there exists a τ_j-open refinement $\mathcal{V} = \{V_1, V_2, \ldots, V_r\}$ of $\mathcal{U}$ such that $ds\ \mathcal{N}(\mathcal{V}) \le 2$ and $\mathcal{V}$ is τ_j-open cover of F. Then $A = V_1 \cup V_2 \cup \ldots \cup V_r$ is a τ_j-open subset of X. Then there exist τ_j-open sets G and H such that $F \subset G, X \setminus A \subset H$ and τ_j-cl $G \cap \tau_j$-cl$H$ = $\varnothing$. Then $X \setminus G$ is a τ_j-closed set with τ_j-dm $(X \setminus G) \le m$. Then there exists a τ_j-open refinement $\mathcal{W} = \{W_1, W_2, \ldots, W_k\}$ of $\mathcal{U}$ such that $ds\ \mathcal{N}(\mathcal{W}) \le m + 1$ and $\mathcal{W}$ covers $X \setminus G$. Now $\mathcal{D} = \{V_1, \ldots, V_r, W_1, \ldots, W_k\}$ is a τ_j-open cover of X, a refinement of $\mathcal{U}$ and $ds\ \mathcal{N}(\mathcal{D}) \le m + 3$. Therefore, $\tau_i\,\tau_j$ dm $X \le m + 2$.

Corollary 1 — Proposition 2 is true even if (X, τ_j) is a regular final compact space.

Remark 1 : If $\tau_i\,\tau_j\, dim\, F \le n\ (m > 0, n > 0)$ in Proposition 2, then $\tau_i\,\tau_j$ dm $X \le n + m + 1$. If in that proposition $m = 0 = n$, then $\tau_i\,\tau_j\, dm\, X \le 3$.

Proposition 3 — Let any p-open cover of a bispace X have a τ_j-open refinement and let $X = A \cup B$ with $\tau_i\,\tau_j$ dm $B \le n\ (n > 0)$. let (X, τ_j) be a normal space. Let A be a τ_i-closed subset of X with τ_i-dim$A$ = 0. Then $\tau_i\,\tau_j\, dm\, X \le n + 2$, for $i \neq j$ and $i, j \in \{1, 2\}$.

PROOF : Let $\mathcal{U} = \left\{ U_i \mid i = 1, \ldots, n \right\}$ be a τ_i-open cover of X. Then there exists a τ_i-open in X refinement $\mathcal{V} = \left\{ V_i \mid i = 1, \ldots, r \right\}$ of $\mathcal{U}$ such that $ds\ \mathcal{N}(\mathcal{V}) \le 2$ and $A \subset \bigcup_1^r V_i$. Then $F = X \setminus \bigcup_1^r V_i$ is a τ_i-closed subset of X with $\tau_i\,\tau_j\, dm\, F \le n$ (because $F \subset B$). Then there exists τ_j-open in X refinement $\mathcal{W} = \left\{ W_i \mid = 1, .., p \right\}$ of $\mathcal{U}$ such that $F \subset \bigcup_1^p W_i$ and $ds\ \mathcal{N}(\mathcal{W}) \le n + 1$. A family $\mathcal{M} = \left\{ V_1, \ldots, V_r, W_1, \ldots, W_p \right\}$ is a p-open cover of X with $ds\ \mathcal{N}(\mathcal{M}) \le n + 3$. A cover $\mathcal{M}$ is refined by a τ_j-open cover $\mathcal{X}$ of $ds\ \mathcal{N}(\mathcal{M}) \le n + 3$. Hence, $\tau_i\,\tau_j\, dm\, X \le n + 2$.

Corollary 2 — Let (X, τ_j) be a regular final compact space. Then Proposition 3 is also true.

Remark 2 : If in Proposition 3 $\tau_i - dm\, A \leq m\ (m > 0)$ then $\tau_i\, \tau_j$ dm $X \leq n + m + 1$. Also if in this Proposition $\tau_i\, \tau_j$ dm $B = 0$ then $\tau_i\, \tau_j\, dm\, X \leq 3$.

Proposition 4 — Let $X = A \cup B$ and let $(X, \tau_k), k = 1, 2$ be a hereditarily normal space with $\tau_i - dm\, A \leq m\ (m > 0)$ and $\tau_i\, \tau_j\, dmB = 0$. Let every p-open cover of bispace X have a τ_j-open refinement. Then $\tau_i\, \tau_j\, dm\, X \leq m + 2$, for $i \neq j$ and $i, j \in \{1, 2\}$.

PROOF : Let $\mathcal{U} = \{U_i \mid i = 1, ..., n\}$ be a τ_i-open cover of X. Then $\mathcal{U}_1 = \left\{ U_i \cap A \mid i = 1, ..., n \right\}$ and $\mathcal{U}_2 = \left\{ U_i \cap B \mid i = 1, ..., n \right\}$ are τ_i-open covers respectively of A and B. Then there exist τ_i-open in A refinement $\mathcal{V}_1 = \left\{ V_i \mid i = 1, ..., r \right\}$ of $\mathcal{U}_t$ and τ_j-open in B refinement $\mathcal{V}_2 = \left\{ V_i \mid j = 1, ..., s \right\}$ of $\mathcal{U}_2$ such that $ds\ \mathcal{N}(\mathcal{V}_1) \leq m + 1$ and $ds\ \mathcal{N}(\mathcal{V}_2) \leq 2$. By Lemma (L) in (3) there exist τ_i-open in X refinement $\mathcal{W}_1 = \left\{ W_i \mid i = 1, .., r \right\}$ of $\mathcal{V}_i$ and τ_j-open in X refinement $\mathcal{W}_2 = \left\{ H_i \mid i = 1, ..., s \right\}$ of $\mathcal{V}_2$ such that $ds\ \mathcal{N}(\mathcal{W}_1) \leq m + 1$ and $ds\ \mathcal{N}(\mathcal{W}_2) \leq 2$. Now family $\mathcal{S} = \left\{ H_1, \ldots, H_s, W_1, .., W_r \right\}$ is p-open cover in X. By hipothesis there exists a τ_j-open refinement $\mathcal{H}$ of $\mathcal{N}$ such that ds $\mathcal{N}(\mathcal{H}) \leq m + 3$. Therefore, $\tau_i\, \tau_j$ dm $X \leq m + 2$.

Corollary 3 — If (X, τ_k), $k = 1, 2$ is a totally normal space [9], the Proposition 4 is also true.

Remark 3 : Let $\tau_i\, dm\, A = 0$, in Proposition 4, then $\tau_i\, \tau_j dmX \leq 3$. Also if $\tau_i\, \tau_j\, dm\, B \leq n\ (n > 0)$ then $\tau_i\, \tau_j\, dm\, X \leq n + m + 1$.

Remark 4 : Let in Proposition 4, A and B be disjoint and let $\tau_i\, dm\, A = \tau_i\, \tau_j\, dm\, B \leq n$, then $\tau_i\, \tau_j\, dm\, X \leq n$.

The proofs of the next results are obtained by copying the proof of the Propositions for dm$X$.

Proposition 5 — Let X be the disjoint union of subsets A and B. Let (X, τ_j) be a hereditarily normal space and let $\tau_i\, \tau_j$ dm $A \leq n\ (n > 0)$ and $\tau_i\, \tau_j$ dm $B = 0$. Then $\tau_i\, \tau_j\, dm\, X \leq n + 2$, for $i \neq j$ and $i, j \in \{1, 2\}$.

Corollary 4 — Let (X, τ_j) be a super normal space [9]. Then, also, Proposition 5, holds.

Remark 5 : Let $\tau_i \tau_j$ dm $A = 0$ in Proposition 5, then $\tau_i \tau_j$ dm $X \leq 3$.

Propsition 6 — Let $X = A \cup B$ with $\tau_i \tau_j\, dm\, A \leq m\ (m > 0)$ and let $\tau_i \tau_j\, dm\, B = 0$. Let (X, τ_j) be a hereditarily normal space. Then $\tau_i \tau_j$ dm $X \leq m + 2$, for $i \neq j$ and $i, j \in \{1, 2\}$.

Remark 6 : Let, in Proposition 6, $m = 0$. Then $\tau_i \tau_j\, dm\, X \leq 3$.

Proposition 7 — Let (X, τ_j) be a normal space in a bitopological space X. Let $X = A \cup B$ with τ_j-closed subset A in X. Let $\tau_i \tau_j$ dm $A = 0 = \tau_i \tau_j$ dm. B and let each τ_j-open cover of X have a τ_i-open refinement. Then $\tau_i \tau_j\, dm\, X \leq 3$ for $i \neq j$ and $i, j \in \{1, 2\}$

Proposition 8 — Let $X = A \cup B$ and let A be a τ_j-closed subset of bispace X with $\tau_j\, dm\, A \leq n\ (n > 0)$ and $\tau_j\, dm\, B = 0$. Let (X, τ_i) be a normal space and let every τ_i-open cover of X have a τ_j-open refinement. Then $\tau_i \tau_j\, dm\, X \leq n + 2$, for $i \neq j$ and $i, j \in \{1, 2\}$.

Remark 7 : If $n = 0$, in Proposition 8, then $\tau_i \tau_j\, dm\, X \leq 3$.

Proposition 9 — Let (X, τ_j) be a normal space of bispace X. If there exists a point $p \in X$ such that $p \notin F$ for each τ_j-closed subset F, F has $\tau_i \tau_j\, dm\, F \leq n\ (n > 0)$ then $\tau_i \tau_j\, dm\, X \leq n + 1$, for $i \neq j$ and $i, j\ \{1, 2\}$.

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