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A SPECTRAL APPROXIMATION OF THE TWO-DIMENSIONAL BURGERS' EQUATION

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In this paper we present a numerical approximation of the two-dimensional Burgers' equation. The solution technique consists of approximating the components of the exact solution by a truncated Fourier series with time dependent coefficients. A system of nonlinear ordinary differential equations is derived and its numerical solution yields the coefficients, which can then be used to construct the approximate solution. Two preliminary examples are given at the end of the paper.

Key Words : Spectral Approximation; 2-dimensional Burgers Equation; Nonlinear Differential Equations; Cole-Hoph Transform

1. INTRODUCTION

The numerical solution of Burger's equation has received a fair amount of attention due to the fact that it is used to model a number of physically important phenomena such as shock waves and acoustic transmission. The resemblance between Burgers' equation and the Navier-Stokes equations for fluid flow makes Burgers' equation an important tool for testing numerical schemes for convection-diffusion phenomena. Burgers' equation is also important because it possesses a large number of exact solution which makes it possible to assess the effectiveness of numerical approximation methods. The Cole-Hoph transform (see Cole[1]) can be used to construct exact solutions of Burgers' equation for arbitrary boundary conditions. Benton and Plazman[2] tabulate a number of known exact solutions of the one-dimensional Burgers' equation.

The numerical solution of the convection dominated Burgers' equation in one space dimension has been studied extensively; see for example Caldwell[3] and Caldwell and Smith[4]. More recently, Mittal and Singhal[5] studied a numerical approximation of the one-dimensional Burgers' equation where they used a truncated Fourier expansion with time dependent coefficients and formulated an approximate problem which consisted of solving a system of non-linear ordinary differential equations for the coefficients. Mittal and Singhal[6] later extended their work to study Burgers' equation with periodic boundary conditions. This paper extends the work of Mittal and Singhal to the two-dimensional Burgers' equation.

Consider the 2-dimensional Burgers' equation

$$\frac{\partial U}{\partial t} + (U \cdot \nabla)\, U - \frac{1}{R} \Delta U = 0, \quad \text{... (1)}$$

where

$$U(x, y, t) = (u, v),\ U \cdot \nabla = u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y},\ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is the 2-dimensional Laplacian, and R is a parameter often called the Reynolds number. Problem (1) is solved in the square $S = (0, 2\pi) \times (0, 2\pi)$ subject to the boundary condition

$$U|_{\partial s} = 0 \quad \text{... (2)}$$

and the initial condition $U(x, y,. 0) = a$ given function on S.

It is well known that the functions

$$u_i(\xi) = \sqrt{\frac{2}{\pi}} \sin(i, \xi),\ i = 1, 2, \ldots. \quad \text{... (3)}$$

form an orthonormal basis for $L^2(0, 2\pi)$ and that the set

$$\varphi_{pq}(x, y) = u_p(x)\, u_q(y), \quad p, q = 1, 2, \ldots, \quad \text{... (4)}$$

is an orthonormal basis for $L^2(S)$.

We seek an approximation of (1) in the form $\tilde{U}(x, y, t) = (\tilde{u}, \tilde{v})$, where

$$\tilde{u} = \sum_{p,q=1}^{N} \alpha_{pq}(t)\, \phi_{pq}(x, y) \text{ and } \tilde{v} = \sum_{p,q=1}^{N} \beta_{pq}(t)\, \phi_{pq}(x, y). \quad \text{... (5)}$$

2. The Approximate Problem

Before formulating the approximate problem we establish below a few identities and introduce some notation necessary for the formulation.

For a fixed natural number N, let $r = (r_1, \ldots, r_N)$ and $s = (s_1, \ldots, s_N)$ be finite sequences of real numbers and consider the expression $\sum_{i=1}^{N} r_i u_i(\xi) \sum_{i=1}^{N} s_j \frac{u_j(\xi)}{d\xi}$. Clearly, the sum is obviously a linear combination of the basis functions $u_2, \ldots, u_{2N}$ and therefore can be written as

$$\sum_{i=1}^{N} r_i u_i(\xi) \sum_{i=1}^{N} s_j \frac{u_j(\xi)}{d\xi} = \sum_{i,j=1}^{N} r_i s_j u_i(\xi) \frac{du_j(\xi)}{d\xi} = \frac{1}{\sqrt{2\pi}}$$

$$\sum_{i,j=1}^{N} j\, r_i\, s_j\, [u_{i+j}(\xi) + u_{i-j}(\xi)], \qquad \dots (6)$$

where the last equality of (6) is obtained using a well known trigonometric identity (see Mittal).

Consider the first term $\sum_{i,j=1}^{N} jr_i s_j u_{i+j}(\xi)$ of the summation on the far right hand side of eq. (6). The sum is obviously a linear combination of the basis functions $u_2, \dots, u_{2N}$ and therefore can be written as

$$\frac{1}{\sqrt{2}\,\pi} \sum_{i,j=1}^{N} jr_i s_j u_{i+j}(\xi) = \frac{1}{\sqrt{2}\,\pi} \left(\sum_{k=2}^{N} P_k u_k(\xi) + \sum_{k=N+1}^{2N} C_k u_k(\xi) \right) \qquad \dots (7)$$

The coefficient P_k of $u_k(\xi)$ is the sum of the contributions of the coefficients $jr_i s_j$ whenever $i + j = k$, thus

$$P_k(r, s) = \sum_{j=1}^{k-1} js_j r_{k-j}. \qquad \dots (8)$$

An exact formula for the coefficients C_k can be obtained but will not be given here because they will not be used in the eventual formulation of the approximate problem.

Consider the second term $\sum_{i,j=1}^{N} jr_i s_j u_{i-j}(\xi)$ of the summation on the far right hand side of eq. (6). This is a linear combination of $u_1(\xi), \dots, u_{N-1}(\xi)$ (note that $u_0(\xi) = 0$, and $u_{-j}(\xi) = -u_j(\xi)$), thus we can write

$$\sum_{i,j=1}^{N} jr_i s_j u_{i-j}(\xi) = \sum_{k+1}^{N-1} G_k u_k(\xi). \qquad \dots (9)$$

To find the coefficients G_k, first note that the sum $\sum_{i,j=1}^{N} jr_i s_j u_{i-j}(\xi)$ can be rearranged as follows

$$\sum_{i,j=1}^{N} jr_i s_j u_{i-j}(\xi) = \sum_{k=1}^{N-1} \left(\sum_{j=1}^{N-k} -(j+k)\, r_j s_{k+j} + jr_{k+j} s_j \right) \cdot u_k(\xi).$$

Thus

$$G_k(r,s) = \sum_{j=1}^{N-k} -(j+k)\, r_j\, s_{k+j}\, s. \qquad \dots (10)$$

Note that the coefficients P_k and G_k given in eqs. (8) and (9) are in agreement with the formulas derived by Mittal and Singhal in the special case when the sequences r and s are identical.

As we mentioned in the introduction, we seek an approximation of u and v in the form

$$\tilde{u}(x,y) = \sum_{i,j=1}^{N} \alpha_{ij}(t)\, u_i(x)\, u_j(y), \; \tilde{v}(x,y) = \sum_{i,j=1}^{N} \beta_{ij}(t)\, u_i(x)\, u_j(y). \qquad \dots (11)$$

Consider the convection term $M\tilde{u} = \tilde{u}\frac{\partial \tilde{u}}{\partial x} + \tilde{v}\frac{\partial \tilde{u}}{\partial y}$ in the first component eq. (1).

Using eqs. (7) and (9) we can write

$$\begin{aligned} M\tilde{u} = \tilde{u}\frac{\partial \tilde{u}}{\partial x} + \tilde{v}\frac{\partial \tilde{u}}{\partial y} &= \left(\sum_{i,j=1}^{N} \alpha_{ij}\, u_i(x)\, u_j(y) \right)\left(\sum_{k,l=1}^{N} \alpha_{kl} \frac{du_k(x)}{dx}\, u_l(y) \right) \\ &+ \left(\sum_{i,j=1}^{N} \beta_{ij}\, u_i(x)\, u_j(y) \right)\left(\sum_{k,l=1}^{N} \alpha_{kl} \frac{du_l(y)}{dy}\, u_k(x) \right) \\ &= \frac{1}{\sqrt{2}\,\pi} \sum_{j,l=1}^{N} \left(\sum_{m=1}^{N-1} G_m(\alpha_{\otimes,j}; \alpha_{\otimes;l})\, u_m(x) + \sum_{m=1}^{N-1} P_m(\alpha_{\otimes,j}; \alpha_{\otimes,l})\, u_m(x) \right. \\ &\left. + \sum_{m=N+1}^{2N} C_m(\alpha_{\otimes,j}; \alpha_{\otimes,l})\, u_m(x) \right) u_j(y)\, u_l(y) \\ &+ \frac{1}{\sqrt{2}\,\pi} \sum_{i,k=1}^{N} \left(\sum_{m=1}^{N-1} G_m(\beta_{i,\otimes}; \alpha_{k,\otimes})\, u_m(y) + \sum_{m=1}^{N-1} P_m(\beta_{i,\otimes}; \alpha_{k,\otimes})\, u_m(y) \right. \\ &\left. + \sum_{m=N+1}^{2N} C_m(\beta_{i,\otimes}; \alpha_{k,\otimes})\, u_m(y) \right) u_i(x)\, u_k(x). \end{aligned} \qquad \dots (12)$$

In the above expressions, coefficients denoted by a double index $\otimes; j$ denote the real sequence obtained by fixing j and varying the index denoted by $\otimes$. Thus for example the symbol $G_q(\beta_{i,\otimes}; \alpha_{k,\otimes})$ stands for $G_q(r,s)$ as given by eq. (10), where $r = (\beta_{i1}, \beta_{i2}, \dots, \beta_{iN})$ and $s = (\alpha_{k1}, \alpha_{k2}, \dots, \alpha_{kN})$.

Note that, unlike the one-dimensional case, the two-dimensional convection operator M is not finitely reproducing in the sense of Campos[7].

Now the approximate problem is obtained by requiring the residual upon substituting $\tilde{u}$ and $\tilde{v}$ to be orthogonal to every basis function ϕ_{pq} for $p, q = 1, 2, ..., N$. Taking the inner product of eq. (11) with ϕ_{pq} then replacing G_p and P_p by the expressions in eqs. (8) and (10) yields the following

$$\langle M\tilde{u}, \phi_{pq}\rangle = \frac{1}{\sqrt{2}\,\pi} \sum_{j,l=1}^{N} (G_p(\alpha_{\otimes,j}; \alpha_{\otimes,l}) + P_p(\alpha_{\otimes,j}; \alpha_{\otimes,l})) \int_0^{\pi} u_j(y)\, u_l(y)\, u_q(y)\, dy$$

$$+ \frac{1}{\sqrt{2}\,\pi} \sum_{i,k=1}^{N} (G_q(\beta_{i,\otimes}; \alpha_{k,\otimes}) + P_q(\beta_{i,\otimes}; \alpha_{k,\otimes}))$$

$$\int_0^{\pi} u_i(x)\, u_k(x)\, u_p(x)\, dx$$

$$= \frac{1}{\sqrt{2}\,\pi} \sum_{j,l=1}^{N}$$

$$\left[\sum_{m=1}^{p-1} m\, \alpha_{p-m,j}\, \alpha_{m,l} + \sum_{m=1}^{N-p} (m\, \alpha_{p+m,j}\, \alpha_{m,l} - (m+p)\, \alpha_{m,j}\, a_{p+m,l}) \right] S_{jlq}$$

$$+ \frac{1}{\sqrt{2}\,\pi} \sum_{i,k=1}^{N}$$

$$\left[\sum_{m=1}^{q-1} m\, \beta_{i,p-m}\, \alpha_{m,m} + \sum_{m=1}^{N-q} (m\, \beta_{i,q+m}\, \alpha_{k,m} - (m+q)\, \beta_{i,m}\, a_{k,q+m}) \right] S_{ikp}\,, \quad \ldots (13)$$

where

$$S_{jlq} = \int_0^{\pi} u_j(\xi)\, u_l(\xi)_q(\xi)\, d\,\xi.$$

The integrals S_{jlq} are clearly elementary integrals and their exact values can be found.

Likewise, one can derive the expression for the $\langle Mv, \phi_{pq}\rangle$ in a similar fashion :

$$\langle M\tilde{u}, \phi_{pq}\rangle = \frac{1}{\sqrt{2}\,\pi} \sum_{j,l=1}^{N} (G_p(\alpha_{\otimes,j}; \beta_{\otimes,l}) + P_p(\alpha_{\otimes,j}; \beta_{\otimes,l})) \int_0^{\pi} u_j(y)\, u_l(y)\, u_q(y)\, dy$$

$$+ \frac{1}{\sqrt{2}\,\pi} \sum_{i,k=1}^{N} (G_q(\beta_{i,\otimes}; \beta_{k,\otimes}) + P_q(\beta_{i,\otimes}; \beta_{k,\otimes}))$$

$$\int_0^{\pi} u_i(x)\, u_k(x)\, u_p(x)\, dx$$

$$= \frac{1}{\sqrt{2}\,\pi} \sum_{j,l=1}^{N}$$

$$\left[\sum_{m=1}^{p-1} m\, \alpha_{p-m,j}\, \beta_{m,l} + \sum_{m=1}^{N-p} (m\, \alpha_{p+m,j}\, \beta_{m,l} - (m+p)\, \alpha_{m,j}\, \beta_{p+m,l}) \right] S_{jlq}$$

$$+ \frac{1}{\sqrt{2}\,\pi} \sum_{i,k=1}^{N}$$

$$\left[\sum_{m=1}^{q-1} m\, \beta_{i,p-m}\, \beta_{m,m} + \sum_{m=1}^{N-q} (m\, \beta_{i,q+m}\, \beta_{k,m} - (m+q)\, \beta_{i,m}\, \beta_{k,q+m}) \right] S_{ikp}. \quad \dots (14)$$

Assembling all the above information yields to the following system of ordinary differential equations for the coefficients α_{pq} and β_{pq}

$$\frac{d\,\alpha_{pq}(t)}{dt} = -\frac{1}{R}(p^2+q^2)\,\alpha_{pq} - \langle M\tilde{u}, \phi_{pq} \rangle \quad \dots (15)$$

and

$$\frac{d\,\beta_{pq}(t)}{dt} = -\frac{1}{R}(p^2+q^2)\,\beta_{pq} - \langle M\tilde{v}, \phi_{pq} \rangle, \quad \dots (16)$$

where $p, q = 1, 2, \dots, N$, and $\langle M\tilde{u}, \phi_{pq} \rangle$ and $\langle M\tilde{v}, \phi_{pq} \rangle$ are given by eqs. (13) and (14).

3. Numerical Examples

The system (1) and (2) was solved for the following set of initial conditions

$$u(x, y, 0) = v(x, y, 0) = \phi_{2,2}. \quad \dots (17)$$

Thus the initial conditions for eqs. (15) and (16) are

$$\alpha_{22}(0) = \beta_{22}(0) = 1,\ \alpha_{ij}(0) = \beta_{ij}(0) = 0 \text{ for } (i,j) \neq (2,2). \quad \dots (18)$$

The initial conditions were chosen for the simplicity of their Fourier coefficients but more complicated initial conditions can be implemented at the cost of computing their Fourier coefficients (via numerical integration). In this example, the two scalar components of the exact solution are identical and the same was reflected perfectly by the components of the computed approximate solutions reported below.

Two test cases for $R = 10$ and $R = 100$ were run for a variety of values of N. For the first case, $R = 10$, the problem is weakly convection-dominated and the solution reported here is

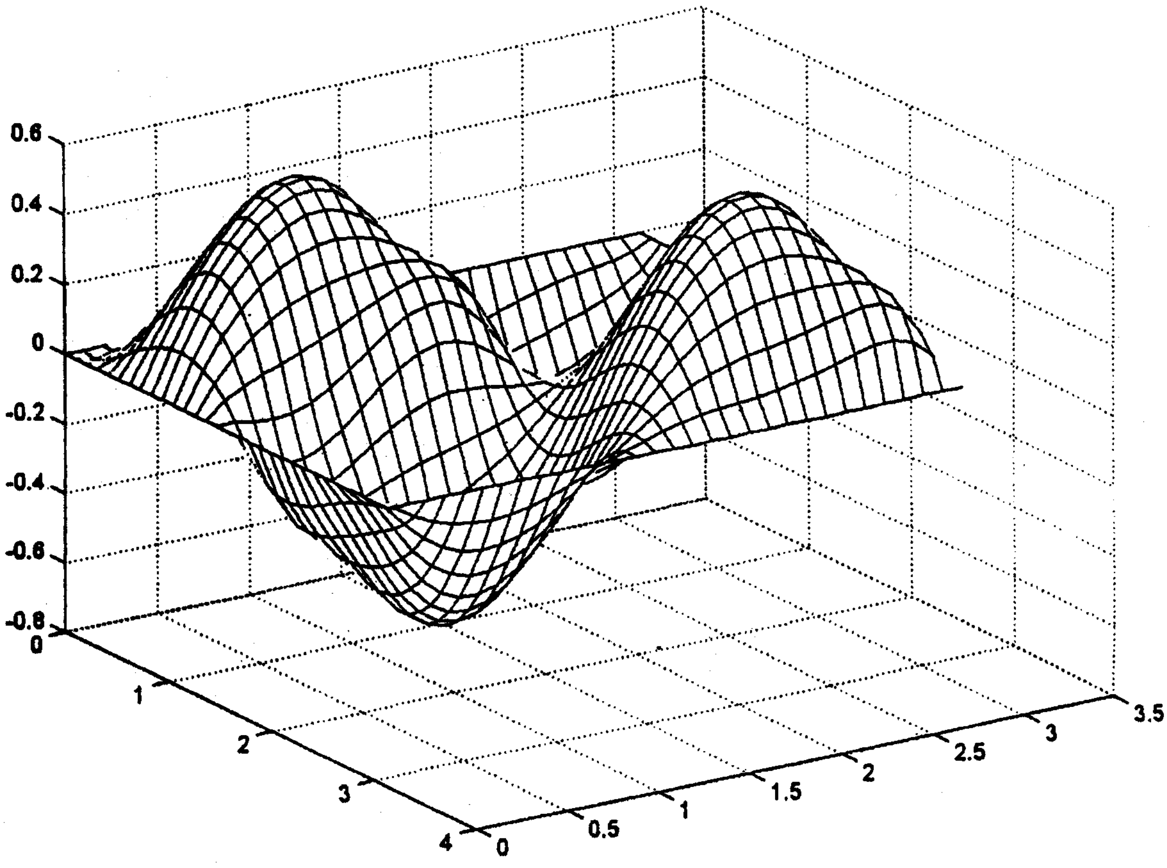

FIG. 1a. $R = 10$; $t = 2$

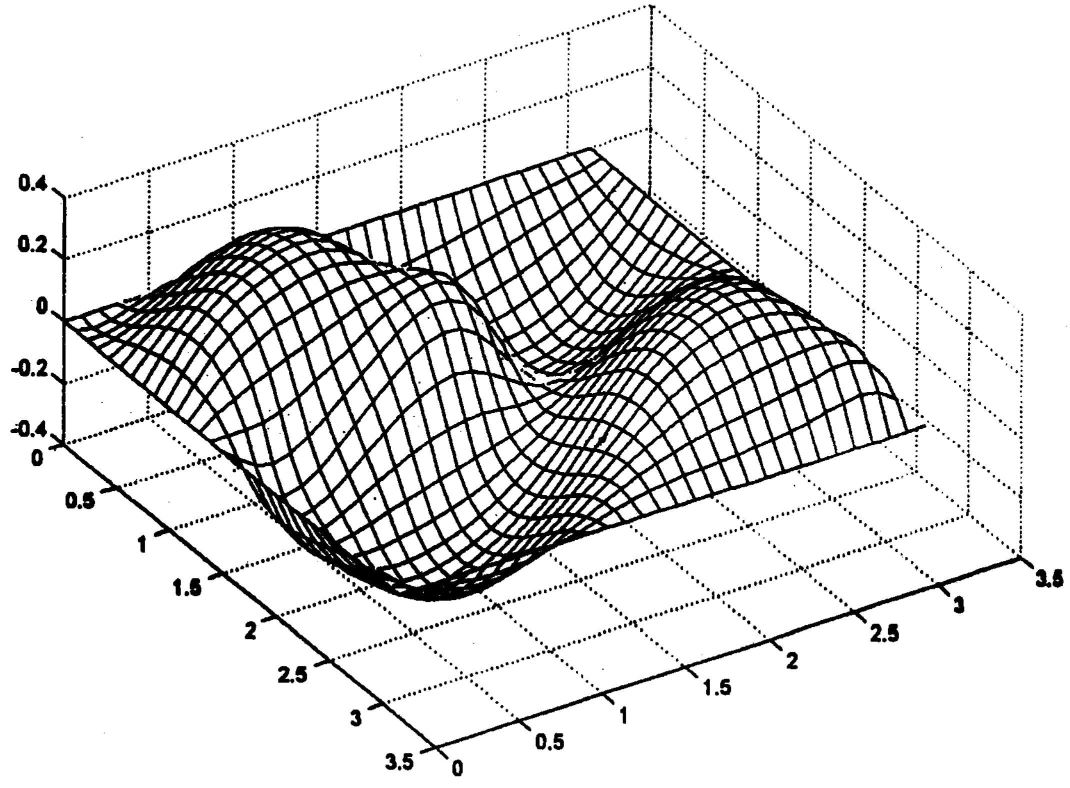

FIG. 1b. $R = 10$; $t = 6$

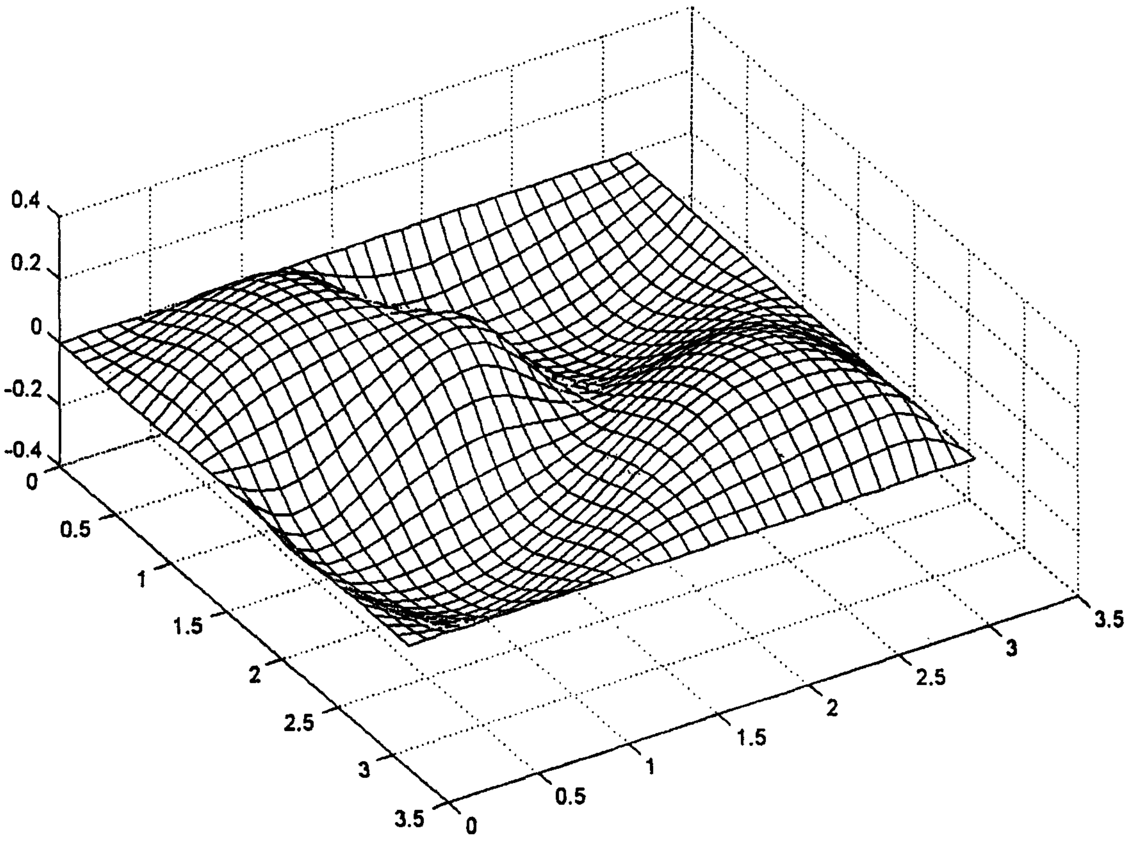

FIG. 1c. $R = 10$; $t = 1$

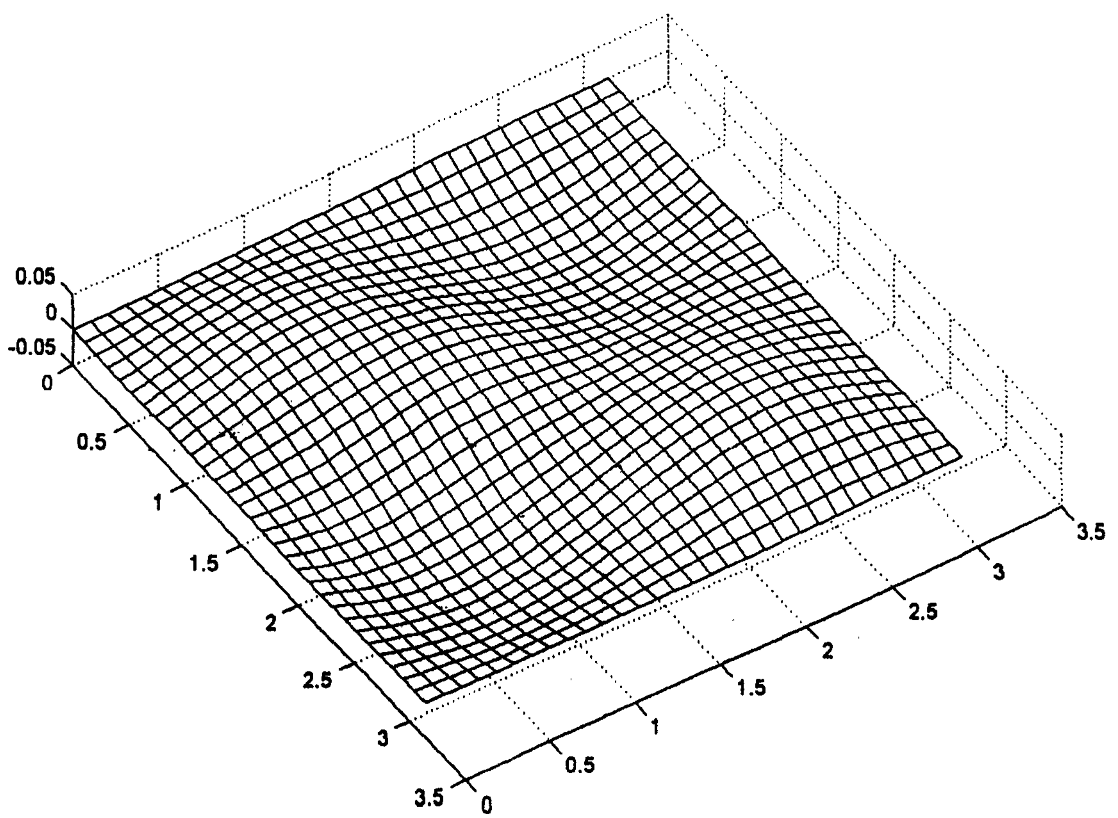

FIG. 1d. $R = 10$; $t = 3$

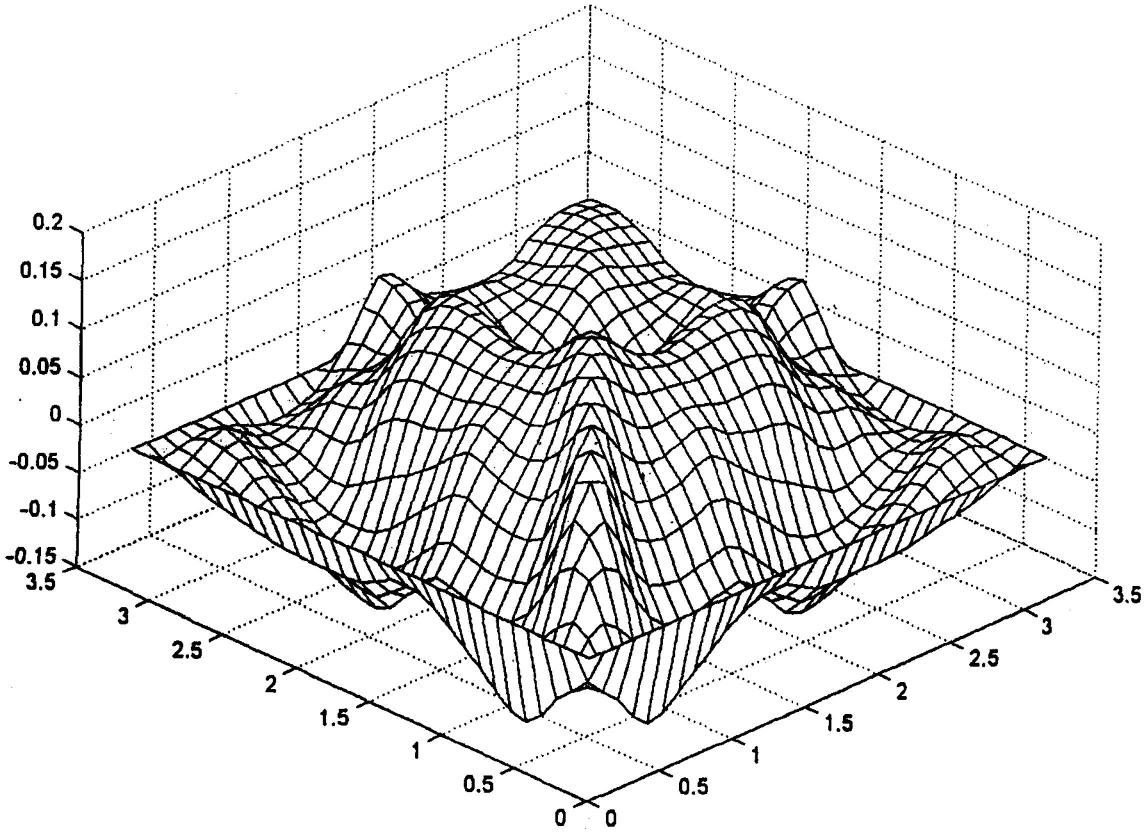

FIG. 2a. $R = 100$; $t = 5$

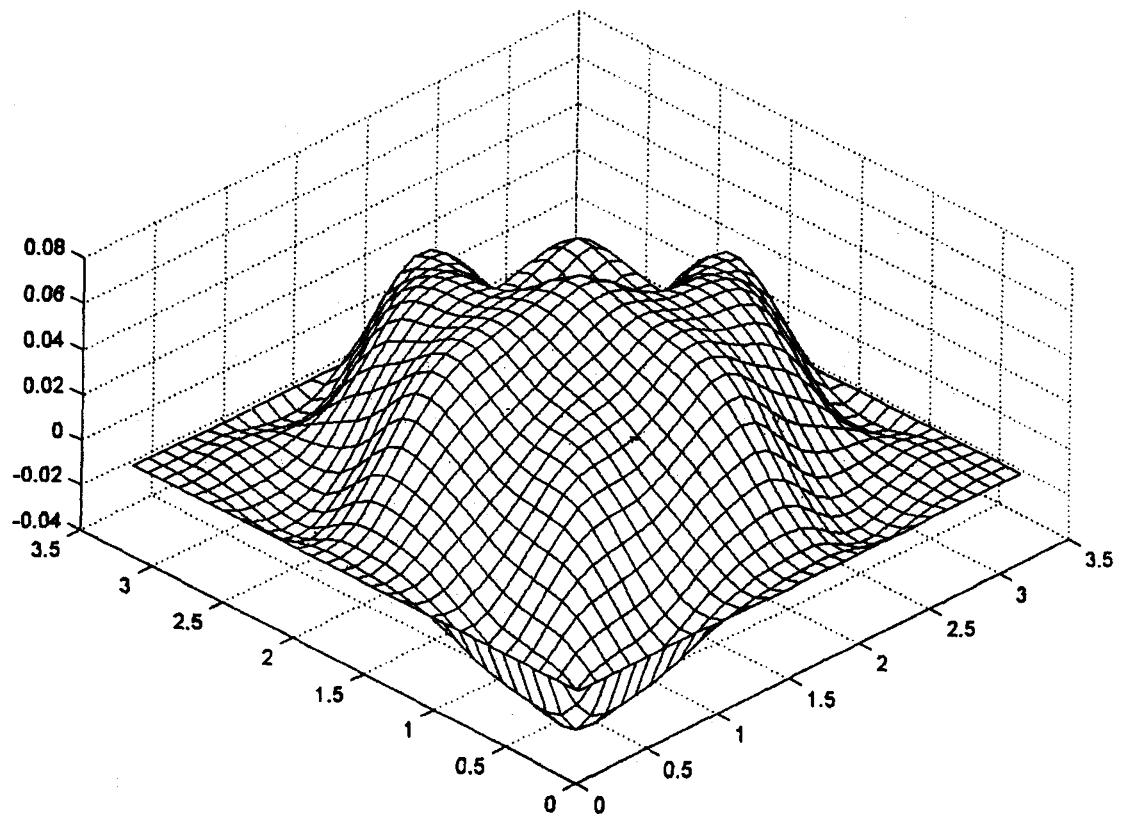

FIG. 2b. $R = 100$; $t = 10$

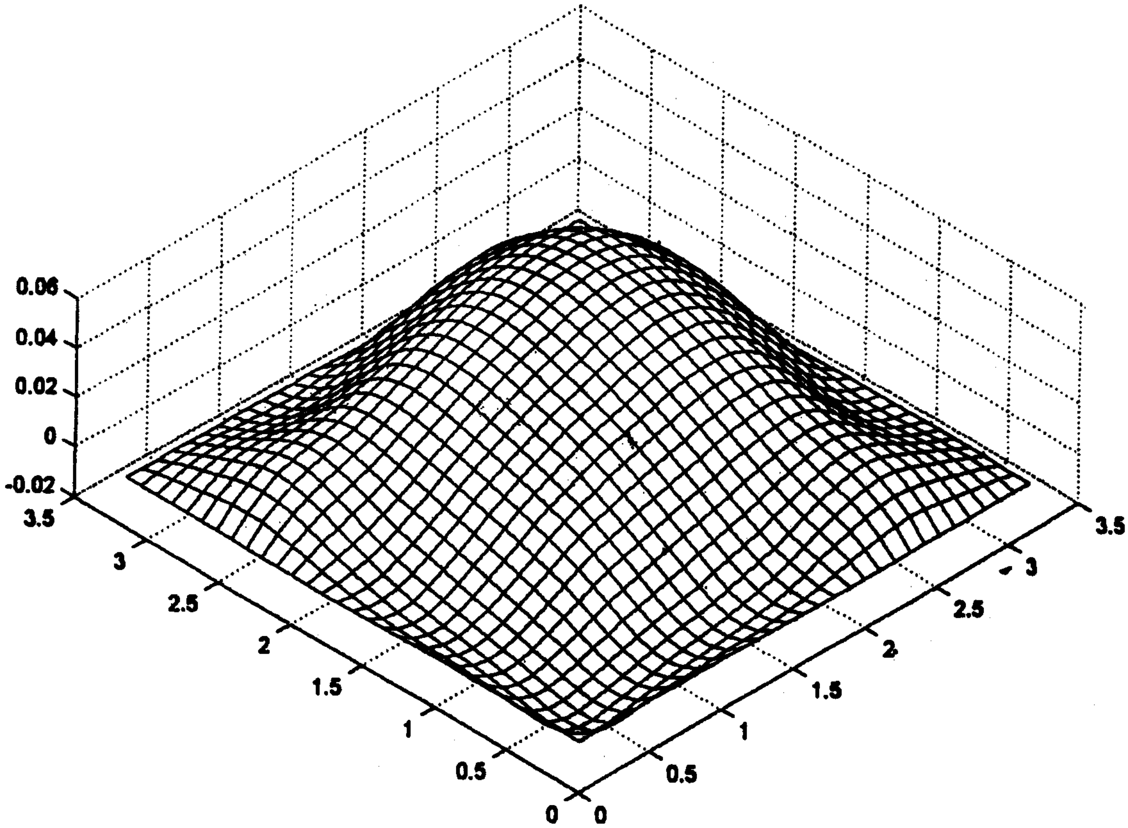

FIG. 2c. $R = 100$; $t = 15$

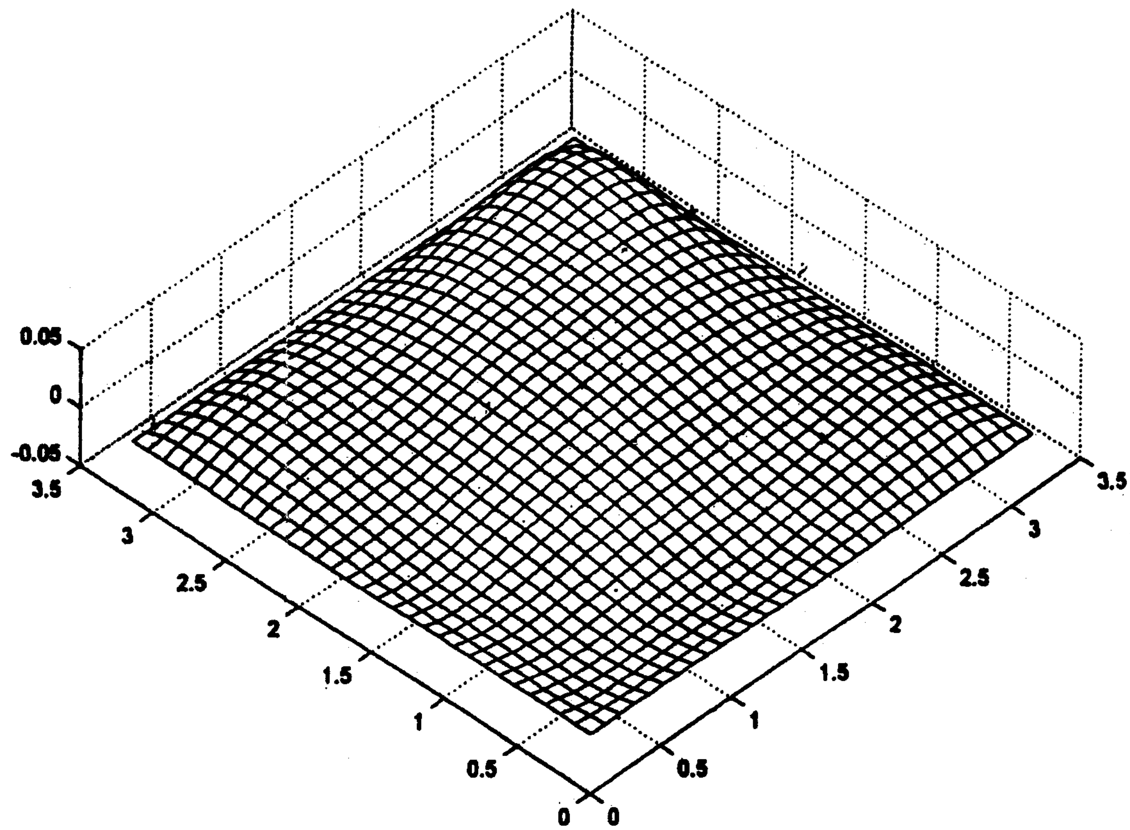

FIG. 2d. $R = 100$; $t = 20$

for $N = 25$ (thus there are 1250 equations involved in the system of ordinary differential eqs. (15) and (16) and the approximate solution was computed for $t = .2, .4, .6, 1, 2$ and 3. The results for some of those times are shown in Fig. 1. The solution shows a straightforward diffusion pattern and the structure of the initial condition is well preserved, as expected.

The same example was implemented for $R = 100$ and, in that case, three runs were made for $N = 20$, 25 and 30. A comparison of the maximum and minimum values of the computed solution for $N = 25$ vs. $N = 30$ are shown in Table I. The difference between the values for $N = 25$ and $N = 30$ are quite small and leads us to believe that the number of eigenfunctions ($N = 30$) used is adequate to capture the solution and therefore gives us confidence in the accuracy of the solution. The results for $R = 100$, $N = 30$ are shown in Fig. 2 for $t = 5$, 10, 15 and 20. Incidentally, a similar comparison was made fo rthe case $R = 10$ and the difference of the maximum and minimum values of the computed solutions for $N = 20$ vs. $N = 25$ are quite miniscule.

TABLE I

	Max. value of u $N = 25$	Max. value of u $N = 30$	Min. value of u $N = 25$	Min. value of u $N = 30$
$T = 5$	.1646	.1639	−.1083	−.1106
$T = 10$	.0666	.0672	−.0311	−.0313
$T = 15$	.0534	.0533	−.0158	−.0160
$T = 20$	.0438	.0438	−.0096	−.0097
$T = 25$	.0344	.0343	−.0061	−.0062

The system of ordinary differential eqs. (15) and (16) for the coefficients was solved numerically using a software package based on an 8th order Runge-Kutta method with two imbedded loer order Runge-Kutta methods. A description of the method can be found in the book by Hairer, Norsett, and Wanner[8].

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