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WEAKER FORMS OF CONTINUITY IN SOSTAK'S FUZZY TOPOLOGY

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We introduce r-fuzzy semi-open, r-fuzzy strongly semi-open, r-fuzzy preopen and r-fuzzy semi-preopn sets in a fuzzy topological spaces in view of the definition of *Sostak*[13]. We invstigate some properties of them. Moreover, we investigate the relationship between fuzzy continuity, fuzzy semi-continuity, fuzzy strong semi-continuity, fuzzy precontinuity and fuzzy semi-precontinuity.

Key Words : r-Fuzzy Semi-open; r-Fuzzy Strongly Semi-Open; r-Fuzzy Semi-preopen; Fuzzy semi-Continuity; Fuzzy Strong Semi-Continuity Fuzzy Semi-Precontinuity

1. INTRODUCTION AND PRELIMINARIES

Sostak[13] introduced the fuzzy topology as an extension of Chang's fuzzy topology[3]. It has been developed in many directions[6, 7 & 12]. Weaker forms of fuzzy continuity between fuzzy topological spaces have been considered by many authors[1, 2, 4 & 5, 9 & 10] using the concepts of fuzzy semi-open sets[1], fuzzy regularly open sets[1], fuzzy preopen sets, and fuzzy strongly semi-open sets[2]. Recently, Shahna[2] introduced and investigated fuzzy strong semi-continuity and fuzzy precontinuity between fuzzy topological spaces, one of which was independent and the other strictly stronger than fuzzy semi-continuity[1]. Park *et al.*[8] introduced the concept of fuzzy semi-preopen sets which is weaker than any of the concepts of fuzzy semi-open or fuzzy preopen. Using this concepts he define and study fuzzy semi-precontinuous mapping between chang's fuzzy topological spaces.

In this paper, we define r-fuzzy semi-open, r-fuzzy semi-closed, r-fuzzy strongly semi-open, r-fuzzy strongly semi-closed, r-fuzzy preopen, r-fuzzy preclosed, r-fuzzy semi-preopen and r-fuzzy semi=preclosed sets in a fuzzy topological space in view of the definition of Sostak.

Using these concepts, we define and study fuzzy semi-continuity, fuzzy strong semi-continuity, fuzzy precontinuity, fuzzy semi-precontinuity, fuzzy semi-open mapping, fuzzy strongly semi-open mapping fuzzy semi-preopen mappings and fuzzy almost continuous mapping.

Throughout this paper, let X be a non-empty set, I = [0, 1] and I_0 = (0, 1]. For $\alpha \in I, \alpha(x) = \alpha \ \forall \ x \in X$. All the other notations and the other definitions are standard in the fuzzy set theory.

Definition 1.1[13] — A function $\tau : I^X \to I$ is called a fuzzy topology on X if it satisfies the following conditions :-

(01) $\tau(0) - \tau(1) = 1$.

(02) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$, for any $\mu_1, \mu_2 \in I^X$.

(03) $\tau(\wedge_{i \in \Gamma} \mu_i) \geq \wedge_{i \in \Gamma} \tau(\mu_i)$, for any $\{\mu\}_{i \in \Gamma} \subset I^X$.

The pair (X, τ) is called a fuzzy topological space (for short, fts).

If $\tau(\lambda) = 1$ for all $\lambda \in I^X$, then τ is called the fuzzy discrete topology on X. Also, if $\tau(0) = \tau(1) = 1$ and $\tau(\lambda) = 0$ otherwise, then τ is called the fuzzy indiscrete topology on X.

Remark 1.2 : Let (X, τ) be a fuzzy topological space. Then, for each $r \in I$, $\tau_r = \{\mu \in I^X \mid \tau(\mu) \geq r\}$ is a Chang's fuzzy topology on X.

***Theorem* 1.3**[12] — *Let (X, τ) be a fuzzy topological space. For each $r \in I_0$, $\lambda \in I^X$, we define an operator $C_\tau : I^X \times I_0 \to I^X$ as follows :-*

$$C_\tau(\lambda, r) = \wedge \{\mu \mid \lambda \leq \mu, \tau(1 - \mu) \geq r\}.$$

For each $\lambda, \mu \in I^X$ and $r, s \in I_0$ it satisfies the following conditions :

(1) $C_\tau(0, r) = 0$,

(2) $\lambda \leq C_\tau(\lambda, r)$,

(3) $C_\tau(\lambda, r) \vee C_\tau(\mu, r) = C_\tau(\lambda \vee \mu, r)$ and $C_\tau(\lambda, r) \wedge C_\tau(\mu, r) \geq C_\tau(\lambda \wedge \mu, r)$,

(4) $C_\tau(\lambda, r) \leq C_\tau(\lambda, s)$ if $r \leq s$

and (5) $C_\tau(C_\tau(\lambda, r), r) = C_\tau(\lambda, r)$.

***Theorem* 1.4** — *Let (X, τ) be a fuzzy topological space. For each $r \in I_0$, $\lambda \in I^X$, we define an operator $I_\tau : I^X \times I_0 \to I^X$ as follows :*

$$I_\tau(\lambda, r) = \vee \{\mu \mid \lambda \geq \mu, \tau(\mu) \geq r\}.$$

For each $\lambda, \mu \in I^X$ and $r, s \in I_0$, it satisfies the following conditions :

(1) $I_\tau(1-\lambda, r) = 1 - C_\tau(\lambda, r)$, and $C_\tau(1-\lambda, r) = 1 - I_\tau(\lambda, r)$

(2) $I_\tau(1, r) = 1$

(3) $I_\tau(\lambda, r) \leq \lambda$,

(4) $I_\tau(\lambda, r) \wedge I_\tau(\mu, r) = I_\tau(\lambda \wedge \mu, r)$ and

$I_\tau(\lambda, r) \vee I_\tau(\mu, r) \leq I_\tau(\lambda \vee \mu, r)$

(5) $I_\tau(\lambda, r) \geq I_\tau(\lambda, s)$ if $r \leq s$,

(6) $I_\tau(I_\tau(\lambda, r), r) = I_\tau(\lambda, r)$.

2. r-Fuzzy Semi-Open, r-Fuzzy Strongly Semi-Open, r-Fuzzy Semi-Preopen, and r-Fuzzy Preopen Sets

Definition 2.1 — Let (X, τ) be a fts, $\lambda \in I^X$ and $r \in I_0$.

(1) λ is called r-fuzzy semi-open (resp. r-fuzzy strongly semi-open) set iff there exists

$\mu \in I^X$ with $\Rightarrow \tau(\mu) \geq r$ such that

$\mu \leq \lambda \leq C_\tau(\mu, r)$, resp. $\mu \leq \lambda \leq I_r(C_r(\mu, r), r))$.

(2) λ is caled r-fuzzy preopen (resp. r-fuzzy semi-preopen) set if

$\lambda \leq I_\tau(C_\tau(\lambda, r), r)$ (resp. $\lambda \leq C_\tau(I_\tau(C_\tau(\lambda, r), r), r)$), and

(3) λ is called r-fuzzy regular open if $\lambda = I_\tau(C_\tau(\lambda, r), r)$.

Definition 2.2 — Let (X, τ) be a fts, $\lambda \in I^X$ and $r \in I_0$.

(1) λ is called r-fuzzy semi-closed (resp. r-fuzzy strongly semi-closed) set iff there exists

$\mu \in I^X$ with $\tau(1-\mu) \geq r$ such that —

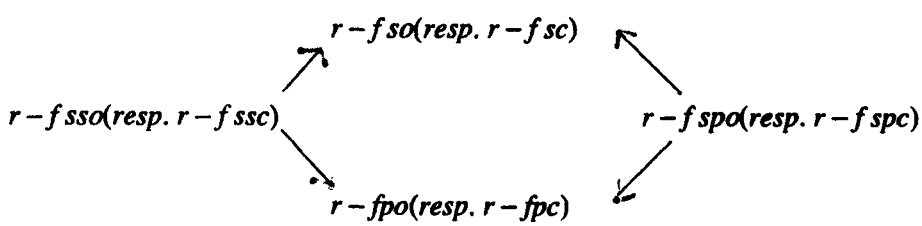
$I_\tau(\mu, r) \leq \lambda \leq \mu$ (resp. $C_\tau(I_\tau(\mu, r), r) \leq \lambda \leq \mu$)

(2) λ is called r-fuzzy preclosed (resp. r-fuzzy semi-preclosed) set if

$C_\tau(I_\tau(\lambda, r), r) \leq \lambda$ $(resp.\ I_\tau(C_\tau(I_\tau(\lambda, r), r), r) \leq \lambda)$.

(3) λ is called r-fuzzy regular closed if $\lambda = C_\tau(I_\tau(\lambda, r), r)$.

Remark 2.3 : From the above definitions it is clear that the following implications are true, for $r \in I_0$,



Where, $r - fso$, $r - fsc$, $r - fsso$, $r - fssc$, $r - fpo$, $r - fspo$ and $r - fspc$ are abbreviated to r-fuzzy semi-open, r-fuzzy semi-closed, r-fuzzy strongly semi-open, r-fuzzy strongly semi-closed, r-fuzzy preopen, r-fuzzy preclosed, r-fuzzy semi-preopen and r-fuzzy semi-preclosed respectively.

The converse of the implications are not true as the following example show

Example 2.4 — Let $X = \{a, b\}$ be a set. Define $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in I^X$ as follows :

$$\lambda_1(a) = 0.5,\ \lambda_1(b) = 0.3,\ \ \lambda_2(a) = 0.5,\ \lambda_2(b) = 0.6,$$

and

$$\lambda_3(a) = 0.6,\ \lambda_3(b) = 0.3,\ \lambda_4(a) = 0.6,\ \lambda_4(b) = 0.7.$$

We define fuzzy topologies $\tau_1, \tau_2 : I^X \to I$ as follows

$$\tau_1(\lambda) = \begin{cases} 1, & \text{if } \lambda \in (0, 1) \\ \frac{1}{2}, & \text{if } \lambda = \lambda_1 \\ 0, & \textit{otherwise}, \end{cases} \qquad \tau_2(\lambda) = \begin{cases} 1, & \text{if } \lambda \in (0, 1) \\ \frac{2}{3}, & \text{if } \lambda = \lambda_2 \\ 0, & \textit{otherwise}, \end{cases}$$

Then,

(1) In (X, τ_1), λ_2 is $\frac{1}{2} - fso$ set which is neither $\frac{1}{2} - fsso$ nor $\frac{1}{2} - fpo$ set.

(2) In (X, τ_1), λ_3 is $\frac{1}{2} - fpo$ set which is neither $\frac{1}{2} - fsso$ nor $\frac{1}{2} - fso$ set.

(3) In (X, τ_1), λ_4 is $\frac{2}{3} - fsso$ set but $\tau_2(\lambda_4) = 0$.

***Theorem* 2.5** — *Let (X, τ) be a fts. For $\lambda \in I^X$ and $r \in I_0$, the following statements are equivalent :-*

(1) λ is $r - f$ so.

(2) $\lambda \le C_\tau(I_\tau(\lambda, r), r)$.

(3) $C_\tau(\lambda, r) = C_\tau(I_\tau(\lambda, r), r)$.

(4) $1 - \lambda$ is $r - fsc$.

(5) $I_\tau(C_\tau(1 - \lambda, r), r) \le 1 - \lambda$.

(6) $I_\tau(C_\tau(1 - \lambda, r), r) = I_\tau(1 - \lambda, r)$.

PROOF : (1) $\Rightarrow$ (2) Let λ be a $r - fso$. There exists $\mu \in I^X$ such that $\mu \le \lambda \le C_\tau(\mu, r)$ with $\tau(\mu) \ge r$. Since $\tau(\mu) \ge r$, by Theorem 1.4, $I_\tau(\mu, r) = \mu$. Since, $\mu \le \lambda$, we have, $\mu = I_\tau(\mu, r) \le I_\tau(\lambda, r)$. It implies

$$C_\tau(\mu, r) \le C_\tau(I_\tau(\lambda, r), r).$$

Since, $\lambda \le C_\tau(\mu, r)$, we have

$$\lambda \le C_\tau(I_\tau(\lambda, r), r).$$

(2) $\Rightarrow$ (3) By the definition of C_τ and (2),

$$C_\tau(\lambda, r) \le C_\tau(I_\tau(\lambda, r), r).$$

Since, $I_\tau(\mu, r) \le \lambda$,

$$C_\tau(I_\tau(\lambda, r), r) \le C_\tau(\lambda, r).$$

Thus, $C_\tau(\lambda, r) = C_\tau(I_\tau(\lambda, r), r)$.

(3) $\Rightarrow$ (1) Put $\mu = I_\tau(\lambda, r)$. By the definition of I_τ from Theorem 1.4, and Definition 1.1 (03), we have $\tau(\mu) \ge r$. Thus, by (3),

$$\mu \le \lambda \le C_\tau(\lambda, r) = C_\tau(I_\tau(\lambda, r), r) = C_\tau(\mu, r).$$

Hence, λ is $r-fso$ set.

(1) $\Leftrightarrow$ (4) It is easily proved from the following :

$$\mu \le \lambda \le C_\tau(\mu, r)$$

$$\Leftrightarrow\ 1 - C_\tau(\mu, r) \le 1 - \lambda \le 1 - \mu$$

$$\Leftrightarrow\ I_\tau(1 - \mu, r) \le 1 - \lambda \le 1 - \mu. \text{ (by Theorem 1.4(1))}$$

(2) $\Leftrightarrow$ (5) and (3) $\Leftrightarrow$ (6) It is easily proved from Theorem 1.4(1) ■

***Theorem* 2.6** — *Let (X, τ) be a fts. For $\lambda \in I^X$ and $r \in I_0$, the following statements are equivalent :-*

(1) λ is $r - fsso$.

(2) $\lambda \le I_\tau(C_\tau(I_\tau(\lambda, r), r), r)$.

(3) $1 - \lambda$ is $r - fssc$.

(4) $C_\tau(I_\tau(C_\tau(1 - \lambda, r), r), r) \le 1 - \lambda$.

PROOF : Similar to the proof of Theorem 2.5. ■

***Theorem* 2.7** — *Let (X, τ) be a fts. Let $\lambda \in I^X$, $r \in I_0$.*

(1) $\tau(\lambda) \ge r$ (resp. $\tau(1 - \lambda) \ge r$) then, λ is $r - fsso$ (resp. $r - fssc$).

(2) $I_\tau(\lambda, r)$ is $r-fsso$.

(3) $C_\tau(\lambda, r)$ is $r-fssc$.

(4) If λ is $r-fso$ and $I_\tau(\lambda, r) \le \mu \le C_\tau(\lambda, r)$, then μ is $r-fso$.

(5) If λ is $r-fsc$ and $I_\tau(\lambda, r) \le \mu \le C_\tau(\lambda, r)$, then μ is $r-fsc$.

(6) If λ is $r-fsso$ then $C_\tau(\lambda, r) = C_\tau(I_\tau(C_\tau(I_\tau(\lambda, r), r), r), r)$.

PROOF : (1) Since $\lambda \le I_\tau(C_\tau(\lambda, r), r)$, it is trivial.

(2) From the definition of I_τ of Theorem 1.4, and Definition 1.1 (O3), since $\tau(I_\tau(\lambda, r)) \ge r$, by (1), $I_\tau(\lambda, r)$ is $r-fsso$.

(3) Since, $1-C_\tau(\lambda, r) = I_\tau(1-\lambda, r)$ from Theorem 1.4 (1) by (2) we have, $\tau(1-C_\tau(\lambda r)) \ge r$. Hence, $1-C_\tau(\lambda, r)$ is $r-fsso$. By Theorem 2.6, $C_\tau(\lambda, r)$ is $r-fssc$.

(4) Since, λ is $r-fso$, there exists $\nu \in I^X$ with $\tau(\nu) \ge r$ such that

$$\nu \le \lambda \le C_\tau(\nu, r).$$

It implies $\nu = I_\tau(\nu, r) \le I_\tau(\lambda, r)$ and $C_\tau(\lambda, r) \le C_\tau(\nu, r)$. Thus,

$$\nu \le \mu \le C_\tau(\nu, r).$$

Hence, μ is $r-fso$.

(5) It is easily proved from (4) and Theorem 2.5 and the following :-

$$I_\tau(\lambda, r) \le \mu \le C_\tau(\lambda, r)$$

$$\Leftrightarrow \quad 1-C_\tau(\lambda, r) \le 1-\mu \le 1-I_\tau(\lambda, r)$$

$$\Leftrightarrow \quad I_\tau(1-\lambda, r) \le 1-\mu \le C_\tau(1-\lambda, r). \text{ (by Theorem 1.4 (1))}$$

(6) By the definition of C_τ and λ is $r-fsso$

$$C_\tau(\lambda, r) \le C_\tau(I_\tau(C_\tau(I_\tau(\lambda, r), r), r), r).$$

But $I_\tau(C_\tau(I_\tau(\lambda, r), r), r) \le C_\tau(\lambda, r)$. Then,

$$C_\tau(I_\tau(C_\tau(I_\tau(\lambda, r), r), r), r) \le C_\tau(\lambda, r).$$

■

***Theorem* 2.8** — *Let* (X, τ) *be a fts and* $\lambda, \mu \in I^X, r \in I_0$. *If* λ *is a* r − fso *set such that* $\lambda \le \mu \le I_\tau(C_\tau(\lambda, r), r)$, *then* μ *is* r − *fsso set.*

PROOF : Since λ is a r − *fso* set, $\lambda \le C_\tau(I_\tau(\lambda, r), r)$. Then,

$$\mu \le I_\tau(C_\tau(\lambda, r), r) \le I_r(C_\tau(I_\tau(\lambda, r), r), r)$$

$$\le I_\tau(C_\tau(I_\tau(\mu, r), r), r),$$

which shows that μ is r − *fsso* set. ■

***Theorem* 2.9** — *Let* (X, τ) *be a fts and* $\lambda \in I^X, r \in I_0$. *Then,*

(1) λ is r−*fspo* iff $1-\lambda$ is r − *fpc*.

(2) λ is r − *fpo* iff $1-\lambda$ is r − *fspc*.

PROOF : Straightforward ■

***Theorem* 2.10** — *Let* (X, τ) *be a fts* $\lambda \in I^X, r \in I_0$. *If* λ *is* r − *fspo (resp.* r − *fspc) and* r − *fsc (reso.* r − *fso), then* λ *is* r − *fso (resp.* r − *fsc).*

PROOF : Since λ is r − *fspo* and r − *fsc* set we have,

$$\lambda \le C_\tau(I_\tau(C_\tau(\lambda, r), r), r) = C_\tau(I_\tau(\lambda, r), r)$$

Hence, λ is r − *fso* set. ■

***Theorem* 2.11** — *Let* (X, τ) *be a fts and* $\lambda \in I^X, r \in I_0$. *Then,* λ *is* r − *fsso set iff it is* r − *fso and* r − *fpo set.*

PROOF : That is r − *fsso* set is r − *fpo* and r − *fso* set from Remark 2.3. Conversely, let λ be both r − *fso* and r − *fpo* set. Then since λ is r − *fso*, $\lambda \le C_\tau(I_\tau(I_\tau(\lambda, r), r)$. This implies

$$C_\tau(\lambda, r) \le C_\tau(C_\tau(I_\tau(\lambda, r), r), r) = C_\tau(I_\tau(\lambda, r), r).$$

Hence, $$I_\tau(C_\tau(\lambda, r), r) \le I_\tau(C_\tau(I_\tau(\lambda, r), r), r).$$

Since, λ is r−*fpo*, $\lambda \le I_\tau(C_\tau(\lambda, r), r)$,. Therefore we have $\lambda \le I_\tau(C_\tau(I_\tau(\lambda, r), r), r)$ which shows that λ is r − *fsso*. ■

***Theorem* 2.12** — *Let* (X, τ) *be a fts and* $r \in I_0$.

(1) Any union of r − *fsso* (resp. r − *fso*, r − *fpo* and r − *fspo*) sets is r − *fsso* (resp. r − *fso*, r − *fpo* and r − *fspo*) set.

(2) Any intersection of r − *fssc* (resp. r − *fsc*, r − *fpc* and r − *fspc*) sets is r − *fssc* (resp. r − *fsc*, r − *fpc* and r − *fspc*) set.

PROOF : We prove only $r-fsso$, and the others are similar

(1) Let $\{\lambda_\alpha \mid \alpha \in \Gamma\}$ be a family of $r-fsso$ sets. For each $\alpha \in \Gamma$, there exists $\mu_\alpha \in I^X$ with $\tau(\mu_\alpha) \geq r$ such that

$$\mu_\alpha \leq \lambda_\alpha \leq I_\tau(C_\tau(\mu_\alpha, r), r)$$

Since, $I_\tau(C_\tau(\mu_\alpha, r), r) \leq I_\tau(C_\tau(\vee_{\alpha\in\Gamma}\mu_\alpha, r), r)$, we have,

$$\tau(\vee_{\alpha\in\Gamma}\mu_\alpha) \geq \wedge_{\alpha\in\Gamma}\tau(\mu_\alpha) \geq r,$$

$$\vee_{\alpha\in\Gamma}\mu_\alpha \leq \vee_{\alpha\in\Gamma}\lambda_\alpha \leq \vee_{\alpha\in\Gamma}(I_\tau(C_\tau(\mu_\alpha, r), r)$$

$$\leq I_\tau(C_\tau(\vee_{\alpha\in\Gamma}\mu_\alpha, r), r).$$

Hence, $\wedge_{\alpha\in\Gamma}\lambda_\alpha$ is $r-fsso$ set.

(2) It is easily proved from

$$I_\tau\left(C_\tau\left(\bigwedge_{\alpha\in\Gamma}\mu_\alpha, r\right), r\right) \leq \bigwedge_{\alpha\in\Gamma} I_\tau(C_\tau(\mu_\alpha, r), r).$$

■

Definition 2.13 — Let (X, τ) be a fts, $\lambda \in I^X$ and $r \in I_0$.

(1) The r-fuzzy semi-interior of λ, denoted by $SI(\lambda, r)$ is defined by

$$SI(\lambda, r) = \vee\{\mu \in I^X \mid \mu \leq \lambda, \mu \text{ is } r-fso\}.$$

(2) The r-fuzzy semi-closure of λ, denoted by $SC(\lambda, r)$, is defined by

$$SC(\lambda, r) = \wedge\{\mu \in I^X \mid \mu \geq \lambda, \mu \text{ is } r-fsc\}.$$

(3) The r-fuzzy strongly semi-interior of λ, denoted by $SSI(\lambda, r)$, is defined by

$$SSI(\lambda, r) = \vee\{\mu \in I^X \mid \mu \leq \lambda, \mu \text{ is } r-fsso\}.$$

(4) The r-fuzzy strongly semi-closure of λ, denoted by $SSC(\lambda, r)$, is defined by

$$SSC(\lambda, r) = \wedge\{\mu \in I^X \mid \mu \geq \lambda, \mu \text{ is } r-fssc\}.$$

The following theorem is easily proved from Definition 2.13 and Theorem 212.

***Theorem* 2.14** — *Let (X, τ) be a fts, $\lambda \in I^X$ and $r \in I_0$.*

(1) λ is $r-fso$ iff $\lambda = SI(\lambda, r)$.

(2) λ is $r-fsc$ iff $\lambda = SC(\lambda, r)$

(3) λ is $r-fsso$ iff $\lambda = SSI(\lambda, r)$

(4) λ is r – *fssc* iff $\lambda = SSC(\lambda, r)$.

***Theorem* 2.15** — *Let* (X, τ) *be a fts. For* $\lambda, \mu \in I^X$ *and* $r \in I_0$ *it satisfies the following statements :-*

(1) $SC(0, r) = 0$ and $SSC(0, r) = 0$.

(2) $I_\tau(\lambda, r) \leq SSI(\lambda, r) \leq SI(\lambda, r) \leq \lambda \leq SC(\lambda, r) \leq SSC(\lambda, r) \leq C_\tau(\lambda, r)$

(3) $\lambda \leq \mu \Rightarrow SSI(\lambda, r) \leq SSI(\mu, r)$, and $SI(\lambda, r) \leq SI(\mu, r)$.

(4) $\lambda \leq \mu \Rightarrow SSC(\lambda, r) \leq SSC(\mu, r)$ and $SC(\lambda, r) \leq SC(\mu, r)$.

(5) $SC(SC(\lambda, r), r) = SC(\lambda, r)$ and $SI(SI(\lambda, r), r) = SI(\lambda, r)$.

(6)$^{\cdot}$ $SSC(SSC(\lambda, r), r) = SSC(\lambda(r)$ and $SSI\, SSI(\lambda, r), r) = SSI(\lambda, r)$.

(7) $C_\tau(SC(\lambda, r), r = SC(C_\tau(\lambda, r), r) = C_\tau(\lambda, r)$.

(8) $C_\tau(SSC(\lambda, r), r) = SSC(C_\tau(\lambda, r), r) = C_\tau(\lambda, r)$.

PROOF : (1), (2), (3), (4), (5) and (6) are easily proved.

(7) From Theorem 2.7, and Theorem 2.14, $SC(C_\tau(\lambda, r), r) = C_\tau(\lambda, r)$. We only show that

$$C_\tau(SC(\lambda, r, r) = C_\tau(\lambda, r)$$

Since $\qquad \lambda \leq SC(\lambda, r),\ C_\tau(\lambda, r) \leq C_\tau(SC(\lambda, r), r),$

suppose that $\quad C_\tau(\lambda, r) \not\geq C_\tau(SC(\lambda, r), r),$

There exist $\lambda \in I^X, x \in X, r \in I_0$ and $\mu \in I^X$ with $\lambda \leq \mu$ and $\tau(1 - \mu) \geq r$ such that

$$C_\tau(SC(\lambda, r), r)(x) > \mu(x) \geq C_\tau(\lambda, r)(x).$$

On the ther hand, since $\mu = C_\tau(\mu, r), \lambda \leq \mu$ implies

$$SC(\lambda, r) \leq SC(\mu, r) = SC(C_\tau(\mu, r), r) = C_\tau(\mu, r) = \mu.$$

Thus, $\qquad C_\tau(SC(\lambda, r), r) \leq \mu.$

It is a contradiction. Hence, $C_\tau(SC(\lambda, r), r) = C_\tau(\lambda, r)$.

(8) Similar to the proof of (5). ■

***Theorem* 2.16** — *Let* (X, τ) *be a fts. For* $\lambda \in I^X$ *and* $r \in I_0$ *we have*

(1) $SI(1 - \lambda, r) = 1 - SC(\lambda, r)$ and $SC(1 - \lambda, r) = 1 - SI(\lambda, r)$.

(2) $SSI(1 - \lambda, r) = 1 - SSC(\lambda, r)$ and $SSC(1 - \lambda, r) = 1 - SSI(\lambda, r)$.

PROOF : We only prove (1) and the others are similar (1). For all $\lambda \in I^X, r \in I_0$ we have the following

$$1 - SC(\lambda, r) = 1 - \wedge \{ \mu \mid \mu \geq \lambda, \mu \text{ is } r - fsc \}$$

$$= \vee \{ 1 - \mu \mid 1 - \mu \leq 1 - \lambda, 1 - \mu \text{ is } r - fso \}$$

$$= SI(1 - \lambda, r).$$

■

3. VARIOUS CONTINUITY IN FUZZY TOPOLOGICAL SPACES

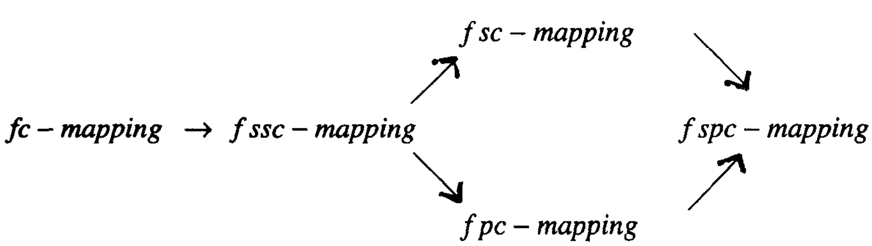
Definition 3.1 — Let (X, τ_1) and (Y, τ_2) be fts's. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a function.

(1) f is called fuzzy continuous[11] iff $\tau_2(\mu) < \tau_1(f^{-1}(\mu))$, for each $\mu \in I^Y$

(2) f is called fuzzy semi-continuous (resp. fuzzy strongly semi-continuous) iff $f^{-1}(\mu)$ is $r - fso$ (resp. $r - fsso$) for each $\mu \in I^Y, r \in I_0$ with $\tau_2(\mu) \geq r$.

(3) f is called fuzzy precontinuous (resp. fuzzy semi-precontinuous) iff $f^{-1}(\mu)$ is $r - fpo$ (resp. $r - fspo$) for each $\mu \in I^Y, r \in I_0$ with $\tau_2(\mu) \geq r$.

The implications contained in the following diagram are true :



Where, fc = fuzzy continuous, $fssc$ = fuzzy strongly semi-continuous, fsc = fuzzy semi-continuous, fpc = fuzzy precontinuous and $fspc$ = fuzzy semi-precontinuous.

From the following example, the converse of these implications are not true.

Example 3.2 — Let $X = \{a, b\}$: Define $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in I^X$ as follows :

$$\lambda_1(a) = 0.5, \lambda_1(b) = 0.6, \lambda_2(a) = 0.6, \lambda_2(b) = 0.7$$

$$\lambda_3(a) = 0.3, \lambda_3(b) = 0.4, \lambda_4(a) = 0.3, \lambda_4(b) = 0.6.$$

Let $\tau_i : I^X \to I$, for each $i = 1, \ldots, 4$, be fuzzy topologies on X defined as :

$$\tau_1(\lambda) = \begin{cases} 1, & \text{if } \lambda \in (0,1) \\ \frac{1}{2}, & \text{if } \lambda = \lambda_1 \\ 0, & \textit{otherwise.} \end{cases} \qquad \tau_2(\lambda) = \begin{cases} 1, & \text{if } \lambda \in (0,1) \\ \frac{1}{2}, & \text{if } \lambda = \lambda_2 \\ 0, & \textit{otherwise.} \end{cases}$$

$$\tau_3(\lambda) = \begin{cases} 1, & \text{if } \lambda \in (0,1) \\ \frac{2}{3}, & \text{if } \lambda = \lambda_3 \\ 0, & \textit{otherwise.} \end{cases} \qquad \tau_4(\lambda) = \begin{cases} 1, & \text{if } \lambda \in (0,1) \\ \frac{1}{2}, & \text{if } \lambda = \lambda_4 \\ 0, & \textit{otherwise.} \end{cases}$$

Then (1) the identity mapping $i_x : (X, \tau_1) \to (X, \tau_2)$ is *fssc*-mapping but not *fc*-mapping.

(2) the identity mapping $i_x : (X, \tau_3) \to (X, \tau_4)$ is *fsc*-mapping but it is neither *fssc*-mapping nor *fpc*-mapping.

(3) An injective mapping from a fuzzy indiscrete space to a fuzzy discrete space is a *fpc*-mapping but it is neither a *fssc*-mapping nor a *fsc*-mapping.

***Theorem* 3.3** — *Let (X, τ_1) and (Y, τ_2) be fts's. The following statements are equivalent :-*

(1) A map f is *fsc*.

(2) $f^{-1}(\mu)$ is r - *fsc* in X for each $\mu \in I^Y, r \in I_0$ with $\tau_2(1-\mu) \geq r$.

(3) $I_{\tau_1}(C_{\tau_1}(f^{-1}(\mu), r), r) \leq f^{-1}(C_{\tau_2}(\mu, r)) \ \forall \ \mu \in I^Y, r \in I_0$.

(4) $f(I_{\tau_1}(C_{\tau_1}(\lambda, r), r)) \leq C_{\tau_2}(f(\lambda), r) \ \forall \ \lambda \in I^X, r \in I_0$.

(5) $f(SC(\lambda, r)) \leq C_{r_2}(f(\lambda), r) \ \forall \ \lambda \in I^X, r \in I_0$.

(6) $SC(f^{-1}(\mu), r) \leq f^{-1}(C_{\tau_2}(\mu, r)) \ \forall \ \mu \in I^Y, r \in I_0$.

(7) $f^{-1}(I_{\tau_2}(\mu, r)) \leq SI(f^{-1}(\mu), r) \quad \forall \ \mu \in I^Y, r \in I_0$.

PROOF : (1) $\Rightarrow$ (2) Let $\mu \in I^Y, r \in I_0$ with $\tau_2(1-\mu) \geq r$. Then by (1), $f^{-1}(1-\mu)$ is $r-$*fso* set of X. But $f^{-1}(1-\mu) = 1 - f^{-1}(\mu)$. Therefore, $f^{-1}(\mu)$ is r - *fsc* set of X.

(2) $\Rightarrow$ (3) Let $\mu \in I^Y, r \in I_0$, since $\tau_2(1 - C_{\tau_2}(\mu, r)) \geq r$. Then, by (2),$f^{-1}(C_{\tau_2}(\mu, r))$ is *fsc* set in X. So,

$$f^{-1}(C_{\tau_2}(\mu, r)) \geq I_{r_1}(C_{\tau_1}(f^{-1}(C_{\tau_2}(\mu, r)), r), r)$$

$$\geq I_{\tau_1}(C_{\tau_1}(f^{-1}(\mu), r), r).$$

(3) $\Rightarrow$ (4) Let $\lambda \in I^X, r \in I_0$. Then by (3) we have

$$I_{\tau_1}(C_{\tau_1}(f^{-1}(f(\lambda)), r), r) \leq f^{-1}(C_{\tau_2}(f(\lambda), r)).$$

This implies $f(I_{\tau_1}(C_{\tau_1}(\lambda, r), r)) \leq C_{\tau_2}(f(\lambda), r)$.

(4) $\Rightarrow$ (1) Let $\mu \in I^Y, r \in I_0$ with $\tau_2(\mu) \geq r$. By (4)

$$f(I_{\tau_1}(C_{\tau_1}(f^{-1}(1-\mu), r), r)) \leq C_{\tau_2}(f(f^{-1}(1-\mu)), r)$$

$$\leq C_{\tau_2}(1-\mu, r) = 1-\mu$$

So, $I_{\tau_1}(C_{\tau_1}(f^{-1}(1-\mu), r), r) \leq f^{-1}(1-\mu)$ i.e., $f^{-1}(1-\mu)$ is $r-fsc$ and $f^{-1}(\mu)$ is $r-fso$ set of X.

(1) $\Rightarrow$ (5) For all $\lambda \in I^X, r \in I_0$, since $\tau_2(1 - C_{\tau_2}(f(\lambda), r)) \geq r$ from the definition of C_{τ_2} of Theorem 1.3, and Definition 1.1 (03),

$$f^{-1}(1 - C_{\tau_2}(f(\lambda), r)) = 1 - f^{-1}(C_{\tau_2}(f(\lambda), r))$$

is $r-fso$. By Theorem 2.5, $f^{-1}(C_{\tau_2}(f(\lambda), r))$ is $r-fsc$. Since,

$$\lambda \leq f^{-1}(f(\lambda)) \leq f^{-1}(C_{\tau_2}(f(\lambda), r))$$

we have $$SC(\lambda, r) \leq f^{-1}(C_{\tau_2}(f(\lambda), r)).$$

Hence, $$f(SC(\lambda, r)) \leq f(f^{-1}(C_{\tau_2}(f(\lambda), r))) \leq C_{\tau_2}(f(\lambda), r)$$

(5) $\Rightarrow$ (6) For all $\mu \in I^Y, r \in I_0$, let $\lambda = f^{-1}(\mu)$. By (5)

$$f(SC(f^{-1}(\mu), r)) \leq C_{\tau_2}(f(f^{-1}(\mu)), r) \leq C_{\tau_2}(\mu, r).$$

It implies $SC(f^{-1}(\mu), r) \leq f^{-1}(C_{\tau_2}(\mu, r))$.

(6) $\Rightarrow$ (7) For $\mu \in I^Y, r \in I_0$, we have

$$SC(1 - f^{-1}(\mu), r) = SC(f^{-1}(1 - \mu), r) \leq f^{-1}(C_{\tau_2}(1 - \mu, r)).$$

Since $SC(1 - f^{-1}(\mu), r) = 1 - SI(f^{-1}(\mu), r)$ and $C_{\tau_2}(1 - \mu, r) = 1 - I_{\tau_2}(\mu, r)$ from Theorem 2.16, and Theorem 1.4(1), respectively we have $f^{-1}(I_{\tau_2}(\mu, r)) \leq SI(f^{-1}(\mu), r)$.

(7) $\Rightarrow$ (1) For each $\mu \in I^Y, r \in I_0$ with $\tau_2(\mu) \geq r$, since $I_{\tau_2}(\mu, r) = \mu$.

$$f^{-1}(\mu) = f^{-1}(I_{\tau_2}(\mu, r)) \leq SI(f^{-1}(\mu), r).$$

Hence, by definition of

$$SI(f^{-1}(\mu), r), f^{-1}(\mu) = SI(f^{-1}(\mu), r).$$

Thus, by Theorem 2.14(1), $f^{-1}(\mu)$ is $r - fso$. Therefore f is *fsc*-mapping. ■

The following theorem is similarly proved from Theorem 3.3.

***Theorem* 3.4** — *Let (X, τ_1) and (Y, τ_2) be fbts's. The following statements are equivalent :-*

(1) *A* map f is *fssc*.

(2) $f^{-1}(\mu)$ is $r - fssc$ in X for each $\mu \in I^Y, r \in I_0$ with $\tau_2(1 - \mu) \geq r$.

(3) $C_{\tau_1}(I_{\tau_1}(C_{\tau_1}(f^{-1}(\mu), r), r), r) \leq f^{-1}(C_{\tau_2}(\mu, r)) \ \forall \ \mu \in I^Y, r \in I_0$.

(4) $f(C_{\tau_1}(I_{\tau_1}(C_{\tau_1}(\lambda, r), r), r)) \leq C_{\tau_2}(f(\lambda), r) \ \forall \ \lambda \in I^X, r \in I_0$.

(5) $f(SSC(\lambda, r)) \leq C_{\tau_2}(f(\lambda), r) \ \forall \ \lambda \in I^X, r \in I_0$.

(6) $SSC(f^{-1}(\mu), r) \leq f^{-1}(C_{\tau_2}(\mu, r)) \ \forall \ \mu \in I^Y, r \in I_0$.

(7) $f^{-1}(I_{\tau_2}(\mu, r)) \leq SSI(f^{-1}(\mu), r) \ \forall \ \mu \in I^Y, r \in I_0$.

***Theorem* 3.5** — *Let $f: (X, \tau_1) \to (Y, \tau_2)$ be a mapping. The following are equivalent :-*

(1) f is *fspc*-mapping.

(2) $f^{-1}(\mu)$ is $r - fspc$ in X for each $\mu \in I^Y, r \in I_0$ with $\tau_2(1 - \mu) \geq r$.

(3) $I_{r_1}(C_{\tau_1}(I_{r_1}(f^{-1}(\mu), r), r), r) \leq f^{-1}(C_{\tau_2}(\mu, r))$ for each $\mu \in I^Y, r \in I_0$

(4) $f(I_{\tau_1}(C_{\tau_1}(I_{\tau_1}(\lambda, r), r)) \leq C_{\tau_2}(f(\lambda), r)$ for each $\lambda \in I^X, r \in I_0$.

PROOF : (1) $\Rightarrow$ (2) It is obvious.

(2) $\Rightarrow$ (3) Let $\mu \in I^Y, r \in I_0$. Then, by (2) we have $f^{-1}(C_{\tau_2}(\mu, r))$ is a $r-fspc$ set of X. So,

$$f^{-1}(C_{\tau_2}(\mu, r)) \geq I_{r_1}(C_{\tau_1}(I_{\tau_1}(f^{-1}(C_{\tau_2}(\mu, r)), r), r), r)$$

$$\geq I_{\tau_1}(C_{\tau_1}(I_{\tau_1}(f^{-1}(\mu), r)\, r), r)$$

(3) $\Rightarrow$ (4) Let $\lambda \in I^X, r \in I_0$. Then by (3) we have

$$I_{\tau_1}(C_{\tau_1}(I_{\tau_1}(f^{-1}(f(\lambda)), r), r), r) \leq f^{-1}(C_{\tau_2}(f(\lambda), r))$$

Hence,

$$f(I_{\tau_1}(C_{\tau_1}(I_{\tau_1}(\lambda, r), r), r), r)) \leq C_{\tau_2}(f(\lambda), r)$$

(4) $\Rightarrow$ (1) Let $\mu \in I^Y, r \in I_0$ with $\tau_2(\mu) \geq r$. By (4),

$$f(I_{\tau_1}(C_{\tau_1}(I_{\tau_1}(f^{-1}(1-\mu), r), r), r)) \leq C_{\tau_2}(f(f^{-1}(1-\mu), r)$$

$$\leq C_{\tau_2}(1-u, r) = 1-\mu.$$

Therefore, $I_{\tau_1}(C_{\tau_1}(I_{\tau_1}(f^{-1}(1-\mu), r), r), r) \leq f^{-1}(1-\mu)$, i.e., $f^{-1}(1-\mu)$ is $r-fspc$ in X. Hence, $f^{-1}(\mu)$ is $r-fspo$ set of X. ■

The following theorem is similarly proved from Theorem 3.3.

***Theorem* 3.6** — *Let $f:(X, \tau_1) \to (Y, \tau_2)$ be a mapping. The following are equivalent :*

(1) f is *fpc*-mapping.

(2) $f^{-1}(\mu)$ is $r-fpc$ in X for each $\mu \in I^Y, r \in I_0$ with $\tau_2(1-\mu) \geq r$.

(3) $C_{\tau_1}(I_{\tau_1}(f^{-1}(\mu), r).\, r) \leq f^{-1}(C_{\tau_2}(\mu, r))$ for each $\mu \in I^Y, r \in I_0$.

(4) $f(C_{\tau_1}(I_{\tau_1}(\lambda, r)) \leq C_{\tau_2}(f(\lambda), r)$ for each $\lambda \in I^X, r \in I_0$.

***Theorem* 3.7** — *Let $f: (X, \tau_1) \to (Y, \tau_2)$ be a* fssc*-mapping. Then, for $r \in I_0$*

(1) $f(C_{\tau_1}(\lambda, r)) \le C_{r_2}(f(\lambda), r)$ for each λ is $r - fpo$ set of X.

(2) $C_{\tau_1}(f^{-1}(\mu), r) \le f^{-1}(C_{\tau_2}(\mu, r))$ for each μ s $r - fpo$ set of Y.

PROOF : (1) Let λ be a $r - fpo$ set of X. Then, $\lambda \le I_{\tau_1}(C_{\tau_1}, (\lambda, r), r)$. so, $C_{\tau_1}(\lambda, r) \le C_{\tau_1}(I_{\tau_1}(C_{\tau_1}(\lambda, r), r), r$ which implies that

$$f(C_{\tau_1}(\lambda, r)) \le f(C_{\tau_1}(I_{\tau_1}(C_{\tau_1}(\lambda, r), r), r))$$

Since, $\tau_2(1 - C_{\tau_2}(f(\lambda), r)) \ge r$, then by Theorem 3.4, we have

$$f^{-1}(C_{\tau_2}(f(\lambda), r)) \ge C_{\tau_1}(I_{\tau_1}(C_{\tau_1}(f^{-1}(C_{\tau_2}(f(\lambda), r), r), r), r)$$

$$\ge C_{\tau_1}(I_{\tau_1}(C_{\tau_1}(f^{-1}(f(\lambda)) r), r), r)$$

$$\ge C_{\tau_1}(I_{\tau_1}(C_{\tau_1}(\lambda, r), r), r).$$

Then, $f(C_{\tau_1}(I_{\tau_1}(\lambda, r), r), r)) \le C_{\tau_2}(f(\lambda), r)$. Hence,

$$f(C_{\tau_1}(\lambda, r)) \le f(C_{\tau_1}(I_{\tau_1}(C_{\tau_1}(\lambda, r), r), r)) \le C_{\tau_2}(f(\lambda), r).$$

(2) Let μ be a $r - fpo$ set of Y. Then, $\mu \le I_{\tau_2}(C_{\tau_2}(\mu, r), r)$ which implies that

$$f^{-1}(\mu) \le f^{-1}(I_{\tau_2}(C_{\tau_2}(\mu, r), r)).$$

Since $\tau_2(I_{\tau_2}(C_{\tau_2}(\mu, r)) \ge r$, then $f^{-1}(I_{\tau_2}(C_{\tau_2}(\mu, r), r))$ is $r - fsso$ which is also r fpo. So,

$$f^{-1}(I_{\tau_2}(C_{\tau_2}(\mu, r), r)) \le I_{\tau_1}(C_{\tau_1}(f^{-1}(I_{\tau_2}(C_{\tau_2}(\mu, r), r)), r), r)$$

$$\le I_{\tau_1}(C_{\tau_1}(f^{-1}(C_{\tau_2}(\mu, r)), r, r).$$

This implies that

$$C_{\tau_1}(f^{-1}(\mu), r) \le C_{\tau_1}(I_{\tau_1}(C_{\tau_1}(f^{-1}(C_{\tau_2}(\mu, r)), r), r), r).$$

Since f is *fssc*-mapping and $\tau_2(1 - C_{\tau_2}(\mu, r)) \ge r$, we have $f^{-1}(C_{\tau_2}(\mu, r))$ is $r - fssc$ set.

Then we have $C_{\tau_1}(I_{\tau_1}(C_{\tau_1}(f^{-1}(C_{\tau_2}(\mu, r)), r), r), r) \le f^{-1}(C_{\tau_2}(\mu, r))$.

Hence,

$$C_{\tau_1}(f^{-1}(\mu), r) \le f^{-1}(C_{\tau_2}(\mu, r)). \qquad \blacksquare$$

***Theorem* 3.8** — *A mapping $f: (X, \tau_1) \to (Y, \tau_2)$ is* fssc*mapping iff it is* fsc-*mapping and* fpc-*mapping.*

PROOF : Follows immediately from Theorem 2.11. $\blacksquare$

4. Semi-Open, Strongly Semi-Open, Preopen and Semi-Preopen Mapping in Fuzzy Topological Spaces

Definition 4.1 — Let (X, τ_1) and (Y, τ_2) be fts's. Let $f: (X, \tau_1) \to (Y, \tau_2)$ be a function.

(1) f is called fuzzy open[11] (resp. fuzzy closed[11]) iff $\tau_1(\lambda) \le \tau_2(f(\lambda))$ (resp. $\tau_1(1-\lambda) \le \tau_2(1-f(\lambda))$ for each $\lambda \in I^X$.

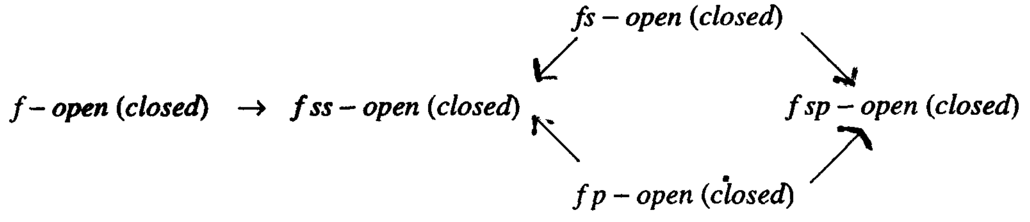
(2) f is called fuzzy semi-open (resp. fuzzy strongly semi-open) iff $f(\lambda)$ is r – *fso* (resp. r – *fsso*) for each $\lambda \in I^X, r \in I_0$ with $\tau_1(\lambda) \ge r$.

(3) f is called fuzzy semi-closed (resp. fuzzy strongly semi-closed) iff $f(\lambda)$ is r – *fsc* (resp. r – *fssc*) for each $\lambda \in I^X, r \in I_0$ with $\tau_1(1-\lambda) \ge r$.

(4) f is called fuzzy preopen (resp. fuzzy semi-preopen) iff $f(\lambda)$ is r – *fpo* (resp. r – *fspo*) for each $\mu \in I^X, r \in I_0$ with $\tau_1(\lambda) \ge r$.

(5) f is called fuzzy preclosed (reso. fuzzy semi-preclosed) iff $f(\lambda)$ is r – *fpc* (resp. r – *fspc*) for each $\mu \in I^X, r \in I_0$ with $\tau_1(1-\lambda) \ge r$.

The implications contained in the following diagram are true :



Where, f – open (closed) = fuzzy open (closed), *fss* – open (closed) = fuzzy strongly semi-open (closed), *fs* – open (closed) = fuzzy semi-open (closed), *fp*–open (closed = fuzzy preopen (preclosed), *fsp* – open (closed) = fuzzy semi-preopen (closed).

From the following example, the converse of these implications are not true.

Example 4.2 — In Example 3.2, we have the following :

(1) the identity mapping $i_X : (X, \tau_2) \rightarrow (X, \tau_1)$ is *fss*-open mappinġ but not *f*-open mapping.

(2) the identity mapping $i_X : (X, \tau_4) \rightarrow (X, \tau_3)$ is *fs*-open mapping but it is neither *fss*-open mapping nor *fp*-open mapping.

(3) An injective mapping from a fuzzy discrete space to a fuzzy indiscrete space is a *fp*-open mapping but it is neither a *fss*-open mapping nor a *fs*-open mapping.

***Theorem* 4.3** — *Let* (X, τ_1) *and* (Y, τ_2) *e fts's. The following statements are equivalent -*

(1) A map f is *fs*-closed.

(2) $f(C_{\tau_1}(\lambda(r)) \geq I_{r_2}(C_{\tau_2}(f(\lambda), r), r)$, for each $\lambda \in I^X$ and $r \in I_0$.

(3) $SC(f(\lambda), r) \leq f(C_{\tau_1}(\lambda, r))$ for each $\lambda \in I^X$ and $r \in I_0$.

PROOF (1) $\Rightarrow$ (2) For all $\lambda \in I^X, r \in I_0$, since $\tau_1(1 - C_{\tau_1}(\lambda, r)) \geq r, f(C_{\tau_1}(\lambda, r))$ is $r - fsc$. From Theorem 2.5,

$$f(C_{\tau_1}(\lambda\, r)) \geq I_{\tau_2}(C_{\tau_2}(f(C_{\tau_1}(\lambda, r)), r), r)$$

$$\geq I_{\tau_2}(C_{\tau_2}(f(\lambda), r)\, r).$$

(2) $\Rightarrow$ (1) For each $\lambda \in I^X, r \in I_0$ with $\tau_1(1 - \lambda) \geq r$, we have $C_{\tau_1}(\lambda, r) = \lambda$. From (2),

$$f(\lambda) = f(C_{\tau_1}(\lambda, r)) \geq I_{\tau_2}(C_{\tau_2}(f(\lambda), r), r).$$

By Theorem 2.5, $f(\lambda)$ is $r - fsc$.

(1) $\Rightarrow$ (3) For all $\lambda \in I^X, r \in I_0$, since $\tau_1(1 - C_{\tau_1}(\lambda, r)) \geq r, f(C_{\tau_1}(\lambda, r))$ is $r - fsc$. Since

$$f(\lambda) \leq SC(f(\lambda), r) \leq f(C_{\tau_1}(\lambda, r)) = f(\lambda).$$

Thus, $f(\lambda) = SC(f(\lambda), r)$. By Theorem 2.14, $f(\lambda)$ is $r - fsc$. ■

The following three theorems are similarly proved from Theorem 4.3.

***Theorem* 4.4** — *Let* (X, τ_1) *and* (Y, τ_2) *be fts's. The following satatements are equivalent :-*

(1) A map f s *fs*[open

(2) $f(I_{\tau_1}(\lambda, r)) \leq SI(f(\lambda), r)$, for each $\lambda \in I^X$ and $r \in I_0$.

(3) $I_{\tau_1}(f^{-1}(\mu), r) \leq f^{-1}(SI(\mu, r))$ for each $\mu \in I^Y$ and $r \in I_0$.

***Theorem* 4.5** — *Let $f: (X, \tau_1) \to (Y, \tau_2)$ be a mapping. The following statements are equivalent :*

(1) f is *fss*-closed

(2) $f(C_{\tau_1}(\lambda, r)) \geq C_{\tau_2}(I_{\tau_2}(C_{\tau_2}(f(\lambda), r, r), r))$, for each $\lambda \in I^X$ and $r \in I_0$.

(3) $SSC\ (f(\lambda), r) \leq f(C_{\tau_1}(\lambda, r))$ for each $\lambda \in I^X$ and $r \in I_0$.

***Theorem* 4.6** — *Let $f: (X, \tau_1) \to (Y, \tau_2)$ be a mapping. The following statements are equivalent*

(1) f is *fss*-open

(2) $f(I_{\tau_1}(\lambda, r)) \leq SSI\ (f(\lambda), r)$, for each $\lambda \in I^X$ and $r \in I_0$.

(3) $I_{\tau_1}(f^{-1}(\mu), r) \leq f^{-1}(SSI\ (\mu, r))$, for each $\mu \in I^Y$ and $r \in I_0$.

***Theorem* 4.7** — *Let $f: (X, \tau_1) \to (Y, \tau_2)$ be an injective mapping. If f is a fssopen mapping then $f^{-1}(C_{\tau_2}(\mu, r)) \leq C_{\tau_1}(f^{-1}(\mu), r)$ for each r – fpo set μ of Y, $r \in I_0$.*

PROOF : Suppose that μ is r – *fpo* set of Y. Since $f^{-1}(\mu) \leq C_{\tau_1}(f^{-1}(\mu)\, r)$ and $\tau_1(1 - C_{\tau_1}(f^{-1}(\mu), r)) \geq r$, then

$$1 - C_{\tau_1}(f^{-1}(\mu), r) \leq 1 - f^{-1}(\mu) = f^{-1}(1 - \mu)$$

$$\Rightarrow f(1 - C_{\tau_1}(f^{-1}(\mu), r)) \leq f(1 - f^{-1}(\mu)) \leq 1 - \mu$$

$$\Rightarrow \mu \leq 1 - f(1 - C_{\tau_1}(f^{-1}(\mu), r)).$$

Let $\nu = 1 - f(1 - C_{\tau_1}(f^{-1}(\mu), r))$. Since f is *fss*-open mapping, then ν is r – *fssc* set of Y. So, we have $C_{\tau_2}(I_{\tau_2}(C_{\tau_2}(\nu, r), r), r) \leq \nu$. Since $\mu \leq \nu$, then

$$C_{\tau_2}(I_{\tau_2}(C_{\tau_2}(\mu, r), r), r) \leq C_{\tau_2}(I_{\tau_2}(C_{\tau_2}(\nu, r), r), r) \leq \nu$$

$$\Rightarrow f^{-1}(C_{\tau_2}(I_{\tau_2}(C_{\tau_2}(\mu, r), r), r)) \leq f^{-1}(\nu).$$

Since μ is $r-fpo$ set in Y, we have $\mu \leq I_{\tau_2}(C_{\tau_2}(\mu, r), r)$. Then

$$C_{\tau_2}(\mu, r) \leq C_{\tau_2}(I_{\tau_2}(C_{\tau_2}(\mu, r), r), r)$$

$$\Rightarrow f^{-1}(C_{\tau_2}(\mu, r)) \leq f^{-1}(C_{\tau_2}(I_{\tau_2}(C_{\tau_2}(\mu, r), r), r)) \leq f^{-1}(\nu).$$

From the injectivity of f, we have $f^{-1}(\nu) = C_{r'}f^{-1}(\mu, r)$. So,

$$f^{-1}(C_{\tau_2}(\mu, r)) \leq C_{\tau_1}(f^{-1}(\mu)\, r). \qquad \blacksquare$$

***Theorem* 4.8** — *Let $f: (X, \tau_1) \to (Y, \tau_2)$ be a fs-open (resp. fss-open and fsp-open) mapping. If $\mu \in I^Y$ and $\lambda \in I^X$, $\tau_1(1-\lambda) \geq r$, $r \in I_0$ such that $f^{-1}(\mu) \leq \lambda$, then there exists a $r-fsc$ (resp. $r-fssc$ and $r-fspc$) set ν of Y, such that $\mu \leq \nu, f^{-1}(\nu) \leq \lambda$.*

PROOF : Let $\nu = 1 - f(1-\lambda)$. Since $f^{-1}(\mu) \leq \lambda$ we have, $f(1-\lambda) \leq 1-\mu$. Since f is *fs*-open (resp. *fss*-open and *fsp*-open) mapping then ν is $r-fsc$ (resp. $r-fssc$ and $r-fspc$) set of Y and $f^{-1}(\nu) = 1 - f^{-1}(f(1-\lambda)) \leq 1-(1-\lambda) = \lambda$. $\blacksquare$

Corollary 4.9 — (1) If $f: (X, \tau_1) \to (Y, \tau_2)$ is a *fs*-open mapping then for each $\mu \in I^Y$, $r \in I_0$.

$$f^{-1}(I_{\tau_2}(C_{\tau_2}(\mu, r), r)) \leq C_{\tau_1}(f^{-1}(\mu), r).$$

(2) If $f: (X, \tau_1) \to (Y, \tau_2)$ is a *fss*-open mapping then for each $\mu \in I^Y$, $r \in I_0$

$$f^{-1}(C_{\tau_2}(I_{\tau_2}(C_{\tau_2}(\mu, r), r)) \leq C_{\tau_1}(f^{-1}(\mu), r).$$

(3) If $f: (X, \tau_1) \to (Y, \tau_2)$ is a *fss*-open mapping then for each $r-fpo$ set μ of Y, $r \in I_0$

$$f^{-1}(C_{\tau_2}(\mu, r), r), r)) \leq C_{\tau_1}(f^{-1}(\mu), r).$$

(4) If $f: (X, \tau_1) \to (Y, \tau_2)$ is a *fsp*-open mapping then for each $\mu \in I^Y$, $r \in I_0$

$$f^{-1}(I_{\tau_2}(C_{\tau_2}(I_{\tau_2}(\mu, r), r), r)) \leq C_{\tau_1}(f^{-1}(\mu), r).$$

(5) If $f: (X, \tau_1) \to (Y, \tau_2)$ is a *fp*-open mapping then for each $\mu \in I^Y$, $r \in I_0$

$$f^{-1}(C_{\tau_2}(I_{\tau_2}(\mu, r), r), r)) \leq C_{\tau_1}(f^{-1}(\mu), r).$$

PROOF : We prove (2) and (3). Other results have similar proofs.

(2) Since $\tau_1(1-C_{\tau_1}(f^{-1}(\mu),r))\geq r$ and $f^{-1}(\mu)\leq C_{\tau_1}(f^{-1}(\mu),r)$ for each $u\in I^Y, r\in I_0$, it follows from Theorem 4.8, that there exists a $r-fssc$ set λ of Y, $\mu\leq\lambda$ such that $f^{-1}(\lambda)\leq C_{\tau_1}(f^{-1}(\mu),r)$. Since $\mu\leq\lambda$,

$$f^{-1}(C_{\tau_2}(I_{\tau_2}(C_{\tau_2}(\mu,r),r),r))\leq f^{-1}(C_{\tau_2}(I_{\tau_2}(C_{\tau_2}(\lambda,r),r),r)).$$

Since λ is $r-fssc$ which implies $C_{r_2}(I_{\tau_2}(C_{\tau_2}(\lambda,r),r),r)\leq\lambda$. Thus we have

$$f^{-1}(C_{\tau_2}(I_{\tau_2}(C_{\tau_2}(\mu,r),r),r))\leq f^{-1}(C_{\tau_2}(I_{\tau_2}(C_{\tau_2}(\lambda,r),r),r))$$

$$\leq f^{-1}(\lambda)\leq C_{\tau_1}(f^{-1}(\mu),r).$$

(3) Since μ is $r-fpo$ set of Y, $C_{\tau_2}(\mu,r)=C_{\tau_2}(I_{\tau_2}(C_{\tau_2}(\mu,r),r),r)$. Using (2), we have

$$f^{-1}(C_{\tau_2}(\mu,r))=f^{-1}(C_{\tau_2}(I_{\tau_2}(C_{\tau_2}(\mu,r),r))\leq C_{\tau_1}(f^{-1}(\mu),r).$$ ■

***Theorem* 4.10** — (1) *If* f *is both* fssc-*mapping and* fp-*open mapping, then the inverse image of each* r − fsso *(resp.* r − fssc*) set of* Y, $r\in I_0$ *is* $r-fsso$ *(resp.* r − fssc *set in* X.

(2) If f is both *fssc*-mapping and *fsp*-open mapping, then the inverse image of each $r-fsso$ (resp. $r-fssc$) set of Y, $r\in I_0$ is $r-fsso$ (resp. $r-fssc$) set in X.

(3) If f is both *fsc*-open mapping and *fp*-mapping, then the inverse image of each $r-f$ so (resp. $r-fsc$ set of Y, $r\in I_0$ is $r-fso$ (resp. $r-fsc$) set in X.

PROOF : We prove (1). Other results have similar proofs.

(1) Let $\mu\in I^Y$ be $r-fsso$, $r\in I_0$. Then,

$$f^{-1}(\mu)\leq f^{-1}(I_{\tau_2}(C_{\tau_2}(I_{\tau_2}(\mu,r),r),r))$$

$$\leq I_{\tau_1}C_{\tau_1}(I_{\tau_1}(f^{-1}(I_{\tau_2}(C_{\tau_2}(I_{\tau_2}(\mu,r),r),r)),r),r),r$$

$$\leq I_{\tau_1}(C_{\tau_1}(I_{\tau_1}(f^{-1}(C_{\tau_2}(I_{\tau_2}(\mu,r),r),r),r.$$

Using Corollary 4.9 (5) we have

$$f^{-1}(C_{\tau_2}(I_{\tau_2}(\mu,r),r))=f^{-1}(C_{\tau_2}(I_{\tau_2}(I_{\tau_2}(\mu,r),r),r))\leq C_{\tau_1}(f^{-1}(I_{\tau_2}(\mu,r)),r)$$

Hence,

$$f^{-1}(\mu) \leq I_{\tau_1}(C_{\tau_1}(I_{\tau_1}(C_{\tau_1}(f^{-1}(I_{\tau_2}(\mu, r), r), r), r) \leq I_{\tau_1}(C_{\tau_1}(f^{-1}(I_{\tau_2}(\mu, r)), r), r).$$

Since f is *fssc*-mapping, $f^{-1}(I_{\tau_2}(\mu, r))$ is $r-fsso$. Therefore,

$$f^{-1}(\mu) \leq I_{\tau_1}(C_{\tau_1}(I_{\tau_1}(f^{-1}(I_{\tau_2}(\mu, r), r), r), r) \leq I_{\tau_1}(C_{\tau_1}(I_{\tau_1}(f^{-1}(I_{\tau_2}(\mu, r)), r), r)$$

i.e., $f^{-1}(\mu)$ is $r-fsso$. ■

***Theorem* 4.11** — *A mapping $f: (X, \tau_1) \to (Y, \tau_2)$ is fss-open iff it is fs-open and fp-open mapping.*

PROOF : Easy from Theorem 2.14. ■

5. Fuzzy Almost Continuous Mappings

Definition 5.1 — A function $f: (X, \tau_1) \to (Y, \tau_2)$ is said to be a fuzzy almost continuous if $\tau_1(f^{-1}(\mu)) \geq r$ for each r-fuzzy regular open set μ of Y, $r \in I_0$.

Remark 5.2 : On may easily verify that

$$\begin{array}{ccc} fc-mapping & \rightarrow & f\,ssc-mapping \\ & \searrow \quad \swarrow & \\ & \textit{fuzzy almost continuous} & \end{array}$$

It is easy to construct examples to show that non of the above implications are reversable, in general.

***Theorem* 5.3** —*Let $f: (X, \tau_1) \to (Y, \tau_2)$ be a mapping. Then the following are equivalent :-*

(1) f is a fuzzy almost continuous mapping.

(2) $\tau_1(1 - f^{-1}(\mu)) \geq r$, for each r-fuzzy regular closed set μ of Y, $r \in I_0$.

(3) $f^{-1}(\mu) \leq I_{\tau_1}(f^{-1}(I_{\tau_2}(C_{\tau_2}(\mu, r), r), r)$ $\forall\ \mu \in I^Y, r \in I_0$ with $\tau_2(\mu) \geq r$

(4) $C_{\tau_1}(f^{-1}(C_{\tau_2}(I_{\tau_2}(\mu, r), r), r) \leq f^{-1}(\mu)$ $\forall\ \mu \in I^Y, r \in I_0$ with $\tau_2(1 - \mu) \geq r$.

PROOF : (1) $\Leftrightarrow$ (2) Follows from the complement of r-fuzzy regular open set is r-fuzzy regular closed (from Theorem 1.4).

(1) $\Rightarrow$ (3) For all $\mu \in I^Y, r \in I_0$ with $\tau_2(\mu) \geq r$. Then $I_{\tau_2}(\mu, r) = \mu$, $\mu \leq I_{\tau_2}(C_{\tau_2}(\mu, r), r)$ and hence, $f^{-1}(\mu) \leq f^{-1}(I_{\tau_2}(C_{\tau_2}(\mu, r), r))$. And, $I_{\tau_2}(C_{\tau_2}(\mu, r), r)$ is r-fuzzy regular open set of Y, hence, $\tau_1(f^{-1}(I_{\tau_2}(C_{\tau_2}(\mu, r), r))) \geq r$. Thus

$$f^{-1}(\mu) \le f^{-1}(I_{\tau_2}(C_{\tau_2}(\mu, r), r)) = I_{\tau_1}(f^{-1}(I_{\tau_2}(C_{\tau_2}(\mu, r), r))).$$

(3) $\Rightarrow$ (1) Let μ be r-fuzzy regular open set of Y, $r \in I_0$. Then, we have

$$f^{-1}(\mu) \le I_{\tau_1}(f^{-1}(I_{\tau_2}(C_{\tau_2}(\mu, r), r)), r) - I_{\tau_1}(f^{-1}(\mu), r).$$

Thus,

$$f^{-1}(\mu) = I_{\tau_1}(f^{-1}(\mu), r) \text{ shows that } \tau_1(f^{-1}(\mu)) \ge r.$$

(2) $\Leftrightarrow$ (4) Can similarly be proved. ■

***Theorem* 5.4** — *If $f : (X, \tau_1 \to Y, \tau_2)$ is both almost continuous and fs-open mapping, then f is fpc-mapping.*

PROOF : Let $\mu \in I^Y, r \in I_0$, with $\tau_2(\mu) \ge r$. Then $I_{\tau_2}(\mu, r) = \mu$. Since f is fs-open mapping, by corollary 4.9(1), we have

$$f^{-1}(\mu) \le f^{-1}(I_{\tau_2}(C_{\tau_2}(\mu, r), r)) \le C_{\tau_1}(f^{-1}(\mu), r).$$

But, since f is almost continuous, we have

$$f^{-1}(\mu) \le f^{-1}(I_{\tau_2}(C_{\tau_2}(\mu, r), r)) = I_{\tau_1}(f^{-1}(I_{\tau_2}(C_{\tau_2}(\mu), r))))$$

$$\le I_{\tau_1}(C_{\tau_1}(f^{-1}(\mu), r), r).$$

Hence, $f^{-1}(\mu)$ is $r - fpo$ set in X and f is fpc-mapping. ■

Corollary 5.5 — If $f : (X, \tau_1) \to (Y, \tau_2)$ is both almost continuous and fss-open mapping, then f is fpc-mapping.

Using similar arguments, as in Theorem 5.4, on may prove the following result.

***Theorem* 5.6** — *If $f : (X, \tau_1) \to (Y, \tau_2)$ is both almost continuous and fp-open mapping, then f is fpc-mapping.*

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