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PHOTOGRAVITATIONAL RESTRICTED THREE-BODY PROBLEM WITH VARIABLE MASS

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The photogravitational restricted three-body problem with variable mass has been studied. The problem is photogravitational in the sense that the more massive primary is a source of radiation and the variable that the mass of the third body varies with time. It has been seen that the equations of motion of the problem differ from the equations of the classical restricted problem by additional terms. The triangular points form ordinary triangles with the primaries and the collinear points remain collinear and lie on the line joining the primaries. The triangular points are stable for $0 < \mu < \mu_C$ and unstable for $\mu_c < \mu < \frac{1}{2}$ where the critical value of the mass parameter μ_c depends upon the radiation factor q_1 ans β, the constant due to the variation in mass governed by Jeans law. It is further seen that the collinear points are in general unstable.

Key Words : Photogravitational/RTBP/Variable Mass

1. INTRODUCTION

The classical restricted three-body problem is unable to discuss the motion of a material point when one of the interacting masses is an intense emitter of radiation. But radzievskii[9] discussed it with the help of a formula in which the radiation repulsive force F_p exerted on a particle is represented in terms of gravitational attraction F_g and a mass reduction factor G_1 as $F_p = F_g(1 - q_1)$, $0 < 1 - q_1 << 1$. The investigation relating to the existence and the linear stability of equilibrium points were carried out by Radzievskii[10], Chernikov[8], Perezhogin[1], Bhatnagar and Chawla[6] and Sharma[7], Singh and Ishwar[4 & 5] discussed the effect of perturbations on the location and the stability of equilibrium points in the restricted problem of three bodies with variable mass under the assumption that the mass of the third body varies with time. They studied the above problem with the help of Jeans' law[3] and space-time transformation comparing it to the transformation of Meshcherskii[2].

2. EQUATIONS OF MOTION

We introduce a synodic coordinate system OXYZ with the origin at the centre of mass of the primaries with masses $m_1 > m_2$, and the axes rotating relative to the inertial space with angular velocity ω about the z-axis. Let the more massive primary be a source of radiation and the mass m of the third body varies with the time t. Let (x, y) be the coordinates of the third body. Its kinetic energy is given by

$$T = T_0 + T_1 + T_2 = \frac{1}{2} m\, \omega^2 (x^2 + y^2) + m\, \omega (x\dot{y} - \dot{x} y) + \frac{1}{2} m (\dot{x}^2 + \dot{y}^2),$$

and the potential energy by

$$V = -km \left(\frac{m_1 g_1}{\rho_1} + \frac{m_2}{\rho_2} \right),$$

where $\rho_1 = [(x+a)^2 + y^2]^{1/2}$ and $\rho_2 = [(x-b)^2 + y^2]^{1/2}$

are distances of the third body from m_1 at $(-a, 0)$ and m_2 at $(b, 0)$. The Lagrangian is given by $L = T_1 + T_2 - U$ with $U = V - T_0$. Then the equations of motion of the third body can be written as

$$\left.\begin{aligned} \frac{\dot{m}}{m}(\dot{x} - \omega y) + \ddot{x} - 2\,\omega \dot{y} &= -\frac{1}{m}\frac{\partial U}{\partial x} \\ \frac{\dot{m}}{m}(\dot{y} + \omega x) + \ddot{y} + 2\,\omega \dot{x} &= -\frac{1}{m}\frac{\partial U}{\partial y} \end{aligned}\right| \quad \text{... (1)}$$

Now from Jeans' law[3]

$$\frac{dm}{dt} = -\alpha m^n, \quad \text{... (2)}$$

where α is a constant coefficient and $0.4 \le n \le 4.4$ for the star of the main sequence.

We introduce the space-time transformation : $(x, y, t) \to (\xi, \eta, \Gamma)$, given by

$$x = \gamma^{-q}\,\xi,\ y = \gamma^{-q}\,\eta,\ dt = \gamma^{-k}\, d\Gamma,\ \rho_1 = \gamma^{\beta} r_1,\ \rho_2 = \gamma^{\beta} r_2$$

with $$\gamma = \frac{m}{m_0} \quad \text{... (3)}$$

where m_0 is the mass of the third body when $t = 0$. From (2) and (3), we have

$$\frac{d\gamma}{dt} = -\beta\, \gamma^{n},$$

where $$\beta + \alpha\, m_0^{n-1} \quad \text{... (4)}$$

Similar to as Singh and Ishwar[4], the equations of motion of the variable mass in dimensionless barycentric-synodic coordinate system (x, y) become

$$\xi'' - 2\,\eta' = \frac{\partial \Omega}{\partial \xi},\quad \eta'' + 2\,\xi' = \frac{\partial \Omega}{\partial \eta}, \quad \text{... (5)}$$

where $$\Omega=\frac{1}{2}(\xi^2+\eta^2),\ \gamma^{3/2} q_1\left(\frac{1-\mu}{r_1}\right)+\frac{\mu\gamma^{3/2}}{r_2}+\frac{\beta^2}{8}(\xi^2+\eta^2) \quad \dots (6)$$

$$r_1^2=(\xi+\mu\gamma^{1/2})^2+\eta^2,\ r_2^2=[\xi-(1-\mu\gamma^{1/2})^2+\eta^2 \quad \dots (7).$$

$$\mu=\frac{m_2}{m_1+m_2},\ 0<\mu<\frac{1}{2}$$

and the coordinates of $m_1=1-\mu$ and $m_2=\mu$ are $(\mu,\ 0)$ and $(1-\mu, 0)$ respectively. The dashes indicate differentiation with respect to Γ.

These equations of motion differ from those of the classical restricted problem by additional terms $\frac{\beta^2}{4}\xi$ and $\frac{\beta^2}{4}\eta$ due to the variation in the mass of the third body and q_1 appearing due to radiation of the more massive primary.

3. Location of Equilibrium Points

The equilibrium points are the solutions of

$$\frac{\partial\Omega}{\partial\xi}=0 \text{ and } \frac{\partial\Omega}{\partial\eta}=0.$$

3.1. Location of Triangular Points

The triangular points are the solutions of

$$\frac{\partial\Omega}{\partial\xi}=0,\ \frac{\partial\Omega}{\partial\eta}=0,\ n\neq 0$$

which give $$r_1=\frac{q_1^{1/3}}{\left(\frac{\beta^2}{4}+1\right)^{1/3}}\gamma^{1/2},\quad r_2=\frac{\gamma^{1/2}}{\left(\frac{\beta^2}{4}+1\right)^{1/3}} \quad \dots (8)$$

$$\xi=\frac{\gamma^{1/2}}{2}\left[1-2\mu+(q_1^{2/3}-1)\left(\frac{\beta^2}{4}+1\right)^{-2/3}\right]$$

$$\eta=\pm\frac{\gamma^{1/2}}{2}\left[\left(\frac{\beta^2}{4}+1\right)^{-2/3}\left[2q_1^{2/3}+2-\left(\frac{\beta^2}{4}+1\right)^{-2/3}\times(g_1^{4/3}-2q_1^{2/3}+1)\right]-1\right]^{1/2} . \quad \dots (9)$$

These points are denoted by L_4 and L_5 and known as triangular points. Eqs. (8) show that these two points form triangles with the primaries different from cases of Bhatnagar and Chawla[6], Singh and Ishwar[4] and Sharma[7] where they form isosceles triangles contrary to the classical case in which they form equilatemal triangles.

3.2. Location of Collinear Points

The collinear points are the solutions of $\frac{\partial \Omega}{\partial \xi}=0, \frac{\partial \Omega}{\partial \eta}=0, \ \eta=0$. Here $n = 0$ and therefore the collinear points lie on the line joining two primaries. In order to obtain their abscissa, we denote the expression obtained from $\left(\frac{\partial \Omega}{\partial \xi}\right)_{n=0}$ by $f(\xi)$. Similar to as in Singh and Ishwar[4], we can show that there are three solutions of $f(\xi)=0$, which correspond to three collinear points L_1, L_2 and L_3.

We find that the positions of equilibrium points are affected by the radiation pressure of the more massive primary and the variation in mass of the third body.

4. Stability of Equilibrium Points

Let (U, V) denote the small displacement in (ξ_0, η_0), the cooedinates of one of the equilibrium points. Putting $\xi=\xi_0+u, \eta=\eta_0+V$ in (5), we get

$$\left.\begin{aligned} u''-2v' &= u\,(\Omega^0_{\xi\xi})+v\,(\Omega^0_{\xi\eta}) \\ v''+2\,u' &= u\,(\Omega^0_{\eta\xi})+v\,(\Omega^0_{\eta\eta}) \end{aligned}\right| \qquad \text{... (10)}$$

The characteristic equation corresponding to (10) is

$$\lambda_4+(4-\Omega^0_{\xi\xi}-\Omega^0_{\eta\eta})\,\lambda^2+(\Omega^0_{\xi\xi}).\,(\Omega^0_{\eta\eta})=0. \qquad \text{... (11)}$$

4.1. Stability of Triangular Points

In this case the characteristic eq. (11) becomes

$$\lambda^4+c\,\lambda^2+d=0, \qquad \text{... (12)}$$

where $$C=1-\frac{3}{4}\beta^2,$$

$$d=\frac{q}{4}\frac{(1-\mu)\,\mu}{g_1^{2/3}}\left(\frac{\beta^2}{4}+1\right)^{10/3}\left[\left(\frac{\beta^2}{4}+1\right)^{-2/3}\right.$$

$$\left\{ 2q_1^{2/3} + 2 - \left(\frac{\beta^2}{4} + 1 \right)^{-2/3} \qquad (g_1^{4/3} - 2q_1^{2/3} + 1) \right\} - 1 \Bigg]$$

The solutions of (12) as a quadratic in $\wedge = \lambda^2$ are

$$\wedge_{1,2} = \frac{1}{2} [-c + (c^2 - 4d)^{1/2}]$$

and hence its four roots

$$\lambda_1 = \wedge_1^{1/2}, \lambda_2 = -\wedge_1^{1/2}, \lambda_3 = \lambda_2^{1/2}, \lambda_4 = -\wedge_2^{1/2}$$

depend not only upon the mass parameter μ but also on the radiation factor g_1 and β, the constant due to the variation in mass. The nature of these roots depends upon the nature of the discriminant Δ of (12) given as

$$\Delta = p_1 \mu^2 - p_1 \mu + p_2$$

where $$p_1 = 36\left[\left(1 - \frac{q_1^{2/3}}{4}\right) + \beta^2\left(\frac{7}{12} - \frac{g_1^{2/3}}{8}\right)\right],\; p_2 = 1 - \frac{3}{2}\beta^2.$$

4.1.1. Critical Mass

The critical value of the mass parameter is

$$\mu_c = \mu_0 - \frac{2}{27\sqrt{\lambda 69}}\varepsilon + \frac{1}{207\lambda\sqrt{69}}\varepsilon^2 - \left(\frac{19}{27\sqrt{69}} - \frac{688}{5589\sqrt{69}}\varepsilon - \frac{26}{16767\sqrt{69}}\varepsilon^2\right)\beta^2 \quad \text{... (13)}$$

where

$$\mu_0 = \frac{1}{2}\left(1 - \frac{\sqrt{69}}{9}\right),$$

and ε is a small quantity such that $q_1 = 1 - \varepsilon$.

Since $(\Delta)_{\mu=0}$ and $(\Delta)_{\mu=1/2}$ are of opposite signs, there is only one value of μ say μ_c in the open intervals $\left(0, \frac{1}{2}\right)$ for which Δ vanishes. Three cases can be discussed. If $0 < \mu < \mu_c$, the four roots of (12) are pure imaginary and the triangular points are stable in the linear sense.

If $\mu = \mu_c$, the solutions of (12) contain secular terms, consequently the triangular points are unstable.

If $\mu_c < \mu \leq \frac{1}{2}$, the real parts of the two roots of (12) are positive and so the triangular points are unstable.

Thus, the triangular points are stable for $0 < \mu < \mu_c$ and unstable for $\mu_c \leq \mu \leq \frac{1}{2}$. The rangte of stability for these points decreases. It may be observed that when $\beta^2 = 0$, the expression (13) is the same up to quadratic terms as in Bhatnagar and Chawla[6] and when $\beta^2 = 0$, $\varepsilon = 0$, it coincides with the expression μ_0 as in the classical problem.

4.2. Stability of Collinear Points

First we consider the point L_1.

$$-\infty \quad \underset{(-\mu,\,0)}{m_1 = 1 - \mu} \quad \underset{0}{} \quad \underset{(1-\mu,\,0)}{m_2 = \mu} \quad \underset{\longleftarrow \rho \longrightarrow}{L_1} \quad +\infty$$

If ρ be the distance from μ to L_1 on the line joining the primaries between ∞ and μ, then

$$q_0 = 1 - \mu + \rho,\ \eta_0 = 0,\ r_1 = 1 + \rho - \mu(1 - \mu^{1/2}),\ r_2 = \rho + (1 - \mu)(1 - \gamma^{1/2}).$$

The characteristic eq. (11) becomes

$$\lambda^4 + g\lambda^2 + f = 0 \qquad \dots (14)$$

where,

$$g = 2 - \frac{\beta^2}{2} - N,\ f = \left(\frac{\beta^2}{4} + 1\right)^2 + \left(\frac{\beta^2}{4} + 1\right)N - 2N^2,$$

$$N = \frac{(1-\mu)\,a_1\,\gamma^{3/2}}{(\xi_0 + \mu\,\gamma^{1/2})^3} + \frac{\mu\,\gamma^{3/2}}{\left\{\xi_0 - (1-\mu)\,\gamma^{1/2}\right\}^3}.$$

We may show from the definition of N, together with any $\mu < \frac{1}{2}$ and the corresponding value of ρ, that $f < 0$. The discriminant of (14) is therefore, positive and the solution $\lambda^2 = \frac{1}{2}[-g + (g^2 - 4f)^{1/2}]$ is positive when the plus sign is taken and negative when the minus sign is used. Hence (14) has two real roots equal in numerical value but opposite in sign and two conjugate pure imaginary roots. This indicates the unbounded motion in the xy-plane, hence L_1 is unstable. The same type of analysis shows that L_2 and L_3 are unstable. Thus, the stability of the collinear points is not affected by the variation of mass and the radiation force. They, in general, remain unstable.

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REFERENCES

1. A. A. Perezhogin *Astron. Zh. Lett.* **2** (1976) 448.
2. I. V. Meshcherskii, *On the Mechanics of bodies of variable mass,* Moscow, Leningrad (in *Russian*), 1949.
3. J. H. Jeans, *Astronomy and Cosmogony*, Cambridge Univ. Press, 1928.
4. J. Singh and B. Ishwar, *Celest. Mech.* **32** (1984) 297.
5. J. Singh and B. Ishwar, *Celest. Mech.* **35** (1985) 201.
6. K. B. Bhatnagar and J. M. Chawla, *Indian J. pure appl. Math.* **10** (1979), 1443.
7. R. K. Sharma, *Astrophys. Space Sci.* **135** (1987), 271-81.
8. V.A. Chernikov, *Astron. Zh.* **47** (1970) 217.
9. V. V. Radzievsky, *Astron. Zh.* **27** (1950) 250.
10. V. V. Radzievsky, *Astron. ZH.* **30** (1953) 265.
11. V. Szebehely, *Theory of Orbits*, Academic Press, New York, 1967.