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ORTHO-REFINABLE SPACES

XU YUMING AND JIANG SHOULI

Department of Mathematics, Shandong University, Jinan, 250 100, *P.R. China*

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In this paper, we study some properties of transitive neighbournets, and make use of them to discuss the properties of ortho-refinable spaces. We prove that ortho-refinable spaces are irreducible and iso-compact; every closed subspace of an ortho-refinable space is ortho-refinable.

Key Words : Neighbournets; Ortho-Refinable; Irreducible; Iso-compact; Open Mapping

1. INTRODUCTION

A base $\mathcal{B}$ for a space X is an ortho-base if for each subcollection $\mathcal{A}$ of $\mathcal{B}$, either (*i*) $\bigcap \mathcal{A}$ is open, or (*ii*) $\bigcap \mathcal{A}$ is a nonisolated singleton $\{x\}$ and $\mathcal{A}$ is a base for the neighbourhoods of x.[1] Junnila and Kunzi[2] make use of neighbournets to give two valuable characterizations of the topological spaces with an ortho-base. Then they introduce the monotonically orthocompact space and the ortho-refinable space. About ortho-refinable spaces, they prove that each metacompact space is ortho-refinable, and each ortho-refinable space is orthocompact. This indicates that, the neighbournet as a tool is nicer then the open cover in studying covering property; and for the other covering properties, it is worth studying the properties of ortho-refinable spaces. We study some properties of transitive neighbournets, and make use of them to discuss the properties of ortho-refinable spaces. We prove that ortho-refinable spaces are irreducible and iso-compact; every closed subspace of an ortho-refinable space is ortho-refinable, etc.

In this paper, all the spaces are T_1-spaces, the mappings are continuous, the closure of A in a space X is denoted by cl (A), and the ordinal numbers are denoted by α, β etc.

2. SOME PROPERTIES OF NEIGHBOURNET

Let X be a set, a subset R of $X \times X$ is called a relation on X. We usually denote the set $\{y : (x, y) \in R\}$ by $R\ \{x\}$. It follows that $R = \bigcup\ \{\{x\} \times R\ \{x\} : x \in X\}$, and thus R is determined by $\{R\ \{x\} : x \in X\}$. it is obvious that if R_1 and R_2 are two relations on X, then $R_1 \subseteq R_2$ iff for each $x \in X$ we have $R_1 \{x\} \subseteq R_2 \{x\}$. For each $A \subseteq X$, we denote $\bigcup \{R\{x\} : x \in X\}$ by $R(A)$. The inverse of a relation R, denoted by R^{-1}, is defined by that for all $x, y \in X, (x, y) \in R$ iff $(y, x) \in R^{-1}$. Suppose R and S row two relations on X, we define the relation $R \circ S = \{x, z) : \exists$ $y \in X$ such that $(x, y) \in R$ and $(y, z) \in S\}$. In particular, for each natural number n, let $R^1 = R, R^{n+1} = R \circ R^n$. It is easy to see that, for each $x \in X$, we have $(R \circ S)\{x\} = R(S\{x\})$, $(R \cap S)\{x\} = R\{x\} \cap S\{x\}$, $(R \cup S\{x\} = R\{x\} \cup S\{x\}$. About the inverse relation, we have $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$, $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$, and $S^{-1} \subseteq R^{-1}$, whenever $S \subseteq R$. A relation R on X is transitive if $R^2 \subseteq R$, i.e., $(x, z) \in R$ whenever $(x, y) \in R$ and $(y, z) \in R$, or $z \in R\{x\}$ whenver $y \in R\{x\}$ and $z \in R\{y\}$.

Definition 2.1 — A relation R on a space X is a neighbournet of X if $R\{x\}$ is a neighbourhood of x for each $x \in X$[3]. R is called a partial neighbournet if for each $x \in X$, $R\{x\} \cup R^{-1}\{x\} = \phi$ or $R\{x\}$ is a neighbourhood of x in X[2].

Obviously, a neighbournet of X is a partial neighbournet.

A neighbournet R of a topological space X is said to be open if for each $x \in X$, the set $R\{x\}$ is open. We know that a transitive neighbournet is open[3]. Similarly, if R is a transitive partial neighbournet of X, then for each $x \in X$ we also we have that $R\{x\}$ is open.

Junnila[3] have studied the properties on neighbournets in details. In this Section, we also give some other properties on transitive partial neighbournets. Let T be a partial neighbournet of a space X, for convenience, we set $\mathcal{J} = \{T\{x\} : x \in X\}$, $ST(x, \mathcal{J}) = (\mathcal{J})\, x = \{T\{y\} : x \in T(y) \in \mathcal{J}\}$, $st\,\{x, \mathcal{J}\} = \bigcup (\mathcal{J})_x$.

Proposition 2.2 — Let T be a transitive partial neighbournet of a space X, Y a subspace of X. Then $R = T \cap (Y \times Y)$ is also a transitive partial neighbournet of Y.

PROOF : For each $y \in Y$, we have that $R\{y\} = (T \cap (Y \times Y))\{y\} = T\{y\} \cap (Y \times Y)\{y\}$ $= T\{y\} \cap Y, R^1\{y\} = (T^{-1} \cap (Y \times Y)^{-1})\{y\} = T^{-1}\{y\} \cap (Y \times Y)^{-1}\{y\} = T^{-1}\{y\} \cap Y$, and thus $R\{y\} \cup R^{-1}\{y\} = (T\{y\} \cap Y) \cup T^{-1}(y) \cap Y) = (T\{y\} \cup T^{-1}\{y\}) \cap Y$. If $R\{y\} \cup R^{-1}\{y\} \neq \phi$, then we have $T\{y\} \cup T^{-1}\{y\} \neq \phi$. It follows that $T\{y\}$ is a neighbourhood of

y in X, and thus $R\ \{y\}$ is a neighbourhood of y in Y. Therefore, R is a partial neighbournet of Y.

Let $(x, y), (y, z) \in R = T \cap (Y \times Y)$, then we have $(x, y), (y, z) \in T$ and $(x, y), (y, z) \in Y \times Y$. Since T is a transitive partial neighbournet, we have $(x, z) \in T$. Note that $(x, z) \in Y \times Y$, then $(x, z) \in T \cap (Y \times Y)$. Therefore, R is a transitive partial neighbournet of Y.

Corollary 2.3 — Let T be a transitive neighbournet of a space X, Y a subspace of X. Then $R = T \cap (Y \times Y)$ is also a transitive neighbournet of Y.

A collection $\mathcal{U}$ of open subsets of a space X is said to be interior preserving if for each $\mathcal{U}' \subseteq \mathcal{U}$, the intersection $\bigcap \mathcal{U}'$ is open in X.

Proposition 2.4 — Let T be a transtive partial neighbournet of a space X, then $\{T\{x\} : x \in X\}$ is an interior preserving open family of X.

PROOF : Since T is a transitive partial neighbournet of X, for each $x \in X$, we have that $T\{x\}$ is an open subset of X.

Suppose $B \subseteq X$, and $\bigcap \{T\ \{x\} : x \in B\} \neq \phi$. Let $y \in \bigcap \{T\ \{x\} : x \in B\}$. Then, for each $x \in B$, we have $y \in T\ \{x\}$, and thus $T\ \{y\} \subseteq T(T\ \{x\}) = T^2\ \{x\} \subseteq T\ \{x\}$. It follows that $T\ \{y\} \subseteq \bigcap \{T\ \{x\}: x \in B\}$. Clearly, $B \neq \phi$. Choose $x \in B$, then $y \in T\ \{x\}$, we have $(x, y) \in T$, and thus $(y, x) \in T^{-1}, x \in T^{-1}\ \{y\}$. It is to say that $T^{-1}\{y\} \neq \phi$. By Definition 2.1, we know that $T\{y\}$ is an open neighbourhood of y. Hence, the intersection $\bigcap \{T\ \{x\} : x \in B\}$ is open in X.

We prove that $\{T\ \{x\} : x \in X\}$ is an interior-preserving open family of X.

Corollary 2.5 — Let T be a transitive neighbournet of a space X, then $\{T\{x\} : x \in X\}$ is an interior preserving open cover of X.

Proposition 2.6 — Let C be an interior-preserving open family of a space X, $R\ \{x\}$ $\bigcap (C)_x$ for each $x \in X$. Then R is a transitive partial neighbornet of X, and $\{R\ \{x\} : x \in \bigcup C\}$ is an open refinement of C.

PROOF : Since C is an interior-preserving open family of X, then for each $x \in X, R\{x\} = \bigcap (C)_x$ is an open subset of X. If $x \in \bigcup C$, we have $x \in R\ \{x\}$, and thus $R\{x\} \neq \phi$. It follows that

$R\ \{x\}$ is an open neighbourhood of x. If $x \notin \bigcup C$, then $R\ \{x\} = \phi$. Suppose $R^{-1}\{x\} \neq \phi$, then there exists some $y \in X$ such that $y \in R^{-1}\{x\}$, and thus $y \in R\ \{x\}$, a contradiction. So, we hve that $R^{-1}\ \{x\} = \phi$, and thus $R\ \{x\} \bigcup R^{-1}\ \{x\} = \phi$. In light of Definition 2.1, R is a partial neighbournet of X.

Let (x, y), $(y, z) \in R$, then $y \in R\ \{x\}$, $z \in R\ \{y\}$. Since $y \in R\ \{x\} = \bigcap (C)_x$, for each $C \in (C)_x$ we have $y \in C$. It follows that $\bigcap (C)_y \subseteq \bigcap (C)_x$, and thus $R\ \{y\} \subseteq R\ \{x\}$. So, we have $z \in R\ \{y\} \subseteq R\ \{x\}$, and thus $(x, z) \in R$. Therefore, R is a transitive partial neighbournet of X.

By the definition of $R\ \{x\}$, it is easy to see that $\{R\ \{x\}: x \in X\ \}$ is an open refinement of C.

Corollary 2.7 — Let C be an interior-preserving open cover of a space X, $R\ \{x\} = \bigcap (C)_x$ for each $x \in X$. Then R is a transitive neighbournet of X, and $\{R\ \{x\} : x \in \bigcup C\}$is an open refinement of C.

Proposition 2.8 — Let T_1, T_2 be transitive partial neighbournets of space X, $T_1 \bigcap T_2 = \phi$. Then $T_1 \bigcup T_2$ is also a transitive partial neighbournet of X.

PROOF : For each $x \in X$, suppose $(T_1 \bigcup T_2)\ \{x\} \bigcup (T_1 \bigcup T_2)^{-1}\{x\} \neq \phi$ or $(T_1 \bigcup T_2)^{-1}\ \{x\} \neq \phi$. If $(T_1 \bigcup T_2)^{-1}\ \{x\} \neq \phi$, then we have that $T_1\ \{x\} \neq \phi$ or $T_2\ \{x\} \neq \phi$. It follows that $T_1\ \{x\}$ or $T_2\ \{x\}$ is a neighbourhood of x, and thys $(T_1 \bigcup T_2)\ \{x\}$ is a neighbourhood of x. If $(T_1 \bigcup T_2)^{-1}\{x\} \neq \phi$, then there exists $y \in X$ such that $y \in (T_1 \bigcup T_2)^{-1}\{x\}$, so we have $(x, y) \in (T_1 \bigcup T_2)^{-1}$, and thus $(y, x) \in T_1 \bigcup T_2$. It follows that $(y, x) \in T_1$ or $(y, x) \in T_2$. Without lost of generality, we can assume $(Y, x) \in T_1$, then $x \in T_1\ \{y\}$ and thus $y \in T_1^{-1}\ \{x\}$. By Definition 2.1, we know that $T_1\ \{x\}$ is a neighbourhood of x, and thus $(T_1 \bigcup T_2)\ \{x\}$ is a neighbourhood of x. We prove that $T_1 \bigcup T_2$ is a partial neighbournet of X.

Let $(x, y)\, (y, z) \in T_1 \bigcup T_2$, then $y \in (T_1 \bigcup T_2)\ \{x\}$, $z \in (T_1 \bigcup T_2)\ \{y\}$. Without lost of generality, we can assume $y \in T_1\ \{x\}$, then $z \in T_1\ \{y\}$. If not, we have $z \in T_2\ \{y\}$, and thus

$T_2\ \{y\} \neq \phi$. It follows that $T_2\ \{y\}$ is a neighbourhood of y. On the other hand, from $y \in T_1\ \{x\}$ we know that $x \in T_1^{-1}\ \{y\}$. By Definition 2.1, we have that $T_1\ \{y\}$ is a neighbourhood of y. Hence, $(T_1 \cap T_2)\ \{y\} \neq \phi$, and thus $(T_1 \cap T_2) \neq \phi$, a contradiction. Therefore, by the transitive of T_1, we have that $z \in T_1\ \{x\}$, and thus $(x, z) \in T_1 \in T_1 \cup T_2$. Consequently, $T_1 \cup T_2$ is a transitive partial neighbournet of X.

Proposition 2.9 — Let C be an open cover of a space X, T a transitive neighbournet of X, and $\mathcal{J} = \{T\ \{x\}\ :\ x \in X\}$ an open refinement of C. Then $A = \{x \in X : x \in st\ \{x, \mathcal{J}\} \subseteq C$ for some $C \in C$ is a closed subset of X.

PROOF: Let $x \in cl(A)$. Since T is a transitive neighbournet on X, $T\{x\}$ is an open neighbourhood of x. So, there is an element $a \in A$ such that $a \in T\{x\}$. By the definition of A, we have that $a \in st\{a, \mathcal{J}\} \subseteq C$ for some $C \in C$. Because for each $y \in X, a \in T\ \{x\} \subseteq T\ \{y\}$ whenever $x \in T\ \{y\}$, we know that $st\ \{x, \mathcal{J}\} \subseteq st\ \{a, \mathcal{J}\}$. It follows that $x \in st\ \{x, \mathcal{J}\} \subseteq C$, and thus $x \in A$. Hence, A is a closed subset of X.

Proposition 2.10 — Let T be a partial neighbournet of a space X. Then $y \notin$ st $\{x, \mathcal{J}\}$ and $x \in$ st $\{y, \mathcal{J}\}$.

PROOF : Without lost of generality, we can assume $y \notin st\left\{x, \mathcal{J}\right\}$ and $x \in st\left\{y, \mathcal{J}\right\}$. Then there exists $z \in X$ such that $x \in T\ \{x\} \in (\mathcal{J})_y$. By the definition of $(\mathcal{J})_y$, we also have $y \in T\ \{z\}$. It follows that $y \in T\ \{x\} \in st\left\{x, \mathcal{J}\right\}$, a contradiction.

3. Some Properties of the Ortho-Refinable Spaces

Junnila and Kunzi[2] make use of the neighbournets to give a valuable characterization of the topological spaces with an ortho-base. Then they introduce the ortho-refinable space, and discuss some properties of it. A topological space X have an ortho-base if and only if there exists a decreasing chain $(T_\alpha)_{\alpha < \delta}$ of transitive partial neighbournets on X such that for each

$$x \in X, \left\{ st\,(x, \mathcal{J}_\alpha) : x \in \bigcup \mathcal{J}_\alpha, \alpha < \delta \right\}$$

is a local base.

Definition 3.1 — A toplogical space X is called ortho-refinable, provided that for each open cover C of X there are an ordinal δ and a decreasing chain $(T_\alpha)_{\alpha<\delta}$ of transitive partial neighbournets on X so that for each $x \in X$ there exists $\alpha < \delta$ such that $x \in st\left\{x, \mathcal{J}_\alpha\right\} \subseteq C$ for some $C \in \mathcal{C}$[2].

A space X is said to be orthocompact if for each open cover C of X, there is an interior preserving open refinement $\mathcal{R}$. Because each ortho-refinable space is orthocompact, each orthocompact submetacompact space is ortho-refinable, and thus each metacompact space is ortho-refinable[2], the ortho-refinability defined by neighbournet is one property between the metacompactness and the orthocompactness. We know that, in many aspects, the orthocompactness is usually unlike with other covering proprties, so the study of the ortho-refinable space has important value both for itself and for orthocompactness.

Definition 3.2 — An open cover C of a space X is said to be minimal, if no proper subfamily of C can cover X. A space X is called irreducible, if each open cover C of it has a minimal open refinement $\mathcal{R}$[4].

***Theorem* 3.3** — *The ortho-refinable space is irreducible.*

PROOF : Let X be an ortho-refinable space, C an open cover of X. Then there are an ordinal δ and a decreasing chain $(T_\alpha)_{\alpha<\delta}$ of transitive partial neighbournets on X so that for each $x \in X$ there exists $\alpha < \delta$ such that $x \in st\left\{x, \mathcal{J}_x\right\} \subseteq C$ for some $C \in C$. For each $\alpha < \delta$, set $A_\alpha = \{x \in X : x \in st\left\{x, \mathcal{J}_x\right\} \subseteq C$ for some $C \in C$, and when $\beta < \alpha$ we have st $\{x, \mathcal{J}_\beta\} \not\subset C$ for any $C \in C\}$. Obviously, we have that $X = \bigcup\left\{A_\alpha : \alpha < \delta\right\}$.

Without lost of generality, we assume that A_α is not empty for each $\alpha < \delta$. Let $x_{1,1} \in A_1$, then there exists $C_{1,1} \in C$ such that $x_{1,1} \in st\left\{x_{1,1} \mathcal{J}_1\right\} \subseteq C_{1,1}$. Assume that $x_{1,\lambda}$ has been chosen for all ordinal $\lambda < \eta$. If possible, choose $x_{1,\eta} \in A_1 \cap \left(X \setminus \bigcup\left\{st\left\{x_{1,\lambda}, \mathcal{J}_1\right\} : \lambda < \eta\right\}\right)$, then there exists $C_{1,\eta} \in C$ such that $x_{1,\eta} \in st\left\{x_{1,\eta}, \mathcal{J}_1\right\} \subseteq C_{1,\eta}$, Obviously, this inductive construction stops at some ordinal $\delta(1)$, when $A_1 \cap (X \setminus \bigcup\left\{st\left\{x_{1,\lambda}, \mathcal{J}_1\right\} : \lambda < \delta(1)\right\} = \phi$. For all $\beta < \alpha$, we assume that $A_\beta \cap (X \setminus \bigcup$ $\{st$ $\{x_{\gamma,\lambda} \mathcal{J}_\gamma\}$: $\lambda \leq \beta, \lambda < \delta(\gamma) \gamma \leq \beta, \lambda < \delta(\beta)\})$ = ϕ. If possible, choose $x_{\alpha,1} \in A_\alpha \cap$ $(X \setminus \bigcup$ $\{st\{x_{\beta,\lambda}, \mathcal{J}_\beta\}$: $\beta < \alpha, \lambda < \delta(\beta)\})$, then there exists $C_{\alpha,1} \in C$ such that $x_{\alpha,1} \in st$ $\{x_{\alpha,1}, \mathcal{J}_\alpha\}$ $\subseteq C_{\alpha,1}$. Like the idunctive construction for α = 1, there exists an ordinal $\delta(\alpha)$ such that

$A_\alpha \cap (X \setminus \bigcup \{st\ \{x_{\beta,\lambda}, \mathcal{J}_\beta\} : \beta \le \alpha, \lambda < \delta(\beta)\} = \phi$. Obviously, this inductive construction stops that most at the ordinal δ, when $\bigcup \{A_\alpha : \alpha < \delta\} \subseteq \bigcup \{st\ \{x_{\alpha,\lambda}, \mathcal{J}_\alpha\} : \alpha < \delta, \lambda < \delta(\alpha)\}$. Hence $\bigcup \{st\ \{x_{\alpha,\lambda}, \mathcal{J}_\alpha\} : \alpha < \delta, \lambda < \delta(\alpha)\} = X$, and thus $\mathcal{D} = \{st\ \{x_{\alpha,\lambda}, \mathcal{J}_\alpha\} : \alpha < \delta, \lambda < \delta(\alpha)\}$ is an open refinement of C. In the next, we will prove that $\mathcal{D}$ is minimal, and thus X is irreducible,

Suppose that $\beta < \alpha < \delta, \mu < \delta(\beta), \lambda < \delta(\alpha)$. From the above construction, we know that $x_{\alpha,\lambda} \notin st\left\{x_{\beta,\mu}, \mathcal{J}_\beta\right\}$. If $x_{\beta,\mu} \in st\left\{x_{\alpha,\lambda}, \mathcal{J}_\alpha\right\}$, then there exists $y \in \bigcup \mathcal{J}_\alpha$ such that $x_{\beta,\mu} \in T_\alpha(y) \in (\mathcal{J})_x$, and thus $x_{\beta,\mu}, x_{\alpha,\lambda} \in T_\alpha(y) \subseteq T_\beta(y)$. It follows that $x_{\alpha,\lambda} \in T_\beta(y) \subseteq st\ \{x_{\beta,\mu}, \mathcal{J}_\beta\}$, a contradiction. For $\alpha < \delta, \mu < \lambda < \delta(\alpha)$, by Proposition 2.10 we can similarly prove that $x_{\alpha,\lambda} \notin st\left\{x_{\alpha,\mu}, \mathcal{J}_\alpha\right\}$ and $x_{\alpha,\mu} \notin st\left\{x_{\alpha,\lambda}, \mathcal{J}_\alpha\right\}$. Therefore, we have that $x_{\alpha,\lambda} \notin st\left\{x_{\beta,\mu}, \mathcal{J}_\beta\right\}$ and $x_{\beta,\mu} \notin st\left\{x_{\alpha,\lambda}, \mathcal{J}_\alpha\right\}$ whenever $\beta \ne \alpha, \mu \ne \lambda$. It follows that, for any $\beta < \delta, \mu < \delta(\beta)$, we have $x_{\beta,\mu} \notin \bigcup \{\{st\backslash \{x_{\alpha,\lambda}, \mathcal{J}_\alpha\} : \lambda < \delta(\alpha), \alpha < \delta\} \setminus st\ \{x_{\beta,\mu}, \mathcal{J}_\beta\}\}$. Consequently, $\mathcal{D}$ is a minimal open refinement of C.

Junnila and Kunzi[2] have proved that a countably compact ortho-refinable space is compact. In fact, this result can also be induced by the above theorem.

Corollary 3.4 — Each countably compact ortho-refinable space is compact.

PROOF : From the fact that countably compact irreducible space is compact, and ortho-refinable space is irreducible, we immediately conclude the result.

About the hereditary property of ortho-refinable spaces, we have the following two theorems.

***Theorem* 3.5** — *Let X be an ortho-refinable space, Y a closed subspace of X. Then Y is also is an ortho-refinable space.*

PROOF : Let C be an arbitrary open cover of Y, then $\mathcal{D} = C \bigcup \{X \setminus Y\}$ is an open cover of X. So, there are an ordinal δ and a decreasing chain $(T_\alpha)_{\alpha<\delta}$ of transitive partial neighbournets on X so that for each $x \in X$ there exists $\alpha < \delta$ such that $x \in st\left\{x, \mathcal{J}_\alpha\right\} \subseteq C$ for some $C \in \mathcal{D}$. For each $\alpha < \delta$, let $R_\alpha = T_\alpha \cap (Y \times Y)$, then by Proposition 2.2 we have that R_α is a transitive partial neighbournet on Y. Since $R_\alpha = T_\alpha \cap (Y \times Y) \subseteq T_\beta \cap (Y \times Y) = R_\beta$ whenever $\alpha < \beta$, it follows that $(R_\alpha)_{\alpha<\delta}$ is a decreasing chain of transitive partial neighbournets on Y.

Let $y \in Y \subseteq X$, then there exist $\alpha(y) < \delta$ and $D \in \mathcal{D}$ such that $y \in st\left\{y, \mathcal{J}_{\alpha(y)}\right\} \subseteq D$. Choose $z \in \bigcup \mathcal{J}_{\alpha(y)}$ such that $y \in T_{\alpha(y)}\{z\} \in (\mathcal{J}_{\alpha(y)})_y$. By $R_{\alpha(y)} = T_{\alpha(y)} \cap (Y \times Y)$, we have $y \in R_{\alpha(y)}\{z\} \subseteq st\{y, \mathcal{R}_{\alpha(y)}\}$. Since $y \notin X \setminus Y$, we have $D \in \mathcal{C}$ and by the condition $R_\alpha \subseteq T_\alpha$ we know that $y \in st\left\{y, \mathcal{R}_{\alpha(y)}\right\} \subseteq st\left\{y, \mathcal{J}_{\alpha(y)}\right\} \in D \in \mathcal{C}$. In light of Definition 2.1, the subspace Y is ortho-refinable.

Definition 3.6 — A space X is called iso-compact, if each countably compact closed subset of it is compact[5].

Corollary 3.7 — Each ortho-refinable space X is iso-compact.

PROOF : By Theorem 3.5, we have that the closed subspace of X is ortho-refinable. By Corollary 3.4, we know that the countably compact closed subset of X is compact. Hence, the space X is iso-compact.

***Theorem* 3.8** — *Suppose each open subspace of X is ortho-refinable, then each subspace of X is ortho-refinable.*

PROOF : Let M be an arbitrary subspace of X, $\mathcal{V} = \{V_\lambda : \lambda \in \Gamma\}$ is an open cover of M, then there exists a family $\mathcal{U} = \{U_\lambda : \lambda \in \Gamma\}$ of open subsets of X such that $V_\lambda = U_\lambda \cap M$ for each $\lambda \in \Gamma$. Set $G = \bigcup \mathcal{U}$, then $\mathcal{U}$ is obviously an open cover of the open subspace G of X. It follows that there are an ordinal δ and a decreasing chain $(T_\alpha)_{\alpha<\delta}$ of transitive partial neighbournets on G so that for each $x \in G$ there exists $\alpha < \delta$ such that $x \in st\left\{x, \mathcal{J}_\alpha\right\} \subseteq U_\lambda$ for some $U_\lambda \in \mathcal{U}$. For each $\alpha < \delta$, let $R_\alpha \triangleq T_\alpha \cap (M \times M)$, then by Proposition 2.2 we have that R_α is a transitive partial neighbournet on M. Since $R_\alpha = T_\alpha \cap (M \times M) \subseteq T_\beta \cap (M \times M) = R_\beta$ whenever $\alpha < \beta$, it follows that $(R_\alpha)_{\alpha<\delta}$ is a decreasing chain of transitive partial neighbournets on M.

Let $x \in M$, then we have $x \in G$. It follows that there exists $\alpha < \delta$ such that $x \in st\left\{x, \mathcal{J}_\alpha\right\} \subseteq U_\mu$ for some $U_\mu \in \mathcal{U}$ $(\mu \in \Gamma)$. Choose $z \in \bigcup \mathcal{J}_\alpha$ such that $x \in T_\alpha\{z\} \in (\mathcal{J}_\alpha)_x$. By $R_\alpha \triangleq T_\alpha \cap (M \times M)$, we have $x \in R_\alpha\{z\} \subseteq st\{x, \mathcal{R}_\alpha\}$. Because for each $y \in \bigcup \mathcal{R}_\alpha$, $R_\alpha\{y\} = (T_\alpha \cap (M \times M))\{y\} = T_\alpha\{y\} \cap (M \times M)\{y\} = T_\alpha\{y\} \cap M$, we have that $x \in st\left\{x, \mathcal{R}_\alpha\right\} = \bigcup\{R_\alpha\{y\} : x \in R_\alpha\{y\}, y \in \bigcup \mathcal{R}_\alpha\} = \bigcup\{T_\alpha\{y\} \cap M : x \in R_\alpha\{y\}, y \in \bigcup \mathcal{R}_\alpha\} \subseteq (\bigcup\{T_\alpha\{y\} : x \in T_\alpha\{y\}, y \in \bigcup \mathcal{R}_\alpha\}) \cap M \subseteq st\left\{x, \mathcal{J}_\alpha\right\} \cap M \subseteq U_\mu \cap M = V_\mu$. By Definition 3.1, the subspace M of X is ortho-refinable.

It is well known that, if adding perfectness, many covering properties are hereditary. So, the following question is valuable.

Question 1 — Is the perfectly ortho-refinable space hereditary ortho-refinable?

Lemma 3.9 — Let T be a transitive partial neighbournet of X, $f: X \to Y$ an open mapping from finite to one. Then $\{f(T(x) : x \in X\}$ is an interior-preserving open families of $f(Y)$.

PROOF : By Proposition 2.4, $\{T\ \{x\}\ :\ x \in X\}$ is an interior preserving open family of X. Since f is an open mapping, we have that $\{f(T\ \{x\})\ :\ x \in X\}$ is an open family of $f\ (Y)$.

For any $A \subseteq X$, let $y \in \bigcap\ \{f(T\{x\})\ :\ x \in A\}$. Because f is a mapping from finite to one, we can assume $f^{-1}(y) = \{x_1, x_2, \ldots, x_n\}$. Let $A_i = \{x \in A : x_i \in T\{x\}\}$, then $A = \bigcup \left\{A_i : 1 \leq i \leq n\right\}$, without lost of generality, for $1 \leq i \leq n$, we can assume that A_i is not empty. Since $\{T\ \{x\}\ :\ x \in X\}$ is an interior preserving open families of X, we have that $\bigcap\ \{T\{x\}\ :\ x \in A_i\}$ is an open subset of X, and $x_i \in \bigcap\ \{T\ \{x\}\ :\ x \in A_i\}$. Thus, $f\ (\bigcap\ \{T\ \{x\}\ :\ x \in A_i\}$ is an open subset of $f\ (Y)$ and $y = f(x_i) \in f(\bigcap\ \{T\ \{x\}\ :\ x \in A_i\}) \subseteq \bigcap\ \{f\ (T\ \{x\}\ :\ x \in A_i\})$. It follows that $y \in \bigcap\ \{f\ (\bigcap\ \{T\{x\}\ :\ x \in A_i\}) : 1 \leq i \leq n\} \subseteq \bigcap\ \{\bigcap\ \{f(T\ \{x\})\ :\ x \in A_i\}\ :\ 1 \leq i \leq n\} \subseteq \bigcap\ \{f(T\ \{x\})\ :\ x \in A\}$. But $\bigcap\ \{f\ (\bigcap\ \{T\ \{x\}\ :\ x \in A_i\})\ :\ 1 \leq i \leq n\}$ is the intersection of finite many open subsets of X, we have that y is the interior point of the set $\bigcap\ \{f\ (T\ \{x\})\ :\ x \in A\}$. Hence, $\bigcap\ \{f(T\{x\}) : x \in A\}$ is an open subset of X. Therefore, $\{f\ (T\ \{x\})\ :\ x \in X\}$ is an interior-preserving open families of f (Y).

***Theorem* 3.10** — *Let X be a space having an ortho-base, $f: X \to Y$ an open mapping from finite to one. Then f (X) is an ortho-refinable space.*

PROOF : We know that the orthocompactness can be preserved by open mapping from finite to one[6], and the space having an ortho-base is orthocompact[2]. So, $f(X)$ is an orthocompact space. Suppose C is an open cover of $f(X)$, clearly we can assume that C is interior preserving, then $f^{-1}(C) = \left\{f^{-1}(C) : C \in C\right\}$ is an open cover of X.

Since the space X has an ortho-base, there exists a decreasing chain $(T_\alpha)_{\alpha < \delta}$ of transitive partial neighbournets on X such that for each $x \in X$, $\{st\ (x, \mathcal{J}_\alpha) : x \in \bigcup \mathcal{J}_\alpha, \alpha < \delta\}$ is a local base. For each $\alpha < \delta$, by Proposition 2.4 we know that $\{T_\alpha\ \{x\}\ :\ x \in \bigcup \mathcal{J}_\alpha\}$ is an interior-preserving open family of X. In light of Lemma 3.9, we conclude that $\{fT_\alpha\{x\} : x \in \bigcup \mathcal{J}_\alpha\}$ is an interior preserving open family of $f(X)$. For each $\alpha < \delta$, set $R_\alpha\{y\} = \bigcap \{f(T_\alpha\{x\}) : y \in f(T_\alpha\{x\})\}$, by Proposition 2.6 we have that R_α is a transitive partial neighbournet on $f(X)$. Because $(T_\alpha)_{\alpha<\delta}$ is a decreasing chain, it is easy to see that $(R_\alpha)_{\alpha<\delta}$ is also a decreasing chain.

For each $y \in f(X)$, we can assume that $f^{-1}(y) = \{x_1, x_2, ..., x_n\}$. Because C is interior presrving, $\bigcap (C)_y$ is an open subset of $f(X)$, so we have $f^{-1}(\bigcap (C)_y)$ is an open subset of X, and $f^{-1}(y) \subseteq f^{-1}(\bigcap (C)_y)$. It follows that, there exists $\alpha(i) < \delta$ such that $x_i \in st(x_i, \mathcal{J}_{\alpha(i)}) \subseteq f^{-1}(\bigcap (C)_y)$ for each $i\,(1 \le i \le n)$. Let $\alpha = \max\{\alpha(i) : 1 \le i \le n\}$, then we have that $st\{y, \mathcal{R}_\alpha\} = \bigcup \{R_\alpha\{z\} : y \in R_\alpha\{z\}\} = \bigcup \{\bigcap \{f(T_\alpha\{x\}) : z \in f(T_\alpha\{x\}) : y \in \bigcap \{f(T_\alpha\{x\}) : z \in f(T_\alpha\{x\})\}\} \subseteq \bigcup \{f(T_\alpha\{x\}) : y \in f(T_\alpha\{x\})\} = f(\bigcup\{T_\alpha\{x\} : y \in f(T_\alpha\{x\})\}) \subseteq f(\bigcup \{T_\alpha\{x\} : x_i \in T_\alpha\{x\}, 1 \le i \le n\}) \subseteq f(\bigcup\{st(x_i, \mathcal{J}_\alpha) : 1 \le i \le n\}) = \bigcup \{f(st(x_i, \mathcal{J}_\alpha)) : 1 \le i \le n\} \subseteq \bigcup \{f(st(x_i, \mathcal{J}_{\alpha(i)})) : 1 \le i \le n\} \subseteq \bigcup \{f(f^{-1}(\bigcap (C)_y)) : 1 \le i \le n\} = \bigcap (C)_y$. To prove that $y \in st\{y, \mathcal{R}_\alpha\}$, we assume $\alpha = \alpha(k)\,(1 \le k \le n)$, then $x_k \in st(x_k, \mathcal{J}_\alpha)$. Choose $x_0 \in X$ such that $x_k \in T_\alpha\{x_0\}$, then $y = f(x_k) \in F(T_\alpha\{x_0\})$, and thus $y \in \bigcap \{f(T_\alpha\{x\}) : y \in f(T_\alpha\{x\})\} = R_\alpha\{y\}$. It follows that $y \in st\{y, \mathcal{R}_\alpha\}$.. Therefore, for each $y \in f(X)$, there exists an ordinal α defined like the above such that $y \in st\left\{y, \mathcal{R}_\alpha\right\} \subseteq \bigcap (C)_y \subseteq C$ for every $C \in (C)_y$. Consequently, $f(X)$ is an ortho-refinable space.

Since the metacompactness and orthocompactness can all be preserved by open mappings from finite to one, we have the following question.

Question 2 — Can the ortho-refinability be preserved by open mappings from finite to one?

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