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# STOKES FLOW BEFORE A PLANE BOUNDARY

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Two-dimensional Stokes flow about a plane boundary is examined in the light of the complex variable theory; and two reflection theorems are established for the calculation of the Stokes flow about a plane wall. The first theorem is expressed in terms of the complex velocity for a slow irrotational flow in an unbounded incompressible viscous fluid. The second theorem gives the general Stokes flow before a plane boundary in terms of the complex velocity for a motion in an unbounded viscous fluid, which is both rotational and irrotational. A few illustrative examples are presented.

**Key Words : Stokes Flow; Stokeslet; Complex Potential; Fourier Transforms and Dirac Delta-Function**

## 1. INTRODUCTION

Milne-Thomson[1] has given the general method for images in a plane wall immersed in a two-dimensional potential flow in terms of the complex potential. The corresponding method for images in a plane wall in two-dimensional Stokes flow is, to the best of our present knowledge, not available in the literature. Here our object is to establish a general method for the calculation of the Stokes flow in the presence of a plane boundary. Before doing this, we first give a brief review of the earlier work on Stokes flow bounded by a plane wall. Ranger[2] has given the stream function for the two-dimensional Stokes flow due to a rotlet before a plane boundary, and compared the flow properties with those of the Stokes flow due to a cylinder rotating in the presence of the same boundary, and showed the results for the cylinder and the rotlet are qualitatively the same. Jeffery[3] and Wakiya[4] have considered somewhat differently the two-dimensionalStokes flow due to the rotation of a cylinder in front of a plane wall in a viscous fluid by making use of the bipolar co-ordinates. Again in the case of three-dimensional Stokes flow Blake[5], and blake and Chwang[6] have solved a number of Stokes flow problems, with the aid of the Fourier transforms, which arise

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owing to the presence of the fundamental singularities, such as Stokeslet, rotlet, stresslet, etc., in a viscous fluid bounded by a plane boundary. Collins[7&8] have considered axi-symmetric Stokes flows before a plane boundary for Stokes's stream functions.

## 2. Solutions of the Stokes Equations

The Stokes equations governing the two-dimensional, slow and steady motion of a viscous incompressible fluid are

$$grad\ p = \mu\, \nabla^2 \hat{q} + \hat{F}, \qquad \text{... (2.1)}$$

and

$$div\ \hat{q} = 0, \qquad \text{... (2.2)}$$

where $\hat{q}$ is the fluid velocity vector, $p$ the pressure, $\mu$ the constant viscosity coefficient and $\hat{F}$ the external force per unit volume.

In order to give the Stokes eqs. (2.1) and (2.2) the complex variable forms we first define a complex function $\upsilon = \upsilon(z, \bar{z}) = u - iv$, so that its complex conjugate becomes $\bar{\upsilon} = \bar{\upsilon}(\bar{z}, z) = u + iv$, where $u$ and $v$ are Cartesian components of the velocity $\hat{q}$ referred to in eq. (2.1) and $i = \sqrt{-1}$. Here we say that $\bar{\upsilon}$ is the complex representation of the velocity $\hat{q}$. The complex variable formulations of the Stokes eqs. (2.1) and (2.2) can now be derived in a straightforward manner and these are found as

$$2\frac{\partial p}{\partial \bar{z}} = 4\mu \frac{\partial^2}{\partial z\, \partial \bar{z}} \bar{\upsilon} + f, \qquad \text{... (2.3)}$$

and

$$\frac{\partial \bar{\upsilon}}{\partial z} + \frac{\partial \upsilon}{\partial \bar{z}} = 0, \qquad \text{... (2.4)}$$

where $f$ is the complex representation of the force $\hat{F}$ referred to in eq. (2.1);

In this note for viscous flow we call $\upsilon(z, \bar{z})$ the complex velocity, after Milne-Thomson [1, P. 153] who first called $\upsilon(z) = u - iv$, the complex velocity for potential flow. Next with the aid of the result, connecting $u$, $v$ and the stream function $\Psi = \Psi(z, \bar{z})$, in [1, P. 174], we obtain a relation between the stream function $\psi$ and the complex velocity $\upsilon$ as

$$\upsilon = -2i \frac{\partial \psi}{\partial z}. \qquad \text{... (2.5)}$$

We are now interested in giving the complex variable solutions of the Stokes eqs. (2.3) and (2.4) for the primary fundamental singularity, i.e., a Stokeslet (singular point force) at the origin,

$$f_s = 4\pi\mu\alpha\ \ \delta(x)\,\delta(y), \qquad \text{... (2.6)}$$

where $\alpha$ is a constant complex quantity, characterizing the strength of the Stokeslet, and $\delta(x)$ and

$\delta(y)$ are the Dirac-delta functions.

If we now suppose $v_s$ and $p_s$ are respectively the complex velocity and pressure due to the Stokeslet of strength $\alpha$ at the origin, the solutions of equations (2.3) and (2.4) with $f=f_s$, are given by

$$v_s(z,\bar{z};\alpha) = -\frac{1}{2}\bar{\alpha}\log z\bar{z} + \frac{1}{2}(\alpha\bar{z}+\bar{\alpha}z)/z, \quad \text{... (2.7)}$$

and

$$p_s(z,\bar{z};\alpha) = \mu\left(\frac{\alpha}{z}+\frac{\bar{\alpha}}{\bar{z}}\right), \quad \text{... (2.8)}$$

Here it is important to note that the expressions (2.6), (2.7) and (2.8) are respectively the complex representations of the two-dimensional Stokeslet, its velocity and pressure in [9, the vector expressions (18), (19a) and (19b)].

We now introduce below the complex variable formulations of the velocity and pressure fields for the two-dimensional fundamental singularities, such as Stokes doublet, rotlet, stresslet, potential doublet, etc.

Clearly, a derivative of any order of $\bar{v}, p_s$ and $f_s$ is a solution of the Stokes eqs. (2.3) and (2.4). Thus we adopt the following method similar to the three-dimensional vector method in Chwang and Wu[9] to obtain the complex velocity and pressure fields of the singularities.

Stokes doublet :

$$v_{SD}(z,\bar{z};\alpha,\beta) = -\left(\beta\frac{\partial}{\partial z}+\bar{\beta}\frac{\partial}{\partial\bar{z}}\right)v_s(z,\bar{z};\alpha),$$

$$= \frac{1}{2}\left((\bar{\alpha}\beta-\alpha\bar{\beta})\frac{1}{z}+\left(\bar{\alpha}\,\bar{\beta}\frac{1}{\bar{z}}+\alpha\beta\frac{\bar{z}}{z^2}\right)\right), \quad \text{... (2.9)}$$

and

$$p_{SD}(z,\bar{z};\alpha,\beta) = -\left(\beta\frac{\partial}{\partial z}+\bar{\beta}\frac{\partial}{\partial\bar{z}}\right)p_s(z,\bar{z};\alpha)$$

$$= \mu\left(\alpha\beta\frac{1}{z^2}+\bar{\alpha}\bar{\beta}\frac{1}{\bar{z}^2}\right), \quad \text{... (2.10)}$$

where

$$-\left(\beta\frac{\partial}{\partial z}+\bar{\beta}\frac{\partial}{\partial\bar{z}}\right)$$

is the complex operator which corresponds tothe three-dimensional vector operator, used in Chwang and Wu[9] to establish the velocity and pressure fields of the three-dimensional analogues of the singularities referred above.

Rotlet :

$$v_R(z,\bar{z};ik) = \frac{1}{2}\left[\left(\alpha\frac{\partial}{\partial z}+\bar{\alpha}\frac{\partial}{\partial\bar{z}}\right)v_s(z,\bar{z};\beta)-\left(\beta\frac{\partial}{\partial z}+\bar{\beta}\frac{\partial}{\partial\bar{z}}\right)v_s(z,\bar{z};\alpha)\right]$$

$$= ik\frac{1}{z} \quad \text{... (2.11)}$$

and

$$p_s(z,\bar{z};ik)=\frac{1}{2}\left[\left(\alpha\frac{\partial}{\partial z}+\bar{\alpha}\frac{\partial}{\partial \bar{z}}\right)p_s(z,\bar{z};\beta)-\left(\beta\frac{\partial}{\partial z}+\bar{\beta}\frac{\partial}{\partial \bar{z}}\right)p_s(z,\bar{z};\alpha)\right], \quad \text{... (2.12)}$$

where $ik=\frac{1}{2}(\bar{\alpha}\beta-\alpha\bar{\beta})$ is a pure complex number.

Stresslet :

$$v_{SS}(z,\bar{z};\alpha,\beta)=\frac{1}{2}\left[\left(\alpha\frac{\partial}{\partial z}+\bar{\alpha}\frac{\partial}{\partial \bar{z}}\right)v_s(z,\bar{z};\beta)-\left(\beta\frac{\partial}{\partial z}+\bar{\beta}\frac{\partial}{\partial \bar{z}}\right)v_s(z,\bar{z};\alpha)\right]$$

$$=\frac{1}{2}\left(\bar{\alpha}\bar{\beta}\frac{1}{z}+\alpha\beta\frac{\bar{z}}{z^2}\right), \quad \text{... (2.13)}$$

and

$$p_{ss}(z,\bar{z};\alpha,\beta)=\mu\left(\alpha\beta\frac{1}{z^2}+\bar{\alpha}\bar{\beta}\frac{1}{\bar{z}^2}\right). \quad \text{... (2.14)}$$

Potential Doublet :

$$v_D(z,\bar{z};\alpha)=-2\frac{\partial}{\partial z\,\partial \bar{z}}v_s(z,\bar{z};\alpha)$$

$$=\alpha\frac{1}{z^2} \quad \text{... (2.15)}$$

and

$$p_D(z,\bar{z};\alpha)=-2\frac{\partial^2}{\partial z\,\partial \bar{z}}p_s(z,\bar{z}\,\alpha)=0. \quad \text{... (2.16)}$$

Here the above complex variable formulations of the two-dimensional Stokeslet, Stokes doublet, rotlet, and stresslet etc. are mathematically concise and believed to be novel in the literature; their usefulness is demonstrated in the following sections.

## 3. Stokes Flow Bounded by a Rigid Plane

In this section we establish two theorems for two-dimensional steady Stokes flow before a plane boundary. For this purpose we need the fundamental results in Milne-Tomson [1, p. 684, eqs. (8) and (10)]; and these are stated somewhat differently for future reference as

$$v(z,\bar{z})=\overline{W}(\bar{z})-\bar{z}\,W'(z)-w(z), \quad \text{... (3.1)}$$

and

$$2i\,\psi=\bar{z}\,W(z)-z\,\overline{W}(\bar{z})-\int\bar{w}(\bar{z})\,d\bar{z}=\int w(z)\,dz, \quad \text{... (3.2)}$$

where $W(z)$ and $w(z)$ are arbitrary functions of $z$. In fact, the expressions (3.1) and (3.2) give respectively the general complex velocity and the stream function for two-dimensional Stokes flow. First, we present a theorem for Stokes flow before a plane boundary in terms of the complex velocity for a slow irrotational flow in an unbounded incompressible viscous fluid.

***Theorem* 1** — *Let $v_0(z)$ be the complex velocity for an irrotational flow in an incompressible viscous fluid with no rigid boundaries. Then if all the singularities of $v_0(z)$ lie at a finite distance from the origin in the region $y \geq 0$, the complex velocity and the stream function for the new flow in the same region, when $y = 0$ is made a rigid boundary, are given respectively by*

$$v(z, \bar{z}) = v_0(z) - v_0(\bar{z}) + (\bar{z} - z)\,\bar{v}_0'(z), \quad \dots (3.3)$$

*and*

$$\psi = \frac{1}{2} i \left\{ \bar{z}\,\bar{v}_0(z) - z\, v_0(\bar{z}) + \int (\bar{z}\, v_0'(\bar{z}) - \bar{v}_0(\bar{z}))\, d\bar{z} - \int (z\,\bar{v}_0'(z) - v_0(z))\, dz \right\}. \quad \dots (3.4)$$

PROOF : If we replace the arbitrary complex functions $W(z)$ and $w(z)$, referred to the general complex velocity (3.1), by the expression $W(z) = -\bar{v}_0(z)$ and $w(z) = -v_0(z) + z\,\bar{v}'(z)$, the particular complex velocity (3.3) is at once obtained.

On the wall $y = 0$, we have $z = \bar{z}$ so that the complex velocity (3.3) yields $v(z, \bar{z}) = 0$, which implies that the fluid speed vanishes on the wall. Since, by hypothesis, the singularities of $v_0(z)$ are in the region $y \geq 0$, the singularities of $v_0(\bar{z})$ and those of $\bar{v}'(z)$ are in the region $y \leq 0$, so that the complex velocity (3.3), i.e., $v(z, \bar{z})$ has the same singularities as $v_0(z)$ in the region $y \leq 0$. Thus the expression (3.3) is the exact complex velocity for a viscous fluid flow about the boundary $y = 0$. It is then obvious that substituting the above expressions for $W(z)$ and $w(z)$ in (3.2) yields the stream function (3.4) for the flow system. This completes the proof the theorem.

*Remark* : The motion of a viscous fluid with the uniform velocity $U$ in an unbounded region, in a direction, say the positive direction of the $x$-axis is governed by the primary complex velocity $v_0(z) = U$, which is due to the combination of a source and a sink, eachof the same infinite strength, at an infinite distance apart, on the $x$-axis. Since the singularities are not at a finite distance from the origin, the theorem does not apply here.

We now present an exact solution of a Stokes flow problem by making use of the Theorem 1.

## ROTLET IN THE PRESENCE OF A PLANE BOUNDARY

Consider a rotlet of strength $ik$ at the point $z_0$ in a viscous fluid. The complex velocity of the primary flow in this case is

$$v_0(z) = ik/(z - z_0). \quad \dots (3.5)$$

If the $x$-axis is now made a rigid boundary, the complex velocity for the new flow field in the presence of the boundary, by the theorem I, becomes

$$v(z, \bar{z}) = ik\left[\frac{1}{z - z_0} - \frac{1}{\bar{z} - z_0} + (\bar{z} - z)\frac{1}{(z - \bar{z}_0)^2}\right], \quad \dots (3.6)$$

which, on the appropriate reduction takes the following standard form as

$$v(z,\bar{z}) = ik\left[\frac{1}{z-z_0} - \frac{1}{z-\bar{z}_0} - \left(\frac{1}{\bar{z}-z_0} - \frac{\bar{z}-z_0}{(z-\bar{z}_0)^2}\right) + (z_0-\bar{z}_0)\frac{1}{(z-\bar{z}_0)^2}\right], \quad \ldots (3.7)$$

where the terms except the first one, give the image system in the region $y \leq 0$, which thus consists of (*i*) a rotlet of strength, $-ik$ at the point $\bar{z}_0$, (*ii*) a stresslet of strength, $-ik$ at the point $\bar{z}_0$, and (*iii*) a potential doublet of strength $ik(z_0 - \bar{z}_0)$ at the same point.

Next the stream function $\psi$, for the flow pattern can be obtained in a straightforward manner by usin the formula (3.4); and when it is expressed in terms of the polar co-ordinates $(r, \theta)$, a useful result emerges as

$$\psi(r,\theta) = k\,[-\log R_1 + \log R_2 + (r^2\cos 2\theta - 2rr_0\, s\,(\theta-\theta_0) + r_0^2\cos 2\theta_0)/R_2^2$$

$$+\ 2r_0\,(r\sin\theta\sin\theta_0 + r_0\sin^2\theta_0)/R_2^2], \quad \ldots (3.8)$$

where $\quad R_1 = |z - z_0|,\ R_2 = |z - \bar{z}_0|,\ r_0 = |z_0|$ and $\theta_0 = arg\, z_0$.

Here it is of interest to mention that Ranger[2] has obtained by a different method the stream function for the viscous flow field due to a rotlet at the point (0.1) inthe presence of the boundary $y = 0$. However one can at once recover the Ranger stream function from our result (3.8).

The flow fields due to other (irrotational) fundamental singularities, such as simple source, potential doublet, etc., in the presence of the boundary can be similarly obtained.

Next we present the general expression for the complex velocity and the stream function for two-dimensional Stokes flow before a plane boundary in terms of the complex velocity for a general flow in an unbounded incompressible viscous fluid.

***Theorem 2*** — *Let there be a two-dimensional steady flow, in an incompressible viscous fluid in the absence of rigid boundaries, characterized by the complex velocity* $v_0(z,\bar{z}) = \overline{W}_0(\bar{z}) - \bar{z}\,\overline{W_0'}(z) - w_0(z)$, *whose singularities are all at a finite distance from the origin in the region* $y \geq 0$. *If* $y = 0$ *is nowmade a rigid boundary, the complex velocity and the stream function for the new flow field in the same region become respectively.*

$$v(z,\bar{z}) = v_0(z,\bar{z}) - \overline{W}_0(z) + \bar{z}\,W_0'(\bar{z}) + w_0(\bar{z}) -$$

$$(\bar{z} - z)\,(\overline{w}_0'(z) + \overline{W}_0'(z) + z\,\overline{W}_0''(z)), \quad \ldots (3.9)$$

and $$\psi = \frac{1}{2}\,i\left\{z\,\overline{W}(\bar{z}) - \bar{z}\,W(z) + \int \overline{w}(\bar{z})\,d\bar{z} - \int w(z)\,dz\right\}, \quad \ldots (3.10)$$

where $$W(z) = W_0(z) + z\,\overline{W}_0'(z) + \overline{w}_0(z), \quad \ldots (3.11)$$

and $$w(z) = \overline{W}_0(z) + w_0(z) - z(\overline{W}_0''(z) + \overline{W}_0'(z) + \overline{w}_0'(z)). \qquad \text{... (3.12)}$$

PROOF : First we do note that the expression (3.9) is a particular complex velocity which is obtained by substituting the complex functions (3.11) and (3.12) in the general complex velocity (3.1). Since on the boundary $y = 0$, we have $z = \overline{z}$, the complex velocity (3.9) clearly vanishes on $y = 0$. Again since, by hypothesis, the singularities of $w_0(z)$ or $\overline{w}_0(\overline{z})$ and $w_0(z)$ lie in the region $y \geq 0$, those of $\overline{W}_0(z)$, $\overline{W}_0'(z)$, $\overline{W}_0''(z)$, $W_0'(\overline{z})$, $w_0(\overline{z})$ and $\overline{w}_0'(z)$ lie outside the same region. Therefore, $\upsilon(z, \overline{z})$ has the same singularities as the primary velocity $\upsilon_0(z, \overline{z})$ in the region $y \geq 0$; and hence the expression (3.9) is the exact complex velocity for a viscous flow about the boundary $y = 0$. Next, by using the complex functions (3.11) and (3.12) in the formula (3.2) we obtain at once the stream function (3.10) corresponding to the complex velocity (3.9) for the flow system about the boundary. This completes the proof of the theorem.

*Corollary* — If we take $W_0(z) = 0$ and $w_0(z) = -\upsilon_0(z)$, then the primary flow with no rigid boundaries becomes irrotational. Obviously, the theorem 2 then reduces to the theorem 1.

We now proceed to demonstrate an application of the theorem 2 by presenting an exact solution of a Stokes flow problem involving a Stokeslet and a plane boundary.

## STOKESLET IN THE PRESENCE OF A PLANE BOUNDARY

Let the basic flow in a viscous fluid be due to a Stokeslet of strength $\beta$ at the point $z_0$. The complex velocity of the flow in this case may be denoted by

$$\upsilon_0(z, \overline{z}) = -\frac{1}{2}\overline{\beta}\log(z - z_0)(\overline{z} - \overline{z}_0) + \frac{1}{2}\left\{\beta(\overline{z} - \overline{z}_0) + \overline{\beta}(z - z_0)\right\}/(z - z_0), \qquad \text{... (3.13)}$$

which can be expressed in the form

$$\upsilon_0(z, \overline{z}) = \overline{W}_0(\overline{z}) - \overline{z}\,W_0'(z) - w_0(z), \qquad \text{... (3.14)}$$

where $$W_0(z) = \frac{1}{2}\beta\log(z - z_0), \qquad \text{... (3.15)}$$

and $$w_0(z) = \frac{1}{2}\overline{\beta}\log(z - z_0) + \frac{1}{2}\beta\overline{z}_0\frac{1}{z - z_0} - \frac{1}{2}\overline{\beta}. \qquad \text{... (3.16)}$$

If $y = 0$ is now made the rigid boundary in the flow system, then the theorem 2 gives the complex velocity for the Stokes flow about the boundary as

$$\upsilon(z, \overline{z}) = \upsilon_0(z, \overline{z}) + \frac{1}{2}\overline{\beta}\log(z, \overline{z}_0) - \frac{1}{2}\beta\frac{\overline{z}}{\overline{z} - z_0} + \frac{1}{2}\overline{\beta}\log(\overline{z} - z_0) + \frac{1}{2}\beta\overline{z}_0\frac{1}{\overline{z} - z_0} - \frac{1}{2}\overline{\beta}$$

$$- (\bar{z}-z)\left(\frac{1}{2}(\beta-\bar{\beta})\frac{1}{z-\bar{z}_0}+\frac{1}{2}\bar{\beta}\frac{z-z_0}{(z-\bar{z}_0)^2}\right), \qquad \ldots (3.17)$$

which, on the appropriate reduction takes the following standard form.

$$\upsilon(z,\bar{z}) = \upsilon_0(z,\bar{z}) + \left[\frac{1}{2}\bar{\beta}\log(z-\bar{z}_0)\overline{(z-z_0)} - \frac{1}{2}(\beta(\bar{z}-z_0)+\bar{\beta}(z-\bar{z}_0))/(z-\bar{z}_0)\right]$$

$$+\left[\frac{1}{2}(\beta+\bar{\beta})(\bar{z}_0-z_0)\frac{1}{z-\bar{z}_0}+\frac{1}{2}\left(\beta(\bar{z}_0-z_0)\frac{1}{\bar{z}-z_0}+\bar{\beta}(z_0-\bar{z}_0)\frac{\bar{z}-z_0}{z-\bar{z}_0)^2}\right)\right]$$

$$+\frac{1}{2}\bar{\beta}(z_0-\bar{z}_0)^2\frac{1}{(z-\bar{z}_0)^2}, \qquad \ldots (3.18)$$

where the terms except $\upsilon_0(z,\bar{z})$, constitute the image system in the region $y \le 0$, which thus consists of (*i*) a Stokeslet of strength, $-\beta$, (*ii*) a Stokes doublet, which is the combination of a rotlet of strength, $\frac{1}{2}(\beta+\bar{\beta})(\bar{z}_0-z_0)$ and a stresslet of strength, $\left|\frac{1}{2}\beta(\bar{z}_0-z_0)\right|$, at the point $\bar{z}_0$, and (*iii*) a potential doublet of strength, $\frac{1}{2}\bar{\beta}(z_0-\bar{z}_0)^2$ at the same point.

The corresponding stream function for the flow pattern of the system can be obtained as

$$\psi = |\beta|\,[(r\sin(\theta-\theta_1)-r_0\sin(\theta_0-\theta_1))\log R_1 -$$

$$(r\cdot\sin(\theta-\theta_1)+r_0\sin(\theta_0+\theta_1))\log R_2$$

$$-\ 2r_0\sin\theta_0\cos\theta_1\log R_2 + r_0\sin\theta_0\,(r^2\cos(2\,\theta+\theta_1)-2rr_0\cos(\theta-\theta_0+\theta_1)$$

$$+\ r_0^2\cos(2\,\theta_0+\theta_1)''/R_2^2 + 2r_0^2(r\sin(\theta+\theta_1)+r_0\sin(\theta_0-\theta_1))\sin^2\theta_0/R_2^2], \qquad \ldots (3.19)$$

where $\quad R_1 = |z-z_0|,\ R_2 = |z-\bar{z}_0|,\ r_0 = |z_0|,\ \theta_0 = arg\,z_0$ and $\theta_1 = arg\,\beta$.

The flow fields due to other motion-generating singularities, such as stresslets and Stokes quadrupoles in the presence of the boundary can be treated in a similar manner.


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