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ENTIRE FUNCTIONS THAT SHARE TWO VALUES WITH THEIR DERIVATIVES*

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In this paper, we investigate the problem of uniqueness when an entire function f and the linear combination of its derivatives $L(f)$ with small functions as its coefficients share two finite values a, b IM, and give the definite expressions of f and $L(f)$, which are improvements of that of Ping Li and C.C. Yang and some other authors. These results remain to be valid if f is a meromorphic function and satisfying $N(r, f) = o(T(r, f))$ as $r \to \infty$ and the values a, b are replaced by small functions of f.

Key Words : Entire Function; Linear Differential Polynomial; Shared Value

1. INTRODUCTION AND MAIN RESULTS

Let $\overline{C}$ denote the extended complex plane. We say that two meromorphic functions f and g share the value $a \in \overline{C}$ CM (IM) provided that $f(z) = a$ if and only if $g(z) = a$ (ignoring multiplicities). It is assumed that the reader is familiar with the usual notations and fundamental results of Nevalinna theory of meromorphic functions, see e.g. [9] or [10]. In the sequel, a meromorphic function $\alpha(z)$ is called a small function of f iff $T(r, \alpha(z)) = o(T(r, f))$ as $r \to \infty$, possibly outside a set of r of finite linear measure.

In 1976, L. Rubel and C. C. Yang proved the following theorem in [1].

Theorem A — *Let f be a nonconstant entire function. If f and f' share two finite, distinct values CM, then $f \equiv f'$.*

Since then the subject of sharing values between a meromorphic function and its derivatives has been extensively studied by many mathematicians, and a lot of interesting results have been obtained (see [2], [3], [4], [5]).

In 1986, G. Frank and G. Weissenborn proved

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Theorem B[6] — *Let f be a nonconstant meromorphic function. If f and $f^{(k)}$ share two finite, distinct values CM, then $f \equiv f^{(k)}$*

In 1996, Bernstein-Chang-Li[7] studied the similar questions about meromorphic functions of several complex variables. As a special case, they proved

Theorem C — *Let f be a nonconstant entire function and*

$$L(f) \equiv b_n f^{(n)} + b_{n-1} f^{(n-1)} + \ldots + b_1 f' + b_0 f,$$

with b_j ($j = 0, 1, \ldots, n$) being small meromorphic functions of f. If f and L(f) share two finite values CM, then $f \equiv L(f)$.

More recently, Ping Li and C. C. Yang have completely resolved the question: What happens when an entire function f and the linear combination of its derivatives $L(f)$ share one value *CM* and another value IM? Their main results can be expressed by the following two theorems.

Theorem D[8] — *Let f be a nonconstant entire function and*

$$g \equiv b_{-1} + \sum_{i=0}^{n} b_j f^{(i)},$$

where b_i ($i = -1, 0, 1, \ldots, n$) are small meromorphic functions of f. Let a_1 and a_2 be two finite distinct constants. If f and g share a_1 CM and a_2 IM, then $f \equiv g$ or f and g have the following expressions :

$$f \equiv a_2 + (a_1 - a_2)(1 - e^{\alpha})^2,$$

and
$$g \equiv 2a_2 - a_1 + (a_1 - a_2)\, e^{\alpha},$$

where α is an entire function.

Theorem E[8] — *Let f be a nonconstant meromorphic function satisfying $N(r, f) = S(r, f)$, and*

$$g \equiv b_{-1} + \sum_{i=0}^{n} b_i f^{(i)},$$

where b_i ($i = -1, 0, \ldots, n$) are small meromorphic functions of f. Let a_1 and a_2 be two distinct small meromorphic functions of f. If f and g share a_1 CM^* and a_2 IM^*, then the conclusion of Theorem D still holds.

The definitions of the notations CM^* and IM^* will be given later in this section.

In [8], Li Ping and C. C. Yang gave an example which shows that the assumption "f and $L(f)$ share a_1 *CM*" in Theorem *D* can not be replaced by "f and $L(f)$ share a_1 *IM*". Regarding Theorem *D*, there remain two natural problems: Whether the conclusion of Theorem *D* holds if the hypothesis that f and $L(f)$ share a_1 CM is relaxed to a certain extent? And furthermore, what can be said if CM-shared value is replaced by IM-shared value in the hypothesis of Theorem D? The main purpose of this paper is to investigate the question: To what extent does the conclusion of Theorem D remain true? We have obtained results showing that we can relax the condition that one of the shared values be shared by CM, but the conclusion of Theorem D and Theorem E also hold. These results generalize Theorem D and Theorem E.

For the statement of our results, we need the following definitions.

Let f and g be two meromorphic functions and $a \in \overline{C}$. We denote by $\overline{N}_E\left(r,\frac{1}{f-a}\right)$ the counting function of those a-points of f where a is taken by f and g with the same multiplicity; and $\overline{N}_i\left(r,\frac{1}{f-a}\right)$ the counting function of those a-points of f where a is taken by f and g regardless of the multiplicity; each point in these counting functions is counted only once.

Definition 1 — A value a is said to be shared by f and g CM^*, if

$$\overline{N}\left(r,\frac{1}{f-a}\right)-\overline{N}_E\left(r,\frac{1}{f-a}\right)=S(r,f),$$

and

$$\overline{N}\left(r,\frac{1}{g-a}\right)-\overline{N}_E\left(r,\frac{1}{g-a}\right)=S(r,g).$$

Similarly, A value a is said to be shared by f and g IM^*, if

$$\overline{N}\left(r,\frac{1}{f-a}\right)-\overline{N}_i\left(r,\frac{1}{f-a}\right)=S(r,f),$$

and

$$\overline{N}\left(r,\frac{1}{g-a}\right)-\overline{N}_i\left(r,\frac{1}{g-a}\right)=S(r,g).$$

Definition 2 — Let f, g be two meromorphic functions and a be a complex number. We define

$$\tau(a,f)=\liminf_{r\to\infty}\frac{\overline{N}_E\left(r,\frac{1}{f-a}\right)}{\overline{N}\left(r,\frac{1}{f-a}\right)},$$

if $\overline{N}\left(r,\frac{1}{f-a}\right)\not\equiv 0$, $\tau(a,f)=1$ otherwise.

Obviously, $\tau(a,f)=\tau(a,g)$ if the value a is shared by f and g. In this case we shall denote it by $\tau(a)$ throughout the paper, for simplicity.

Remark 1 : In above two definitions, the value a can be replaced by a small function $a(z)$ of f and g.

Now, we can state our results.

Theorem 1— *Let f be a nonconstant entire function, k be a positive integer, and*

$$g \equiv a_{-1} + \sum_{i=0}^{k} a_i f^{(i)},$$

where a_i $(i = -1, 0, 1, ..., k)$ are small meromorphic functions of f. Suppose that f and g share two finite, distinct values a and b IM and that $\tau(a) > \frac{k+2}{k+3}$ for one of the shared values, say a, then $f \equiv g$ or f and g have the following expressions:

$$f \equiv b + (a-b)(e^G - 1)^2,$$

and $$g \equiv 2b - a + (a-b)e^G,$$

where G is an entire function.

From Theorem 1 and Theorem E we can immediately obtain the following corollary :

Corollary — Let f be a nonconstant entire function, k be a positive integer, and

$$g \equiv a_{-1} + \sum_{i=0}^{k} a_i f^{(i)},$$

where a_i $(i = -1, 0, 1, ..., k)$ are small meromorphic functions of f. Suppose that, f, g share two finite, distinct values a and b *IM* and for one of the shared values, say a, satisfy one of the following conditions :-

$$\text{(I)}\ \overline{N}\left(r, \frac{1}{f-a}\right) = S(r,f),$$

$$\text{(II)}\ \overline{N}_E\left(r, \frac{1}{f-a}\right) \geq \lambda \overline{N}\left(r, \frac{1}{f-a}\right) + S(r,f) \text{ for } r \geq r_0, \text{ if } \overline{N}\left(r, \frac{1}{f-a}\right) \neq S(r,f),$$

where r_0 and λ are two constants, $\lambda \geq \frac{k+2}{k+3}$. Then the conclusion of Theorem 1 is still valid.

Theorem 2 — *Let f be a nonconstant meromorphic function satisfying $N(r, f) = S(r, f)$, and*

$$g \equiv a_{-1} + \sum_{i=0}^{k} a_i f^{(i)},$$

where a_i $(i = -1, 0, 1, ..., k)$ are small meromorphic functions of f. Let a and b be two distinct small meromorphic functions of f. Suppose that f and g share a and b IM and that $\tau(a,f) > \frac{k+2}{k+3}$ for* *one of the shared small functions, then $f \equiv g$ or f and g have the following expressions.*

$$f \equiv b + (a-b)\,(e^G - 1)^2$$

$$g \equiv 2b - a + (a-b)\,e^G$$

where G is an entire functions.

Remark 2 : We mention that $\tau(a) = 1$ for a CM-shared value a or for a CM-shared function $a(z)$. So we can say that Theorem 1 is an extensions of Theorem D, simply note that $\frac{k+2}{k+3} < 1$ for all positive integers k. Moreover, if a is a CM^*-shared function of f and g such that $\overline{N}\left(r, \frac{1}{f-a}\right) \neq S(r,f)$, then for any constant $\lambda < 1$, we can derive that $\overline{N}_E\left(r, \frac{1}{f-a}\right) \geq \lambda \overline{N}\left(r, \frac{1}{f-a}\right) + S(r,f)$ for $r \geq r_1$, where r_1 is a positive constant, which gives that $\tau(a,f) \geq \lambda$. Therefore, Theorem 2 improves Theorem E.

2. Lemmas

Let f be a meromorphic function and $a \in \overline{C}$ be a complex number. We denote by $\overline{N}_{m)}\left(r, \frac{1}{f-a}\right)$ the counting function of those a-points of f whose multiplicities are less than or equal to m and by $\overline{N}_{(m+1}\left(r, \frac{1}{f-a}\right)$ the counting function of those a-points of f whose multiplicities are greater than m; each point in these counting functions being counted only once.

Lemma 1[8] — Let f be a transcendental meromorphic function, k and m be two positive integers, $P_m(f)$ denote a polynomial of f of degree m, and c_i $(i = 1, 2, \ldots, n)$ denote finite pairwise distinct constants. Set

$$F \equiv \frac{P_m(f)\, f^{(k)}}{(f-c_1)(f-c_2)\ldots(f-c_n)}.$$

If $m < n$, then $m(r, F) = S(r,f)$.

Lemma 2[8] — Let f be a nonconstant entire function and

$$g \equiv a_{-1} + \sum_{i=0}^{k} a_i f^{(i)},$$

where a_i $(i = -1, 0, 1, ..., k)$ are small meromorphic functions of f. Let a and b be two finite distinct constants. if f and g share a and b IM, then

$$T(r,f) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}\left(r, \frac{1}{f-b}\right) + S(r,f),$$

and $$T(r,f) \leq 2T(r,g) + S(r,f),$$

provided that $f \not\equiv g$.

From Lemma 2 and the definition of g, one can easily see that $S(r, f) = S(r, g)$. Throughout the paper, we will denote it by $S(r)$ for simplicity.

Lemma 3 — Let f, g, a and b be the same as that of Lemma 2. If f and g share a and b IM, then

$$\overline{N}\left(r, \frac{1}{f-a}\right) < \left(\frac{1}{1+\tau(a)} + o(1)\right) T(r,f),$$

provided that $f \not\equiv g$.

PROOF : Set

$$\varphi = \frac{f'(f-g)}{(f-a)(f-b)}. \qquad \text{... (2.1)}$$

Since f and g share a and b IM, we can deduce that $N(r, \varphi) = S(r)$. It follows from Lemma 1 that $m(r, \varphi) = S(r)$, thus $T(r, \varphi) = S(r)$.

Set $$\beta \equiv \frac{g'}{g-b} - \frac{f'}{f-b}, \qquad \text{... (2.2)}$$

then we have $$T(r, \beta) \leq \overline{N}\left(r, \frac{1}{f-b}\right) + S(r). \qquad \text{... (2.3)}$$

From (2.1), we get

$$\varphi \frac{f-a}{f'} \equiv 1 - \frac{g-b}{f-b}.$$

By taking the derivative in both sides of the above identity and using (2.2), we deduce that

$$\varphi' \frac{f-a}{f} + \varphi\left[1 - \frac{(f-a)f''}{(f')^2}\right] \equiv \beta\left(\varphi \frac{f-a}{f'} - 1\right),$$

which gives

$$(\varphi + \beta)\frac{f'}{f-a} - \varphi \frac{f''}{f'} + \varphi' - \beta\varphi \equiv 0. \qquad \text{... (2.4)}$$

We first suppose that $N_{1)}\left(r,\frac{1}{f-a}\right)\neq S(r)$. Let z_0 be a simple a-point of f, from (2.2) we see that $\beta(z_0)\neq\infty$, and from (2.4) we can get $\varphi(z_0)+\beta(z_0)=0$. If $\varphi+\beta\not\equiv 0$, then we obtain from (2.3) that

$$N_{1)}\left(r,\frac{1}{f-a}\right)\leq N\left(r,\frac{1}{\varphi+\beta}\right)\leq T(r,\varphi+\beta)+O(1)$$

$$\leq T(r,\beta)+S(r)\leq \overline{N}\left(r,\frac{1}{f-b}\right)+S(r). \qquad \text{... (2.5)}$$

From (2.1) it can be seen that "almost all" the multiple a-points of f are the simple a-points of g, thus we have

$$\overline{N}_{(2}\left(r,\frac{1}{f-a}\right)\leq \overline{N}\left(r,\frac{1}{f-a}\right)-\overline{N}_E\left(r,\frac{1}{f-a}\right)+S(r)$$

$$<(1-\tau(a)+o(1))\,\overline{N}\left(r,\frac{1}{f-a}\right)+S(r). \qquad \text{... (2.6)}$$

By (2.5), (2.6) and Lemma 2, we get

$$\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}\left(r,\frac{1}{f-b}\right)=N_{1)}\left(r,\frac{1}{f-a}\right)+\overline{N}_{(2}\left(r,\frac{1}{f-a}\right)+\overline{N}\left(r,\frac{1}{f-b}\right)$$

$$\leq 2\,\overline{N}\left(r,\frac{1}{f-b}\right)+(1-\tau(a)+o(1))\,\overline{N}\left(r,\frac{1}{f-a}\right)$$

$$= T(r,f)+(-\tau(a)+o(1))\,\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}\left(r,\frac{1}{f-b}\right)+S(r),$$

That is,

$$\overline{N}\left(r,\frac{1}{f-a}\right)\leq\left(\frac{1}{1+\tau(a)}+o(1)\right)T(r,f). \qquad \text{... (2.7)}$$

This inequality can hold possibly outside a set of r of finite linear measure.

If $\varphi+\beta\equiv 0$, this implies

$$T(r,\beta)=T(r,\varphi)+O(1)=S(r). \qquad \text{... (2.8)}$$

From (2.2) we get

$$N(r,\beta)=\overline{N}\left(r,\frac{1}{f-b}\right)-\overline{N}_E\left(r,\frac{1}{f-b}\right)+S(r). \qquad \text{... (2.9)}$$

Combining (2.8) and (2.9), we obtain

$$\overline{N}\left(r,\frac{1}{f-b}\right)-\overline{N}_E\left(r,\frac{1}{f-b}\right)=S(r). \qquad \text{... (2.10)}$$

Since f, g share the value b, it follows from (2.10) that the value b is shared CM^* by f and g. From Theorem E and the fact $f\not\equiv g$, we have

$$\overline{N}_E\left(r,\frac{1}{f-a}\right)=0 \text{ and } \overline{N}\left(r,\frac{1}{f-a}\right)=\frac{1}{2}T(r,f)+S(r),$$

which gives $\tau(a)=0$. Therefore, (2.7) also holds in this case.

Now we suppose that $N_{1)}\left(r,\frac{1}{f-a}\right)=S(r)$, then we have

$$\overline{N}\left(r,\frac{1}{f-a}\right)=\overline{N}_{(2}\left(r,\frac{1}{f-a}\right)+S(r)\le\frac{1}{2}N_{(2}\left(r,\frac{1}{f-a}\right)+S(r)$$

$$\le\left(\frac{1}{2}+o(1)\right)T(r,f)\le\left(\frac{1}{1+\tau(a)}+o(1)\right)T(r,f). \quad \text{... (2.11)}$$

Therefore, the conclusion of Lemma 3 follows from (2.7) and (2.11).

3. Proof of Theorem 1

Suppose that $f\not\equiv g$. Set

$$\eta\equiv\frac{f'}{f-a}-\frac{g'}{g-a} \quad \text{... (3.1)}$$

If $\eta\equiv 0$, we obtain from (3.1) that $\frac{f-a}{g-a}\equiv c$, where $c\neq 0$ is a constant. Therefore, f and g share a *CM*, the conclusion of our Theorem 1 follows from Theorem D. Now we suppose that $\eta\not\equiv 0$. Since a is shared by f and g, it follows from (3.1) that

$$T(r,\eta)\le\overline{N}\left(r,\frac{1}{f-a}\right)-\overline{N}_E\left(r,\frac{1}{f-a}\right)+S(r)$$

$$<(1-\tau(a)+o(1))\,\overline{N}\left(r,\frac{1}{f-a}\right)+S(r). \quad \text{... (3.2)}$$

By rewriting (2.1) as

$$\varphi\frac{f-b}{f}\equiv 1-\frac{g-a}{f-a}.$$

By taking the derivative in both sides of the above identity and using it again, we deduce that

$$\varphi'\left(\frac{f-b}{f}\right)+\varphi\left[1-\frac{(f-b)f'}{(f')^2}\right]\equiv\left(1-\varphi\frac{f-b}{f'}\right),$$

that is
$$(\varphi-\eta)\frac{f'}{f-b}-\varphi\frac{f''}{f'}+\varphi'+\eta\,\varphi\equiv 0. \quad \text{... (3.3)}$$

If $\varphi \equiv \eta$, then by (3.3) we get

$$-\frac{f''}{f'} + \frac{\varphi'}{\varphi} + \eta \equiv 0.$$

From (3.1) and the above identity, we have

$$\frac{f-g}{(f-b)(g-a)} \equiv K, \qquad \text{.. (3.4)}$$

where $K \neq 0$ is a constant.

On the other hand, it follows from Lemma 2, Lemma 3 and the assumptions of Theorem 1 that

$$\overline{N}\left(r, \frac{1}{f-b}\right) = T(r,f) - \overline{N}\left(r, \frac{1}{f-a}\right) + S(r)$$

$$\geq \left(1 - \frac{1}{1+\tau(a)} + o(1)\right) T(r,f) > \frac{2}{7} T(r,f). \qquad \text{... (3.5)}$$

Suppose that z_0 is a b-point of f and that $f - b$ and $g - b$ have the following Laurent expansions at z_0 :

$$f - b \equiv \alpha_p (z - z_0)^p + \alpha_{p+1} (z - z_0)^{p+1} + \ldots, \qquad \text{... (3.6)}$$

and $$g - b \equiv \beta_q (z - z_0)^q + \beta_{q+1} (z - z_0)^{q+1} + \ldots, \qquad \text{... (3.7)}$$

where α_p and β_q are two nonzero constants. From (3.4), one can easily see that $p \leq q$. If $p < q$, we obtain from (3.4), (3.6) and (3.7) that

$$K = \frac{1}{a-b}. \qquad \text{... (3.8)}$$

Again by (3.4) and (3.8), we get $(f - a)(g - b) \equiv 0$, this is a contradiction. Hence, we have $p = q$, which implies that f and q share b *CM*. Similarly, if $\overline{N}\left(r, \frac{1}{f-a}\right) \neq S(r)$, then we can deduce that f and g share a *CM*, too. Therefore, it follows that $f \equiv g$ from Theorem *C*, a contradiction. If $\overline{N}\left(r, \frac{1}{f-a}\right) = S(r)$, which implies that f, g share the value a CM^*. Theorem 1 follows from this and Theorem *E*.

Now we may suppose that $\varphi \not\equiv \eta$. It can be seen from (3.5) that $\overline{N}\left(r, \frac{1}{f-b}\right) \neq S(r)$. For the convenience of discussions, all the b-points of f will be divided into four pairwise disjoint sets:

$$S_1 = \{z : f - b = 0, \varphi - \eta = 0 \text{ and } a_i \neq \infty \text{ for } i = -1, 0, 1, \ldots, k\},$$

$$S_2 = \{z : f - b = 0,\ \varphi - \eta \neq 0, \varphi' + \eta\,\varphi = \infty, \text{ and } a_i \neq \infty \text{ for } i = -1, 0, 1, \ldots, k\},$$

$$S_3 = \{z : f - b = 0,\ \varphi - \eta \neq 0, \varphi' + \eta\,\varphi \neq \infty \text{ and } a_i \neq \infty \text{ for } i = -1, 0, 1, \ldots, k\}$$

and
$$S_4 = \{z : f - b = 0, \text{ and possible } \prod_{i=-1}^{k} a_i = \infty\}.$$

We denote by S_j^m the subset of those b-points of f with multiplicities m in S_j, one can see that $S_j = \bigcup_{m=1}^{\infty} S_j^m$ for $j = 1, 2, 3, 4$. Let S be a set of complex numbers, we denote by $\overline{N}(r, S)$ the counting function of the points of the set S. By (2.1) and the first fundamental theorem, it is easy to get

$$\overline{N}(r, S_1) \leq N\left(r, \frac{1}{\varphi - \eta}\right) \leq T(r, \eta) + S(r), \quad \ldots (3.9)$$

and
$$\overline{N}(r, S_2) \leq N(r, \varphi' + \varphi\,\eta) \leq T(r, \eta) + S(r). \quad \ldots (3.10)$$

Let z_0 be a b-point of f with multiplicities m, and let $z_0 \in S_3^m$, by (3.3), it can be seen that $\varphi(z_0) - m\,\eta(z_0) = 0$. Now we shall consider the following two cases :

Case 1 — $\varphi - m\,\eta \not\equiv 0$ for all $m \geq 1$.

If $m = 1$, from (3.3) we have that $S_3^1 = \phi$, therefore $\overline{N}(r, S_3^1) = 0$. If $2 \leq m \leq k+1$, it follows from (2.1), (3.3) and the first fundamental theorem that

$$\overline{N}(r, S_3^m) \leq N\left(r, \frac{1}{\varphi - m\,\eta}\right) \leq T(r, \eta) + S(r).$$

Noting that $\overline{N}(r, S_3^1) = 0$, so we have

$$\overline{N}\left(r, \bigcup_{m=1}^{k+1} S_3^m\right) \leq k\,T(r, \eta) + S(r). \quad \ldots (3.11)$$

If $m \geq k+2$, from the definition of g we can deduce that $b = a_{-1}(z_0) + a_0(z_0)\,b$. Now suppose that $a_{-1} + a_0\,b \not\equiv b$, then we have

$$\overline{N}\left(r, \bigcup_{m=k+2}^{\infty} S_3^m\right) = S(r). \quad \ldots (3.12)$$

If $a_{-1} + a_0 b \equiv b$, it follows the definition of g that

$$g - f \equiv (a_0 - 1)(f - b) + \sum_{i=1}^{k} a_i f^{(i)}.$$

Hence, z_0 is a multiple zero of $f - g$ and thus a zero of φ. But $T(r, \varphi) = S(r)$, therefore (3.12) remains to be valid. By (3.11) and (3.12) we have

$$\overline{N}(r, S_3) \le kT(r, \eta) + S(r) \quad \ldots (3.13)$$

Furthermore, it is easy to see that

$$\overline{N}(r, S_4) = S(r), \quad \ldots (3.14)$$

we can deduce from (3.2), (3.9), (3.10), (3.13) and (3.14) that

$$\overline{N}\left(r, \frac{1}{f-b}\right) \le (k+2)\, T(r, \eta) + S(r)$$

$$< (k+2)(1 - \tau(a) + o(1))\, \overline{N}\left(r, \frac{1}{f-a}\right) + S(r).$$

Combining this and Lemma 2 and 3, we obtain

$$T(r,f) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}\left(r, \frac{1}{f-b}\right) + S(r)$$

$$< [k + 3 - (k+2)\tau(a) + o(1)]\, \overline{N}\left(r, \frac{1}{f-a}\right) + S(r)$$

$$< \left[\frac{k+3-(k+2)\,\tau(a)}{1+\tau(a)} + o(1)\right] T(r,f).$$

This inequality holds possibly out side a set of r of finite linear measure. By this and the hypothesis $\tau(a) > \frac{k+2}{k+3}$, we can get a contradiction.

Case II — There exist an integer m such that $\varphi - m\,\eta \equiv 0$. From this and (3.1) we have

$$N(r, \varphi) = N(r, \eta) = \overline{N}\left(r, \frac{1}{f-a}\right) - \overline{N}_E\left(r, \frac{1}{f-a}\right) + S(r). \quad \ldots (3.15)$$

Noting that $T(r, \varphi) = S(r)$ and by (3.15), we have

$$\overline{N}\left(r,\frac{1}{f-a}\right)-\overline{N}_E\left(r,\frac{1}{f-a}\right)=S(r). \qquad \text{... (3.16)}$$

Since f, g share the value a, it follows from (3.16) that f and g shared a CM^*. Again by Theorem E, we can deduce that the conclusion of Theorem 1 holds. The proof of Theorem 1 is thus completed.

Remark 3 : From Theorem E and the proof of Lemma 2 (cf. [8]), Lemma 3 and Theorem 1, we can easily seen that the result of Theorem 1 is still valid for a nonconstant meromorphic function f sharing two finite, distinct values a and b IM^* with g, and satisfying $N(r,f)=S(r,f)$ together with one of the following conditions :

$$\text{(I) } \overline{N}\left(r,\frac{1}{f-a}\right)=S(r)$$

and

$$\text{(II) } \overline{N}_E\left(r,\frac{1}{f-a}\right)\geq \lambda\overline{N}\left(r,\frac{1}{f-a}\right)+S(r) \text{ for } r\geq r_0, \text{ if } \overline{N}\left(r,\frac{1}{f-a}\right)\neq S(r),$$

where r_0, λ are two constants such that $\lambda>\frac{k+2}{k+3}$ and a is one of the IM^* shared values.

In fact, under the assumption of Remark 3, one can see from the proof of Lemma 3 and Theorem 1 that Lemma 3 still valid when the term $\tau(a)$ is replaced by the constant λ, and that the conclusions of Theorem 1 remains true, only noting that each term $\tau(a)$ can be replaced by the constant λ throughout the proof of Theorem 1.

PROOF OF THEOREM 2

Set

$$f_1\equiv\frac{f-a}{b-a},\ g_1\equiv\frac{g-a}{b-a}. \qquad \text{... (4.1)}$$

It can be seen that f_1 and g_1 share 0 and 1 IM^* and that g_1 still has the form $g_1=\alpha_{-1}+\sum_{i=0}^{k}\alpha_i f_1^{(i)}$, where α_i for i = - 1, 0, 1, ..., k are small meromorphic functions of f_1. Furthermore, by (4.1) we have

$$\overline{N}\left(r,\frac{1}{f_1}\right)=\overline{N}\left(r,\frac{1}{f-a}\right)+S(r) \text{ and } \overline{N}_E\left(r,\frac{1}{f_1}\right)=\overline{N}_E\left(r,\frac{1}{f-a}\right)+S(r). \qquad \text{... (4.2)}$$

The condition $\tau(a,f)>\frac{k+2}{k+3}$ implies that there exist two constants r_1 and $\lambda\left(>\frac{k+2}{k+3}\right)$ such that

$$\overline{N}_E\left(r,\frac{1}{f-a}\right)\geq\lambda\overline{N}\left(r,\frac{1}{f-a}\right)+S(r) \text{ for } r\geq r_1. \qquad \text{... (4.3)}$$

From (4.2) and (4.3) it follows that

$$\overline{N}_E\left(r, \frac{1}{f_1}\right) \geq \lambda \overline{N}\left(r, \frac{1}{f_1}\right) + S(r).$$

Therefore, according to Remark 3, we can deduce that $f_1 \equiv g_1$ or

$$f_1 \equiv 1 - (e^G - 1)^2$$

and $$g_1 \equiv 2 - e^G,$$

where G is an entire function. Hence we have $f \equiv g$ or

$$f \equiv b + (a - b)\,(e^G - 1)^2$$

and $$g \equiv 2b - a + (a - b)\,e^G.$$

The proof of Theorem 2 is completed.

Final Remark : We suspect that Theorem 1 is still valid if the condition $\tau(a) > \frac{k+2}{k+3}$ is replaced by $\tau(a) > \frac{1}{2}$.

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