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LOCAL AND GLOBAL APPROXIMATION THEOREMS FOR BASKAKOV TYPE OPERATORS

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In this paper, we study the local and global theorems for the Baskakov type operators by using Ditzian-Totik moduli $\omega_{\varphi^\lambda}^r(f, t)$ of r order, where $r \in N, 0 \le \lambda < 1$ and φ is the Baskakov weight function. Both direct and converse theorems are derived. These results bridge the gap between the pointwise conclusions and global conclusions.

Key Words : Baskakov Operators; Direct Theorem; Inverse Theorem; Derivatives; Pointwise Approximation

1. INTRODUCTION

Let $C[0, \infty)$ be the space of continuous and bounded functions defined on $[0, \infty)$, its norm is defined by $\|f\| = \max\limits_{x \in [0, \infty)} |f(x)|$. We denote by $\varphi(x) = \sqrt{x(1+x)}, x \in [0, \infty)$, the Baskakov weight function. For $0 \le \lambda \le 1$, the Ditzian-Totik moduli of r order are defined by (see [1])

$$\omega_{\varphi^\lambda}^r(f, t) = \sup_{0 < h \le t} \|\Delta_{h\varphi^\lambda}^r f\|,$$

where

$$\Delta_{h\varphi^\lambda}^r f(x) = \begin{cases} \sum\limits_{i=0}^{r} (-1)^i C_r^i f(x + (r/2 - i) h \varphi^\lambda(x)), & x - rh\varphi^\lambda(x)/2 \in [0, \infty), \\ 0, & \text{otherwise.} \end{cases}$$

The Baskakov operators are defined by

$$V_n(f, x) = \sum_{k=0}^{\infty} C_{n+k-1}^k x^k (1+x)^{-(n+k)} f\left(\frac{k}{n}\right).$$

The linear combinations of the Baskakov operators are given by (see [1])

$$B_n(f, r, x) = \sum_{i=0}^{r-1} a_i(n)\, V_{n_i, 1}(f, x),\ r \in N,$$

where n_i and $a_i(n)$ satisfy

(a) $n = n_0 < n_1 < .. < n_{r-1} \le Cn$;

(b) $\sum_{i=0}^{r-1} a_i(n) = 1$;

(c) $\sum_{i=0}^{r-1} |a_i(n)| \le C$;

and (d) $\sum_{i=0}^{r-1} a_i(n)\, n_i^{-k} - 0, \quad k = 1, 2, ..., r-1.$

Here and hereafter C denotes a constant independent of f, n and x, its value may be different in each occurrence. It is clear that $B_n(f, 1, x) = V_n(f, x)$. Usually, such linear combinations operators have higher order of approximation (see [1], [4]).

For Bernstein operators, Felten[2] characterized their behaviour of approximation for the continuous functions by using the Ditzian-Totik moduli $\omega_{\varphi^\lambda}^2(f, t)$ of second order and gave the direct and inverse theorems of approximation. In [3], he further investigated the situation for other positive linear operators. In particular, the direct and inverse estimates for the Bernstein type operators and some exponential-type operators were obtained. However, similar questions for the Baskakov type operators were not solved. The main reason for this is that the Baskakov weight function $\varphi^2(x) = x(1+x)$ is non-concave on $[0, \infty)$. In this paper, we address the questions. We will study the situation more generally for the linear combinations of the Baskakov operators. Meanwhile, we will investigate the relationship between the derivatives of higher order of the Baskakov operators and the smoothness of functions. We believe that the method of this paper is fit for other operators, such as the Baskakov-Kantorovich operators, the Baskakov-Durrmeyer operators and the Szász type operators, etc.

The main result of this paper are as follows.

***Theorem* 1** — *Let* $0 \le \lambda \le 1, 0 < \alpha < r, r \in N,$ *and* $f \in C[0, \infty)$, *we have*

$$|B_n(f, r, x) - f(x)| = O\left(n^{-1/2} A_n^{1-\lambda}(x)\right)^{\alpha}$$

if and only if

$$\omega_{\varphi^\lambda}^r(f, t) = O(t^{\alpha}),$$

where $A_n(x) = \varphi(x) + 1/\sqrt{n} \sim \max(\varphi(x), 1/\sqrt{n})$.

Theorem 2 — *If* $0 \le \lambda \le 1, 0 < \alpha < r, r \in N$, *then for* $f \in C[0, \infty)$ *and* $\omega_{\varphi^\lambda}^r(f, t) = O(t^\beta)$ *with certain* $\beta > 0$, *we have*

$$\varphi^{r\lambda}(x) \,|\, V_n^{(r)}(f, x) \,| = O\left(\min\left(n^{2-\lambda}, \frac{n}{\varphi^2(1-\lambda)}(x)\right)\right)^{(r-\alpha)/2}$$

if and only if

$$\omega_{\varphi^\lambda}^r(f, t) = O(t^\alpha).$$

It is not difficult to see that the cases $\lambda = 0$ in the above results give the classical local conclusions whereas $\lambda = 1$ give the global norm estimates developed by Ditzian and Totik[2]. Therefore, these results bridge the gap between the pointwise conclusion and global conclusion for the Baskakov operators.

2. Direct Theorems

In this part, we give two direct theorems, which will imply the sufficiencies of Theorem 1 and Theorem 2, respectively.

Theorem 3 — *Let* $0 \le \lambda \le 1, r \in N$, *and* $f \in C[0, \infty)$, *we have*

$$|\, B_n(f, r, x) - f(x) \,| \le C\, \omega_{\varphi^\lambda}^r(f, n^{-1/2} A_n^{1-\lambda}(x)).$$

Theorem 4 — *Let* $f \in [0, \infty), r \in N, 0 < \alpha < r, 0 \le \lambda \le 1$, *we have*

$$|\, \varphi^{r\lambda}(x)\, V_n^{(r)}(f, x) \,| \le C n^{r/2} A_n^{r(\lambda-1)}(x)\, \omega_{\varphi^\lambda}^r(f, n^{-1/2} A_n^{1-\lambda}).$$

To prove Theorem 3 and Theorem 4, we first give some lemmas.

Lemma 1 — Let $0 < h < 1, x \in [0, \infty), r \in N, 0 < \beta \le r$, we have

$$\int_0^h \cdots \int_0^h \varphi^{-\beta}\left(x + \sum_{i=1}^r u_i\right) du_1 \ldots du_r \le C h^r \varphi^{-\beta}(x + rh). \quad \ldots (2.1)$$

Proof : First, we prove (2.1) is valid when $\beta = r$. For $r = 1$, we discuss two cases.

(1) If $x \ge 1$, then

$$\int_0^h \varphi^{-1}(x + u_1)\, du_1 \le \frac{1}{\sqrt{1+x}} \int_0^h \frac{1}{\sqrt{x + u_1}}\, du_1$$

$$\le \frac{2h}{\sqrt{x+h}\,(\sqrt{x+h} + \sqrt{x})}.$$

We notice that when $x \geq 1$, inequality $\sqrt{x+h}+\sqrt{x} \geq \sqrt{1+x+h}$ holds. Then

$$\int_0^h \varphi^{-1}(x+u_1)\,du_1 \leq \frac{2h}{\varphi(x+h)}.$$

(2) If $0 \leq x < 1$, then we use

$$\frac{1}{\sqrt{x+u_1}\,\sqrt{1+x+u_1}} \leq 4\left(\frac{1}{\sqrt{x+u_1}} - \frac{1}{\sqrt{x+u_1+1}}\right)$$

and get that

$$\int_0^h \varphi^{-1}(x+u_1)\,du_1 \leq 4\left(\frac{h}{\sqrt{x+h}+\sqrt{x}} - \frac{h}{\sqrt{1+x+h}+\sqrt{1+x}}\right)$$

$$\leq 4h\left(\frac{1}{\sqrt{x+h}} - \frac{1}{2\sqrt{1+x+h}}\right)$$

$$\leq 4\sqrt{3}\,h\,\varphi^{-1}(x+h).$$

Now, we prove the conclusion is true for $r = m + 1$ based on the conclusion being valid in the case $r = m$ $(m \geq 1)$. Since

$$\int_0^h \cdots \int_0^h \varphi^{-(m+1)}\left(x + \sum_{i=1}^{m+1} u_i\right) du_1 \ldots du_{m+1}$$

$$\leq \int_0^h \varphi^{-i}(x+u_{m+1}\,du_{m+1} \int_0^h \cdots\cdots \int_0^h \varphi^{-m}\left(x + \sum_{i+1}^{m} u_i\right) du_1 \ldots du_m$$

$$\leq C\frac{h^{m+1}}{\varphi(x+h)\,\varphi^m(x+mh)}$$

$$= Ch^{m+1}\,\varphi^{-(m+1}(x+(m+1)h)\left(\left(1+\frac{h}{x+mh}\right)\left(1+\frac{h}{1+x+mh}\right)\right)^{m/2}$$

$$\times\left(\left(1+\frac{mh}{x+h}\right)\left(1+\frac{mh}{1+x+h}\right)\right)^{1/2}$$

$$\leq Ch^{m+1}\,\varphi^{-(m+1)}(x+(m+1)h),$$

we derive the conclusion is true by using the mathematical induction. For $0 < \beta < r$, the proof can be completed by using Hölder inequality.

Lemma 2 — If $x, t \in [0, \infty), r \in N, 0 \leq \lambda \leq 1$, then

$$\left| \int_x^t |t-u|^{r-1} \varphi^{-r\lambda}(u)\, du \right| \leq |t-x|^r (\varphi^{r\lambda}(x) + (x(1+t))^{r\lambda/2}).$$

PROOF : Let $u = t + \tau(x-t), 0 \leq \tau \leq 1$, then

$$\left| \int_x^t |t-u|^{r-1} \varphi^{-r\lambda}(u)\, du \right|$$

$$\leq \left| \int_x^t \frac{|t-u|^{r-1}}{u^{r\lambda}}\, du \right| ((1+x)^{-r\lambda/2} + (1+t)^{-r\lambda/2})$$

$$\leq |t-x|^r \int_0^1 \tau^{r-1-r\lambda/2}\, d\tau (\varphi^{-r\lambda}(x) + (x(1+t))^{-r\lambda/2})$$

$$\leq |t-x|^r (\varphi^{-r\lambda}(x) + (x(1+t))^{-r\lambda/2}).$$

The proof of Lemma 2 is complete.

Lemma 3 — If $f \in C[0, \infty), r \in N, 0 \leq \lambda \leq 1$, then

$$\| \varphi^{r\lambda} V_n^{(r)}(f) \| \leq Cn^{r/2} (\max(n^{-1/2} \varphi(x)))^{r(\lambda-1)} \|f\|.$$

PROOF : Similar to section 9.4 of [1], it can be proved that

$$\| \varphi^r V_n^{(r)}(f) \| \leq Cn^{r/2} \|f\|.$$

If $\varphi(x) > 1/\sqrt{n}$, then $\max(n^{-1/2}, \varphi(x)) = \varphi(x)$ and

$$\| \varphi^{r\lambda} V_n^{(r)}(f) \| \leq Cn^{r/2} \varphi^{r(\lambda-1)}(x) \|f\|.$$

If $\varphi(x) \leq 1/\sqrt{n}$, then $\max(n^{-1/2}, \varphi(x)) = n^{-1/2}$ and

$$\| \varphi^{r\lambda} V_n^{(r)}(f) \| \leq C n^{r/2} n^{r(\lambda-1)}(x) \|f\|.$$

The proof of Lemma 3 is finished.

Lemma 4 — Let $f^{(r-1)} \in A\,C_{\cdot loc}, r \in N, 0 \leq \lambda \leq 1$, we have

$$\| \varphi^{r\lambda} V_n^{(r)}(f) \| \leq C \| \varphi^{r\lambda} f^{(r)} \|. \qquad \text{... (2.3)}$$

PROOF : If $\lambda = 0$, then (2.3) follows from the fact that

$$V_n^{(r)}(f,x)$$

$$= \frac{(n+r-1)!}{(n-1)!} \sum_{k=0}^{\infty} P_{n+r,k}(x)\, \Delta_{1/n}^{r} f\left(\frac{k}{n}\right)$$

$$= \frac{(n+r-1)!}{(n-1)!} \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_0^{1/n} \cdots \int_0^{1/n} f^{(r)}\left(\frac{k}{n} + \sum_{i=1}^{r} u_i\right) du_1 \ldots du_r.$$

If $0 < \lambda \le 1$, then for $k \ge 1$ we have

$$n^r \left| \Delta_{1/n}^{r} f\left(\frac{k}{n}\right) \right| \le \| \varphi^{r\lambda} f^{(r)} \| \, n^r \int_0^{1/n} \cdots \int_0^{1/n} \frac{du_1\, du_2 \ldots du_r}{\varphi^{r\lambda}\left(\frac{k}{n} + \sum_{i=1}^{r} u_i\right)}$$

$$\le \| \varphi^{r\lambda} f^{(r)} \| \, \varphi^{-r\lambda}\left(\frac{k}{n}\right).$$

For $k = 0$, it follows from the similar discussion to [1, pp. 154] that

$$n^r \,|\, \Delta_{1/n}^{r} f(0) \,| \le \int_0^{(r+1)/n} u^{r-1} \,|\, f^{(r)}(u) \,|\, du$$

$$\le n^r \| \varphi^{r\lambda} f^{(r)} \| \int_0^{(r+1)/n} u^{r-1-r\lambda/2}\, du$$

$$\le C n^{r\lambda/2} \| \varphi^{r\lambda} f^{(r)} \|.$$

Therefore,

$$|\, V_n^{(r)}(f,x) \,| \le C \| \varphi^{r\lambda} f^{(r)} \| \sum_{k=0}^{\infty} P_{n+r,k}(x) \left(\frac{n^2}{(k+1)(n+k)} \right)^{r\lambda/2}$$

$$\le C \| \varphi^{r\lambda} f^{(r)} \| \left(\sum_{k=0}^{\infty} \frac{P_{n+r,k}(x)}{((K+1)/n)^r} \right)^{\lambda/2} \left(\sum_{k=0}^{\infty} \frac{P_{n+r,k}(x)}{((k+n)/n)^r} \right)^{\lambda/2}$$

Using the facts (see also [1])

$$\sum_{k=0}^{\infty} \frac{n^r}{(k+1)^r} P_{n+r,k}(x) \le C x^{-r}, \quad \sum_{k=0}^{\infty} \frac{n^r}{(k+n)^r} P_{n+r,k}(x) \le C (1+x)^{-r},$$

we have $\varphi^{r\lambda}(x)\,|\,V_n^{(r)}(f,x)\,|\le C\,\|\,\varphi^{r\lambda}f^{(r)}\,\|.$

The proof of Lemma 4 is complete.

Now, we prove Theorem 3. We first define two K-functionals :

$$K_{\varphi^\lambda}^r(f,t)=\inf_{g^{(r-1)}\in A.C_{.loc}}\left\{\|f-g\|+t^r\|\varphi^{r\lambda}g^{(r)}\|\right\}$$

and

$$\overline{K}_{\varphi^\lambda}^r(f,t)=\inf_{g^{(r-1)}\in A.C_{.loc}}\left\{\|f-g\|+t^r\|\varphi^{r\lambda}g^{(r)}\|+t^{r/(1-\lambda/2)}\|g^{(r)}\|\right\}.$$

It was shown in [1] that

$$\omega_{\varphi^\lambda}^r(f,t)\sim K_{\varphi^\lambda}^r(f,t)\sim \overline{K}_{\varphi^\lambda}^r(f,t). \qquad \text{... (2.4)}$$

Then, by the similar way to Lemma 2, it follows that

$$\left|\int_t^x (t-x)^{r-1}g^{(r)}(u)\right|\le\|A_n^{r\lambda}g^{(r)}\|\left|\int_t^x |t-x|^{r-1}A_n^{-r\lambda}(u)\,du\right|$$

$$\le\|A_n^{r\lambda}g^{(r)}\|\left|\int_t^x |t-x|^{r-1}(\max(\varphi(u)),1/\sqrt{n}))^{-r\lambda}du\right|$$

$$\le\|A_n^{r\lambda}g^{(r)}\|\left|\int_t^x |t-x|^{r-1}(\min(\varphi^{-1}(u)),\sqrt{n}))^{-r\lambda}du\right|$$

$$\le C\|A_n^{r\lambda}g^{(r)}\|\,|t-x|^r\Big((\min(\varphi^{-1}(x)+x(1+t),\sqrt{n}))^{-r\lambda}\Big)$$

Using (see [1, (9.6.3.)])

$$V_n((1+t)^{-r},x)\le C(1+x)^{-r},$$

we obtain that

$$V_n(|t-x|^r(x(1+t))^{-r\lambda/2},x)\le(V_n(|t-x|^{2r},x))^{1/2}x^{-r\lambda/2}(V_n((1+t)^{-r},x))^{\lambda/2}$$

$$\le C\left(\frac{\varphi^2(x)}{n}+\frac{1}{n^2}\right)^{r/2}\varphi^{-r\lambda}(x).$$

Now, for any $g^{(r-1)}\in A.C_{.loc}$, applying the fact that $B_n((t-x)^k,r,x)=0$, $k=1,2,\ldots,r-1$ shown in [1], Lemma 2 and Taylor formula, we have

$$|B_n(g,r,x)-g(x)| \le \frac{1}{(r-1)!} B_n\left(\left|\int_x^t (t-u)^{r-1} g^{(r)}(u)\,du\right|, r, x\right)$$

$$\le C \| A_n^{\lambda r} g^{(r)} \| A_n^{-\lambda r}(x) \sum_{i=0}^{r-1} |a_i(n)| \left(\frac{\varphi^2(x)}{n_i} + \frac{1}{n_i^2}\right)^{r/2}$$

$$\le C n^{-r/2} \| A_n^{\lambda r} g^{(r)} \| A_n^{r(1-\lambda)}(x).$$

Similarly, we can prove that

$$|B_n(g,r,x)-g(x)| \le C n^{-r/2} \| \varphi^{\lambda r} g^{(r)} \| A_n^r(x)\, \varphi^{-r\lambda}(x).$$

Thus, for $\varphi(x) \ge 1/\sqrt{n}, f \in C[0,\infty)$, then $A_n(x) \sim \varphi(x)$ and

$$|B_n(f,r,x)-f(x)| \le C\left\{\|f-g\| + n^{-r/2} A_n^r(x)\, \varphi^{-r\lambda}(x) \| \varphi^{\lambda r} g^{(r)} \|\right\}$$

$$\le C\left\{\|f-g\| + n^{-r/2} A_n^{r(1-\lambda)}(x)\, \varphi^{-r\lambda}(x) \| \varphi^{\lambda r} g^{(r)} \|\right\},$$

which implies that

$$|B_n(f,r,x)-f(x)| \le CK_{\varphi^\lambda}^r(f, n^{-1?2} A_n^{1-\lambda}(x))$$

$$\le C\, \omega_{\varphi^\lambda}^r(f, n^{-1/2} A_n^{1-\lambda}(x)),$$

here (2.4) is used.

For $\varphi(x) \le 1/\sqrt{n}$, then $A_n(x) \sim 1/\sqrt{n}$ and

$$|B_n(f,r,x)-f(x)|$$

$$\le C\left\{\|f-g\| + n^{-r/2} A_n^{r(1-\lambda)}(x) \| A_n^{\lambda r} g^{(r)} \|\right\}$$

$$\le C\left\{\|f-g\| + n^{-r/2} A_n^{r(1-\lambda)}(x) \left(\| \varphi^{\lambda r} g^{(r)} \| + n^{-(r\lambda)/2} \| g^{(r)} \| \right)\right\}$$

$$\le C\left\{\|f-g\| + n^{-r/2} A_n^{r(1-\lambda)}(x) \| \varphi^{\lambda r} g^{(r)} \| + (n^{-1/2} A_n^{1-\lambda}(x))^{\frac{r}{1-\lambda/2}} \| g^{(r)} \|\right\}.$$

From (2.4) it follows that

$$|B_n(f,r,x)-f(x)| \le C\, \omega_{\varphi^\lambda}^r\left(f, n^{-1/2} A_n^{1-\lambda}(x)\right).$$

The proof of Theorem 3 is complete. Therefore, the sufficiency of Theorem 1 can be directly derived from Theorem 3.

Now, from Lemma 3 and Lemma 4 it follows that

$$\varphi^{r\lambda}(x)\,|\,V_n^{(r)}(f,x)\,|$$

$$\leq \varphi^{r\lambda}(x)\,|\,V_n^{(r)}(f-g,x)\,|+\varphi^{r\lambda}(x)\,|\,V_n(g,x)\,|$$

$$\leq C\,n^{r/2}\,(\min(n^{1/2},\varphi^{-1}(x)))^{r(1-\lambda)}\,\|f-g\|+C\,\|\varphi^{r\lambda}g^{(r)}\|,$$

which follows Theorem 4 from the definition of K-functional and (2.4) and the sufficiency of Theorem 2 is derived.

3. Inverse Theorems

In this paper, we prove the inverse parts of Theorem 1 and Theorem 2.

Since the sufficiency of Theorem 1 can be directly derived from Theorem 3, we only need to prove the inverse part. For $d>0$, we can choose $g_d^{(r-1)}\in A\cdot C_{.loc}$, such that

$$\|f-g_d\|\leq C\,\omega_{\varphi^\lambda}^r(f,d)\,\|\,\varphi^{r\lambda}g^{(r)}\,\|\leq C\,d^{-r}\,\omega_{\varphi^\lambda}^r(f,d).$$

For $0<h<\frac{1}{2^{\lambda+1}}$ and $x>rh\,\varphi^\lambda(x)/2$ we have

$$|\,\Delta_{h\varphi^\lambda}^r f(x)\,|\leq|\,\Delta_{h\varphi^\lambda}^r(f(x)-B_n(f,r,x))\,|+|\,\Delta_{h\varphi^\lambda}^r(B_n(f-g_d,r,x))\,|$$

$$+\,|\,\Delta_{h\varphi^\lambda}^r B_n(g_d,r,x)\,|=J_1+J_2+J_3.$$

For J_1 we have

$$J_1\leq C\,(n^{-1/2}A_n^{1-\lambda}(x+rh\,\varphi^\lambda(x)/2))^\alpha\leq C\,(n^{-1/2}A_n^{1-\lambda}(x))^\alpha$$

For J_2, from Lemma 1 and Lemma 3 we have

$$J_2\leq C\,n^{r/2}\,\|f-g_d\|\int_{-h\varphi^\lambda(x)/2}^{h\varphi^\lambda(x)/2}\int_{-h\varphi^\lambda(x)/2}^{h\varphi^\lambda(x)/2}\varphi^{-r}\left(x+\sum_{i=1}^r u_i\right)du_1\ldots du_r$$

$$\leq C\,n^{r/2}\,h^r\,A_n^{r(\lambda-1)}(x)\,\|f-g_d\|$$

$$\leq C\,h^r\,(n^{-1/2}A_n^{1-\lambda}(x))^{-r}\,\omega_{\varphi^\lambda}^r(f,d).$$

From Lemma 4 and Lemma 1 it follows that

$$J_3 \le C \| \varphi^{r\lambda} g_d^{(r)} \| \int_{-h\varphi^{\lambda}(x)/2}^{h\varphi^{\lambda}(x)/2} \cdots \int_{-h\varphi^{\lambda}(x)/2}^{h\varphi^{\lambda}(x)/2} \varphi^{-r\lambda}\left(x + \sum_{i=1}^{r} u_i \right) du_1 \ldots du_r$$

$$\le C h^r \| \varphi^{r\lambda} g_d^{(r)} \| \le C h^r d^{-r} \omega_{\varphi^\lambda}^{r}(f, d).$$

Recalling that for

$$n \ge 2,\ n^{-1/2} A_n^{1-\lambda}(x) < (n-1)^{-1/2} A_{n-1}^{1-\lambda}(x) \le 2n^{-1/2} A_n^{1-\lambda}(x),$$

we can choose $n \in N$, such that for $0 < d < \sqrt{h}, x > \frac{1}{2} rh\,\varphi^{\lambda}(x)$

$$n^{-1/2} A_n^{1-\lambda}(x) < d \le 2n^{-1/2} A_n^{1-\lambda}(x).$$

Hence,

$$| \Delta_{h\varphi^\lambda}^{r} f(x) | \le C\,(d^{\alpha} + h^r d^{-r} \omega_{\varphi^\lambda}^{r}(f, d)). \qquad \ldots (4.1)$$

Now, we suppose $C > 2^{(\lambda+1)(r-\alpha)/3}$ in (4.1). For $0 < \alpha < r$, we can choose $k = 0$, such that $\frac{1}{k} < r - \alpha$. Let $h_m = C^{-km}$, we take $h = h_m, d = h_{m-1}$ in (4.1) and get for m = 3, 4, 5, ..., that

$$\omega_{\varphi^\lambda}^{r}(f, h_m) \le \sum_{i=1}^{m-2} C^i \left(\frac{h_m}{h_{m-i+1}} \right)^r h_{m-i}^{\alpha} + C^{m-2} \left(\frac{h_m}{h_2} \right)^r \omega^r\, \omega_{\varphi^\lambda}^{r}(f, h_2)$$

$$\le C_1 \sum_{i=1}^{m-1} C^i \left(\frac{h_m}{h_{m-i+1}} \right)^r h_{m-i}^{\alpha} = C_1 C^{rk} \sum_{i=1}^{m-1} C^{-[(r-\alpha)k-1]i} h_m^{\alpha},$$

where $\qquad C_1 = \max\left\{ \frac{\omega_{\varphi^\lambda}^{r}(f, h_2)}{C h_1^{\alpha}}\ 1 \right\}$. Recalling that $(r-\alpha)k - 1 > 0$, we have

$$\omega_{\varphi^\lambda}^{r}(f, h_m) \le \frac{C_1 C^{rk}}{1 - C^{-(r-\alpha)k+1}} h_m^{\alpha} = C h_m^{\alpha}.$$

For $0 < h < h_3$, we choose $m \in N$, such that $h_{m+1} \le h < h_m$. Thus, we have

$$\omega_{\varphi^\lambda}^{r}(f; h) \le \omega_{\varphi^\lambda}^{r}(f, h_m) \le C h_m^{\alpha} \le C\,C^{k\alpha} h_{m+1}^{\alpha} \le C\,C^{k\alpha} h^{\alpha}.$$

The proof of necessity of Theorem 1 is complete.

By the similar methods to the above discussion and Theorem 4.1 in [4], we can prove the inverse part of Theorem 2. We omit the details.

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