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ON SEQUENCE-COVERING *K*-MAPPINGS*

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This paper gives internal characterizations of the 1-sequence-covering *k*-images of metric spaces and the sequence-covering *k*-images of metric spaces by means of *sn*-covers and *cs*-cover, respectively.

Key Words : 1-Sequence-Covering Mapping; Sequence-Covering Mapping; *sn*-Cover; *cs*-Cover; *K*-Mapping

In 1957, E. Halfar introduced the concept of *k*-mapping[1], which is stronger than a compact-covering mapping and weaker than a perfect mapping. In 1994, Chuan Liu gave an internal characterization of the *k*-mapping images of metric spaces[2]. S. Lin and C. Liu characterized the sequence-covering *s*-images of metric spaces by virtue of point-countable cs-networks[3]. This suggests the questions: what are nice characterizations for the 1-sequence-covering *k*-images of metric spaces and the sequence-covering *k*-images of metric spaces? The purpose of this paper is to give an answer to this question in terms of *sn*-covers and *cs*-covers introduced by S. Lin and P. Yan[4].

All spaces in this paper are assumed to be regular and T_1, and mappings are continuous and surjective.

We recall some basic definitions.

Definition 1 — Let $f: X \to Y$ be a map.

(1) f is a *k*-map[1] if $f^{-1}(K)$ is compact in X for any compact subset K of Y.

(2) f is a 1-sequence-covering map[5] if for each $y \in Y$, there exists $x \in f^{-1}(y)$ such that for each covergent sequence $\{y_n\}$ in Y with $y_n \to y$, there eexists $x_n \in f^{-1}(y_n)$ with $x_n \to x$.

(3) f is a sequence-covering map[6] if each convergent sequence of Y is the image of some convergent sequence of X.

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Obviously, a 1-sequence-covering map is a sequence-covering map.

S. Lin and C. Liu proved the following result by means of sequence-covering maps[3] :

A space X has a point-countable cs-network if and only if it is a sequence-covering s-image of a metric space.

S. Lin proved the following result[5] :

A space X has a point-countable weak base if and only if it is a 1-sequence-covering quotient s-image of a metric space.

C. Liu showed that the following are equivalent for a space X :

(1) X has a sequence $\{\mathcal{U}_i\}$ of compact-finite covers of k-closed subsets of X such that for each $x \in X$, $\{st(x, \mathcal{U}_i) : i \in N\}$ is a network at x.

(2) X is the k-mapping image of a metric space.

Quite recently, J. Li and S. Jiang proved the following[7] :

A space X has a locally countable weak base if and only if it is a 1-sequence-covering quotient strong s-image of a metric space.

The results above show 1-sequence-covering maps, sequence-covering maps and k-maps are very important maps in General Topology.

Definition 2 — Let X be a space, and let $\mathcal{P}$ be a cover of X.

(1) $\mathcal{P}$ is a network if, whenever $x \in U$ with U open in X, then $x \in P \subset U$ for some $P \in \mathcal{P}$.

A subfamily of $\mathcal{P}'$ of $\mathcal{P}$ is a network at $x \in X$ if $x \in \bigcap \mathcal{P}'$ and whenver $x \in U$ with U open in X, then $P \subset U$ for some $P \in \mathcal{P}'$.

(2) $\mathcal{P}$ is a cs-network[8], if, whenever $\{x_n\}$ is a sequence converging to a point $x \in X$ and U is a neighbourhood of x, then $\{x\} \bigcup \left\{x_n : n \geq m\right\} \subset P \subset U$ for some $m \in N$ and some $P \in \mathcal{P}$.

(3) cs-cover is the same as (2), but without requiring $P \subset U$ (See [4]).

(4) Let $\mathcal{P} = \bigcup \left\{\mathcal{P}_x : x \in X\right\}$ be a cover of a space X such that for each $x \in X$, $\mathcal{P}_x$ is a network at x and if $U, V \in \mathcal{P}_x$, $W \subset U \bigcup V$ for some $W \in \mathcal{P}_x$. Then $\mathcal{P}$ is a sequential-neighbourhood network for X[5] if for any $P \in \mathcal{P}_x$, P is a sequential nbd of x in X for each $x \in X$. (i.e., whenever $\{x_n\}$ is a sequence converging to x, then $\{x\} \bigcup \left\{x_n : n \geq m\right\} \subset P$ for some $m \in N$).

(5) $\mathcal{P}$ is an sn-cover[4] if for each $P \in \mathcal{P}$, P is a sequential nbd of a point $x \in X$, and for each $x \in X$, there exists $P \in \mathcal{P}$ such that P is a sequential nbd of x.

(6) Let $\mathcal{P} = \bigcup \left\{\mathcal{P}_n : n \in N\right\}$ be a cover of X, where $\mathcal{P}_n$ is also a cover of X. $\mathcal{P}$ is a sequence of compact-finite sn-covers if each $\mathcal{P}_n$ is both a compact-finite cover and an sn-cover. If we replace

"sn-cover" by "cs-cover", then $\mathcal{P}$ is called a sequence of compact-finite cs-covers of X.

Definition 3 — For a space X, $A \subset X$ is a k-closed subset of X if for any compact subset K of X, $A \bigcap K$ is closed in X.

It is easy to show the following lemmas :

Lemma 1 — Let $\mathcal{P} = \bigcup \left\{ \mathcal{P}_n : n \in N \right\}$ be a sequence of compact-finite sn-covers (resp. cs-covers). If each $\{st(x, \mathcal{P}_n) : n \in N\}$ is a network at $x \in X$, then $\mathcal{P}$ is a point-countable sequential nbd network (resp. cs-network).

Lemma 2 — Let X be a paracompact space, and $\mathcal{P}$ be an open cover of X, then there exists a locally finite open cover $\mathcal{U}$ of X such that $\overline{\mathcal{U}} = \left\{ \overline{U} : U \in \mathcal{U} \right\}$ refines $\mathcal{P}$.

Theorem 1 — *The following are equivalent for a space X.*

(1) X is the 1-sequence-covering k-image of a metric space.

(2) X has a sequence $\{\mathcal{P}_n\}$ *of compact-finite sn-covers of k-closed subsets of X such that for each* $x \in X$, $\{st(x, \mathcal{P}_n) : n \in N\}$ *is a network at x.*

PROOF : (1) $\Rightarrow$ (2) Assume $f : M \rightarrow X$ is both a 1-sequence-covering mapping and a k-mapping, where M is a metric space. Then M has a sequence $\{w_n : n \in N\}$ of open covers such that whenever $K \subset U$ with K compact and U open in M, then $st(K, \mathcal{W}_i) \subset U$ for some $i \in N$ by [9, Jones's Theorem]. Since M is paracompact, by Lemma 2, each w_i has a locally finite open cover $\mathcal{B}_i$ of M such that $\overline{\mathcal{B}}_i$ refines w_i. For each $x \in X$, there exists $\beta_x \in f^{-1}(x)$ satisfying Definition 1 (2). Let $\mathcal{P}_x^{(i)} = \left\{ f(B) : \beta_x \in \mathcal{B}_i \right\}$, $\mathcal{U}_i = \bigcup \left\{ \mathcal{P}_x^{(i)} : x \in X \right\}$.

(i) Each $\mathcal{U}_i$ is a compact-finite cover of X.

For each compact subset K of X, $f^{-1}(K)$ is compact in M because f is a k-mapping. Since $\overline{\mathcal{B}}_i$ is a locally finite cover of M,

$$\left| \left\{ \overline{B} \in \overline{\mathcal{B}}_i : \overline{B} \bigcap f^{-1}(K) \phi 0 \right\} \right| < \omega,$$

so $$\left| \left\{ f(\overline{B}) \in \mathcal{U}_i : f(\overline{B}) \bigcap K \phi 0 \right\} \right| < \omega.$$

(*ii*) Each $\mathcal{U}_i$ consists of k-closed subsets of X.

For each $G \in \mathcal{U}_i$, there is $\overline{B} \in \overline{\mathcal{B}}_i$ with $G = f(\overline{B})$. $G \cap H = f(\overline{B} \cap f^{-1}(H))$ for each compact subset H of X, and $\overline{B} \cap f^{-1}(H)$ is compact in M by virtue of f being a k-mapping and $\overline{B}$ being closed in M, so $G \cap H$ is closed in X. This shows G is k-closed in X.

(*iii*) For $\mathcal{U}_i$ is an *sn*-cover of X.

For each $G \in \mathcal{U}_i$, there exists $x \in X$ with $x \in G = f(\overline{B})$ and $\beta_x \in B \in \mathcal{B}_i$, where $\beta_x \in f^{-1}(x)$. For any sequence $\{x_n\}$ in X with $x_n \to x \in X$, there exists $y_n \in f^{-1}(x_n)$ with $y_n \to \beta_x$ in M. Because B is ope n in M, $\{\beta_x\} \cup \left\{ y_n : n \geq m \right\} \subset B$ for some $m \in N$, so $\{x\} \cup \left\{ x_n : n \geq m \right\} \subset f(B) \subset f(\overline{B}) = G$. Thus G is a sequential *nbd* of x in X. This shows $\mathcal{U}_i$ is an *sn*-cover of X.

(*iv*) For each $x \in X, \left\{ st(x, \mathcal{U}_i : i \in N \right\}$ is a network at x.

Let G be open in X with $x \in G$; then $f^{-1}(x) \subset f^{-1}(G)$. By the construction of $\{\overline{\mathcal{B}}_i\}$, $st\ (f^{-1}(x), \overline{\mathcal{B}}_i) \subset f^{-1}(G)$ for some $i \in N$. Thus $st(x, \mathcal{U}_i) \subset G$ for some $i \in N$. Hence $\{st(x, \mathcal{U}_i) : \iota \in N\}$ is a network at x.

(2) $\Rightarrow$ (1). Assume that X has a sequence $\{\mathcal{P}_n\}$ of compact-finite *sn*-covers of k-closed subsets of X such that for each $x \in X, \left\{ st(x, \mathcal{P}_n) : n \in N \right\}$ is a network at x. Put $\mathcal{P} = \bigcup \left\{ \mathcal{P}_n : n \in N \right\}$. By Lemma 1, $\mathcal{P}$ is a point-countable sequential *nbd* network for X. Let $\mathcal{P}_n = \left\{ P_\alpha : \alpha \in A_n \right\}$, then $\mathcal{P}_n$ is point-finite. The set A_n is endowed with the discrete topology. Define

$$M = \left\{ \beta = (\alpha_i) \in \prod_{i \in N} A_i : \{P_{\alpha_i}\} \text{ is a network of some } x(\beta) \text{ in } X \right\}.$$

and give M the subspace topology induced from the product topology. Then M is a metric space. For each $\beta \in M, x(\beta)$ is unique in X because X is a T_1-space. Define $f : M \to X$ by $f(\beta) = x(\beta)$. It is easy to check that f is a mapping.

(*i*) f is 1-sequence-covering.

For each $x \in X$, denote $\mathcal{P}_x = \left\{ P_{\alpha_i} : i \in N \right\} \subset \mathcal{P}$, where $P_{\alpha_i} \in (\mathcal{P}_i)_x$ and $\alpha_i \in A_i$. Then $\{P_{\alpha_i} : i \in N\}$ is a sequential *nbd* network of x. Put $\beta = (\alpha_i)$; then $\beta \in f^{-1}(x)$. Denote $B_n \left\{ (\gamma_i) \in M : \text{if } i \le n, \textit{then } \gamma_i = \alpha_i \right\}$. Then $\{B_n : n \in N\}$ is a monotonic decreasing *nbd* base of β of in M. For each $n \in N$, it is easy to check that $f(B_n) = \bigcap_{i=1}^{n} P_{\alpha_i}$. For a convergent sequence $\{x_j\}$ of X with $x_j \to x$, since $f(B_n)$ is a sequential *nbd* of x in X, there exists $i(n) \in N$ such that if $i \ge i(n)$, then $x_i \in f(B_n)$. Thus $f^{-1}(x_i) \cap B_n \ne \phi$. We may assume $1 < i(n) < i(n+1)$. For each $j \in N$, let

$$\beta_j \in \begin{cases} f^{-1}(x_j), & \text{if } j < i(1), \\ f^{-1}(x_j) \cap B_n, & \text{if } i(n) \le j < i(n+1), n \in N \end{cases}$$

Then it is easy to show that the sequence $\{\beta_i\}$ converges to β in M. Hence f is a 1-sequence-covering mapping.

(*ii*) f is a k-mapping.

For each compact set $K \subset X$, put $A_n' = \left\{ \alpha \in A_n : P_\alpha \in \mathcal{P}_n, P_\alpha \cap K \ne 0 \right\}$. Since each $\mathcal{P}_n$ is a compact-finite cover of X, so $|A_n'| < \omega$ for each $n \in N$, and so $\prod_{n \in N} A_n'$ is a compact in $\prod_{n \in N} A_n$.

Claim 1 — $f^{-1}(K) = \prod_{n \in N} A_n' \cap M$.

For each $\beta = (\beta_n) \in f^{-1}(K), f(\beta) \in \bigcap_{n \in N} P_{\beta_n} = \{x(\beta)\}$, so $P_{\beta_n} \cap K \ne \phi$ for each $n \in N$. By the definition of A_n', $\beta_n \in A_n'$, so $\beta \in \prod_{n \in N} A_n'$, and hence $f(K) \subset \pi\ n \in N A_n' \cap M$. Suppose that $(\beta = \beta_n) \in \prod_{n \in N} A_n' \cap M$.. Then $\{P_{\beta_n} : n \in N\}$ is a network of some $x(\beta)$ in X. Because $\beta_n \in A_n'$, we have $P_{\beta_n} \cap K \ne 0$ for each $n \in N$ and $f(\beta) = x(\beta)$. Suppose that $x(\beta) \notin K$, then $x(\beta) \in X \setminus K$. Thus $x(\beta) \in P_{\beta_k} \subset X \setminus K$ for some $k \in N$ because $\{P_{\beta_n} : n \in N\}$ is a network of $x(\beta)$. Then $P_{\beta_k} \cap K = 0$, a contradiction. Thus $x(\beta) \in K$, and $\beta \in f^{-1}(K)$. So $\prod_{n \in N} A_n' \cap M \subset f^{-1}(K)$.

Claim 2 — $f^{-1}(K)$ is closed in $\prod_{n\in N} A_n'$.

For each $\alpha=(\alpha_n)\in \prod_{n\in N} A_n'\backslash f^{-1}(K)$, we have $\bigcap_{n\in N} P_{\alpha_n}=\phi$ by Claim 1. Thus $\bigcap_{n\in N}(P_{\alpha_n}\cap K)=\phi$. Since each $\mathcal{P}_n$ consists of k-closed subsets of X,

$\bigcap_{i=1}^{m}(P_{\alpha_i}\cap K)=\phi$ for some $m\in N$. Put

$W=\left\{\beta=(\beta_n)\in \prod_{n\in N} A_n' : \beta_i=\alpha_i \text{ for } i=1,2,\ldots,m\right\}$. Then W is open in $\prod_{n\in N} A_n'$ with $\alpha\in W$ and $W\cap f^{-1}(K)=\phi$. Thus, $f^{-1}(K)$ is closed in $\prod_{n\in N} A_n'$.

By Claim 2, $f^{-1}(K)$ is compact in $\prod_{n\in N} A_n'$, and hence in $\prod_{n\in N} A_n$. Thus, $f^{-1}(K)$ is compact in M, and so f is a k-mapping.

As in the proof of Theorem 1, the following holds :

Theorem 2 — *The following are equivalent for a space.*

(1) X is the sequence-covering k-image of a metric space.

(2) X has a sequence $\{\mathcal{P}_n\}$ *of compact-finite cs-covers of k-closed subsets of X such that for each* $x\in X$, $\{st\,(x,\mathcal{P}_n): n\in N\}$ *is a network at x.*

Example 1 — A mapping which is k-mapping, but not sequence-covering, with metric domain.

Let

$$X=\{(0,0)\}\cup\left\{\left(0,\frac{1}{n}\right): n\in N\right\}\cup\{(1,0)\}\cup\left\{\left(0,\frac{1}{n}\right): n\in N\right\},$$

and X be the subspace of the real line R. Let $Y=X/A$, where $A=\{(0,0),(1,0)\}$, and $f: X\to Y$ be the natural quotient map. Then f is a k-mapping (indeed, f is a perfect map). It is easy to show that f is not a sequence-covering map.

Example 2 — A 1-sequence-covering, but not k-mapping with metric domain. Let $Y=\beta N$, the Stone-Čech compactification of the integers. Let $X=\beta N$ be a discrete topological space, and $f: X\to Y$ be the natural map. Then f is a 1-sequence-covering map, but not a k-map.

Is the sequence-covering k-image of a metric space the 1-sequence-covering k-image of a metric space? In other words, are the conditions in Theorem 1 equivalent to the conditions in Theorem 2? By Lemma 3 in [12], we affirmatively answer the question, that is, the following holds.

***Theorem* 3** — *The following are equivalent for a space X.*

(1) X is the 1-sequence-covering k-image of a metric space.

(2) X is the sequence-covering k-image of a metric space.

(3) X has a sequence $\{\mathcal{P}_n\}$ *of compact-finite cs-covers of k-closed subsets of X such that for each* $x \in X$, $\left\{ st(x, \mathcal{P}_n) : n \in N \right\}$ *is a network at x.*

(4) X has a sequence $\{\mathcal{P}_n\}$ *of compact-finite sn-covers of k-closed subsets of X such that for each* $x \in X$, $\left\{ st(x, \mathcal{P}_n) : n \in N \right\}$ *is a network at x.*

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