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INITIAL VALUE PROBLEMS OF SECOND ORDER NONLINEAR IMPULSIVE INTEGRO-DIFFERENTIAL EQUATIONS IN BANACH SPACES*

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In this paper, by using a new comparison result and the monotone iterative technique, we studied the existence of minimal and maximal solutions of initial value problem for nonlinear second order impulsive integro-differential equations which contains differential argument in Banach spaces and some existence theorems are obtained.

Key Words : Extremal Solutions; Impulsive Integro-Differential Equation; Kuratowski Noncompact Measure; Monotone Iterative Technique

1. INTRODUCTION

In [1], Dajun Guo established the existence theorems of extremal solutions of initial value problem (IVP) for second order differential equations in which f don't contain differential argument in Banach spaces. The author discussed[2 & 3] the IVP of second order nonlinear impulsive integro-differential equations in which f don't contain differential argument in finite domain and in infinite domain respectively. And in [4] and [5], the IVP of second order integro-differential equations in which f contain differential argument but don't contain impulsive argument in finite domain and in infinite domain were investigated respectively. Further, in the special case where f contain n order differential argument was discussed in paper[6]. In this paper, we will consider the IVP of second integro-differential impulsive equations :

$$\begin{cases} u'' = f(t, u, u', Tu), & \forall t \in J, t \neq t_i, \\ \Delta u|_{t=t_i} = L_i u'(t_i), & \\ \Delta u'|_{t=t_i} = I_i(u(t_i), u'(t_i)) & (i = 1, 2, ..., m), \\ u(0) = x_0, u'(0) = x_1, & \end{cases} \quad \text{... (1)}$$

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where E is a real Banach space, $J = [0, a]$ $(a > 0)$, $x_0, x_1 \in E, f \in C[J\times E\times E, E]$, $Tu(t) = \int_0^t k(t,s)u(s)ds, k \in C[D, R^+]$, $D = \{(t,s) \in J\times J : t \geq s\}$, $k_0 = \max \{k(t,s) : (t,s) \in D\}$, $0 < t_1 < t_2 < \ldots < t_m < a, L_i\, (i = 1, 2, ..., m)$, are constants, $I_i \in C[E\times E, E], \Delta u|_{t=t_i}$ denotes the jump of $u(t)$ at $t = t_i$, i.e., $\Delta u|_{t=t_i} = u(t_i^+) - u(t_i^-)$, where $u(t_i^+)$ and $u(t_i^-)$ denote the right and left limits of $u(t)$ at $t = t_i$ respectively. $\Delta u'|_{t=t_i}$ has a similar meaning for $u'(t)$. In this paper, the method used in paper [1-3] isn't suitable, for f contain differential argument u'. And because the equation is impulsive, the method used in paper [4-6] can't resolve this problem too. By using a new comparison result and the monotone iterative technique, the existence theorem of extremal solution of IVP^1 were obtained.

Let $PC[J, E] = \{u : u$ is a map from J into E such that $u(t)$ is continuous at $t \neq t_i$, and $u(t_i^+)$ exists, $i = 1, 2, ..., m\}$, $PC^1[J, E] = \{u \in PC[J, E] : u'(t)$ is continuous at $t \neq t_i$, and $u'(t_i^-), u'(t_i^+)$ exist, $i = 1, 2, ..., m\}$. Evidently, $PC[J, E]$ is a Banach spaces with norm

$$\| u \|_{PC} = \sup_{t\in J} \| u(t) \|.$$

For $u \in PC^1[J, E]$, by virtue of the mean value theorem, we have .

$$u(t_i) - u(t_i - h) \in h\,\overline{co}\left\{u'(t) : t_i - h < t < t_i\right\} \;(h > 0),$$

it is easy to see that the left derivative $u'_-(t_i)$ of $u(t_i)$ exists and

$$u'_-(t_i) = \lim_{h\to 0+} \frac{u(t_i) - u(t_i - h)}{h} = u'(t_i^-).$$

In (1) and the following, $u'(t_i)$ is understood as $u'_-(t_i)$. Therefore, when $u \in PC^1[J, E]$, we have $u' \in PC[J, E]$, and it is easy to see, $PC^1[J, E]$ is a Banach space with norm

$$\| u \|_{PC^1} = \max\left\{\| u \|_{PC}, \| u' \|_{PC}\right\}.$$

Let $J' = J \backslash \{t_1, t_2, \ldots, t_m\}, \tau = \max\{t_i - t_{i-1} : i = 1, 2, ..., m+1\}$ (where $t_0 = 0, (t_{m+1} = a)$ $J_0 = [0, t_1], J_1\,(t_1, t_2], \ldots, J_{m-1} = (t_{m-1}, t_m], J_m = (t_m, a]$. $C[I]$ denote all the continuous functions from

I to E, $C^1[I]$ denote all the continuously differentiable functions from I to E, $u \in PC^1[J, E] \cap C^2[J', E]$ is called a solution of IVP (1), if it satisfies (1).

2. Several Lemmas

Let E be a partially ordered space by a cone P of E, i.e., $x \leq y$ if and only if $y - x \in P$. The cone P is said to be normal if there exists a positive constant N such that $\theta \leq x \leq y$ implies $\| x \| \leq N \| y \|$, where θ denote the zero element of E, and the cone P is regular if $x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots \leq y$, implies $\| x_n - x \| \to 0$ $(n \to \infty)$ for some $x \in E$. It is well known that the regularity of P implies the normality of P. For details on cone theory see [7-9]. α denote Kuratowski noncompact measure, for $H \subset PC[J, E]$, let $H(t) = \{x(t) \mid x \in H\} \subset E$, $(\forall t \in J)$.

Definition — Let $u_0, v_0 \in PC^1[J, E] \cap C^2[J', E]$, $u_0(t) \leq v_0(t)$, $u_0'(t) \leq v_0'(t)$, $(\forall t \in J)$, we said u_0, v_0 are lower and upper solutions of IVP (1) respectively, suppose that the following equations hold

$$\begin{cases} u_0'' = f(t, u_0, u_0', Tu_0), \ \forall t \in J, t \neq t_i, \\ \Delta u_0 |_{t=t_i} = L_i u_0'(t_i), \\ \Delta u_0' |_{t=t_i} = I_i(u_0(t_i), u_0'(t_i)) \quad (i = 1, 2, \ldots, m), \\ u(0) = x_0, u'(0) \leq x_1, \end{cases} \qquad \ldots (2)$$

$$\begin{cases} v_0''(t) \geq f(t, v_0, v_0', Tv_0), \ \forall t \in J, t \neq t_i, \\ \Delta v_0 |_{t=t_i} = L_i v_0'(t_i), \\ \Delta v_0' |_{t=t_i} \geq I_i(v_0(t_i), v_0'(t_i)) \quad (i = 1, 2, \ldots, m), \\ v(0) = x_0, v'(0) \geq x_1, \end{cases} \qquad \ldots (3)$$

Lemma[7] — Let $H \subset PC^1[J, E]$ is a bounded set, the functions of H' are equicontinuous on J_i $(i = 1, 2, \ldots, m)$, then

$$\alpha_{PC^1} H = \max\left\{ \sup_{t \in J} \alpha(H(t)),\ \sup_{t \in J} \alpha(H'(t)) \right\},$$

where α_{PC^1} denotes the noncompact measure in $PC^1[J, E]$.

Lemma 2[7] — If $H \subset PC[J, E]$ is countable and bounded, then $\alpha(H(t)) \in L[J, R^+]$, and

$$\alpha\left(\left\{ \int_{t_0}^{t} x(s)\, ds : x \in H \right\}\right) \leq 2 \int_{t_0}^{t} \alpha(H(s))\, ds.$$

Lemma 3 — (Comparison theorem) Assume $p \in PC^1[J, E] \cap C^2[J', E]$ satisfies

$$\begin{cases} p''(t) \leq -M(t)\, p(t) - L(t)\,(Tp)(t) - N(t)\, p'(t),\ \forall\, t \in J, t \neq t_i, \\ \Delta p\,|_{t=t_i} = L_i\, p'(t_i), \\ \Delta p'\,|_{t=t_i} \leq L_i^*\, p(t_i),\ (i = 1, 2, \ldots, m), \\ p(0) = \theta,\ p'(0) \leq \theta \end{cases} \quad \ldots (4)$$

where $M(t)$, $N(t)$, $L(t)$ are bounded integrable nonnegative functions on J, $L_i \geq 0, L_i^* \leq 0$, $(i = 1, 2, \ldots, m)$ and one of the following four conditions hold :

$$(1)\ N_* = 0,\ \left(a + \sum_{i=1}^{m} L_i\right) \leq \frac{1}{a(M_* + L_* k_0 a) - \sum_{i=1}^{m} L_i^*}$$

$$(2)\ N_* > 0,\ \left(N_*^{-1}(e^{N_* a} - 1) + \sum_{i=1}^{m} L_i e^{N_* a}\right) \leq \frac{1}{a(M_* + L_* k_0 a) - \sum_{i=1}^{m} L_i^*} \quad \ldots (5)$$

$$(3)\ N_* > 0\ \left(a + \sum_{i=1}^{m} L_i\right) \leq \frac{1 - a N_*}{a(M_* + L_* k_0 a) - \sum_{i=1}^{m} L_i^*}$$

where

$M_* = \max\{M(t) : t \in J\}$, $N_* = \max\{N(t) : t \in J\}$, $L_* = \max\{L(t) : t \in J\}$, then

$$p(t) \leq 0,\ p'(t) \leq 0,\ \forall\ t \in J.$$

PROOF : For any $g \in P^*$, let $u(t) = (p(t))$, then $u \in PC^1[J, R[\cap C^2[J', R]$ and

$$u''(t)\, g(p''(t)), u'(t) = g(p'(t)), (Tu)(t) = g((Tp)(t)), \ \forall t \in J.$$

By (4) we have

$$\begin{cases} u'' \le -m(t)\, u - N(t)\, u' - L(t)\,(Tu), \ \forall\, t \in J, t \ne t_i, \\ \Delta u\,|_{t=t_i} = L_i\, u'(t_i), \\ \Delta u'\,|_{t=t_i} \le L_i^*(u(t_i), \qquad (i = 1, 2, ..., m), \\ u(0)\ 0, u'(0) \le 0. \end{cases} \quad ...\ (6)$$

Let $u_1(t) = u'(t)$, then $u_1 \in PC[J, R] \cap C^1[J', R]$ and

$$u(t) = u(0) + \int_0^t u_1(s)\, ds + \sum_{0<t_i<t} \Delta u\,|_{t=t_i} = \int_0^t u_1(s)\, ds + \sum_{0<t_i<t} L_i\, u_i(t_i).$$

Let $u(t)$, $u'(t)$ apply to (6), we have

$$\begin{cases} u_1'(t) \le -\int_0^t \left[M(t) + L(t) \int_s^t k(t, r)\, dr \right] u_1(s)\, ds - N(t)\, u_1(t) - M(t) \sum_{0<t_i<t} L_i\, u_1(t_i) \\ \qquad - L(t) \int_0^t \left[k(t, s) \sum_{0<t_i<s} L_i\, u_1(t_i) \right] ds, \ \forall\, t \in J, t \ne t_i, \\ \Delta u_1\,|_{t=t_i} \le L_i^* \left[\int_0^{t_i} u_1(s)\, ds + \sum_{k=1}^{i} L_k\, u_1(t_k) \right], (i = 1, 2, ..., m), \\ u_1(0) \le 0, \end{cases} \quad ...\ (7)$$

Now, we prove $u_1(t) \le 0$, $\forall t \in J$.

In cse of condition (3) holding, suppose $u_1(t) \le 0$ is not true, then there is a $t^* \in (0, a]$ such that $u_1(t^*) > 0$. Let $t^* \in J_j$, $\inf_{0 \le t \le t^*} u_1(t) = -\lambda$, then $\lambda \ge 0$.

If $\lambda = 0$, by (7) we have $u_1'(t) \le 0$, $\Delta u_1|_{t=t_i} \le 0$. Consequently, $u_1(t)$ is nonincreasing in $[0, t^*]$, so $u_1(t^*) \le u_1(0) \le 0$, which contradicts $u_1(t^*) > 0$.

If $\lambda > 0$, there exists a $J_l (l \le j)$, such that $u_1(t_*) = -\lambda$ for some $t_* \in J_l$ or $u_1(t_l^+) = -\lambda$. Let $u_1(t_*) = -\lambda$, (when $u_1(t_l^+) = -\lambda$, the proof is similar). From (7) we have

$$u_1'(t) \le a(M_* + L_* k_0 a)\lambda + N_* \lambda + (M_* + L_* k_0 a) \sum_{i=1}^{m} L_i \lambda = M_0 \lambda, \ \forall t \in [0, t^*],$$

$$\Delta u_1|_{t=t_i} \le -L_i^* \left(a + \sum_{k=1}^{m} L_k \right) \lambda \qquad \dots (8)$$

where $M_0 = (M_* + L_* k_0 a)\left(a + \sum_{i=1}^{m} L_i \right) + N_*$. By mean value theorem we have

$$\begin{cases} u_1(t^*) - u_1(t_j^+) = u_1'(\xi_j)(t^* - t_j), t_j < \xi_j < t^* \\ u_1(t_j) - u_1(t_{j-1}^+) = u_1'(\xi_{j-1})(t_j - t_{j-1}), \ t_{j-1} < \xi_{j-1} < t_j, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ u_1(t_{l+2}) - u_1(t_{l+1}^+) = u_1'(\xi_{l+1}(t_{l+2} - t_{l+1}), \ t_{l+1} < \xi_{l+1} < t_{l+2}, \\ u_1(t_{l+1}) - u_1(t_*) = u_1'(\xi_l)(t_{l+1} - t_*), \ t_* < \xi_l < t_{l+1}, \end{cases} \qquad \dots (9)$$

and then, by (8) we see

$$u_1(t_i^+) = u_1(t_i) + \Delta u_1|_{t=t_i} \le u_1(t_i) - L_i^* \left(a + \sum_{k=1}^{m} L_k \right) \lambda, \ \forall t_i \le t^*, \qquad \dots (10)$$

hence, by (8) (9) (10) we have

$$\begin{cases} u_1(t^*) - u_1(t_j) \le -L_j^* \left(a + \sum_{k=1}^{m} L_k \right) \lambda + \lambda M_0 (t^* - t_j), \\ u_1(t_j) - u_1(t_{j-1}) \le -L_{j-1}^* \left(a + \sum_{k=1}^{m} L_k \right) \lambda + \lambda M_0 (t_j - t_{j-1}), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ u_1(t_{l+2}) - u_1(t_{l+1}) \le -L_{l+1}^* \left(a + \sum_{k=1}^{m} L_k \right) \lambda + \lambda M_0 (t_{l+2} - t_{l+1}), \\ u_1(t_{l+1}) + \lambda \le -L_l^* \left(a + \sum_{k=1}^{m} L_k \right) \lambda + \lambda M_0 (t_{l+1} - t_*). \end{cases}$$

Adding together we obtain

$$\lambda < u_1(t^*) + \lambda \le -\left(a + \sum_{k=1}^{m} L_k \right) \sum_{i=l+1}^{j} L_i^* \lambda + \lambda M_0 (t^* - t_*)$$

$$< -\left(a + \sum_{k=1}^{m} L_k \right) \sum_{i=1}^{m} L_i^* \lambda + \lambda M_0 a,$$

i.e.,
$$1 \le -\sum_{i=1}^{m} L_i^* \left(a + \sum_{k=1}^{m} L_k \right) + M_0 a,$$

which contradicts conditin (3), so $u_1(t) \le 0, \ \forall t \in J$.

In case of conditions (1) (2) holding, let

$$w(t) = u_1(t)\, e^{\int_0^t N(s)\, ds},$$

apply it to (7) we have

$$\begin{cases} w'(t) \le -\int_0^t \left[M(t) + L(t) \int_s^t k(t,r)\,dr \right] e^{\int_s^t N(\xi)\,d\xi} w(s)\,ds - M(t) \sum_{0<t_i<t} L_i e^{\int_{t_i}^t n(s)\,ds} w(t_i) \\ -L(t) \int_0^t \left[k(t,s) \sum_{0<t_i<s} L_i e^{\int_{t_i}^t n(\xi)\,d\xi} w(t_i) \right] ds, \ \forall t \in J, t \ne t_i, \\ \Delta w|_{t=t_i} \le L_i^* \left[\int_0^{t_i} e^{\int_s^{t_i} N(\xi)\,d\xi} w(s)\,ds + \sum_{k=1}^{i} L_k e^{\int_{t_k}^{t_i} N(s)\,ds} u_1 t_k) \right], \ (i = 1, 2, \ldots, m), \\ w(0) \le 0, \end{cases}$$

by the similar proof process to above, we can obtain $w(t) \le 0$, $\forall t \in J$. And so $u_1(t) \le 0$, $\forall t \in J$. Consequently, $h(t) = \int_0^t u_1(s)\,ds + \sum_{<t_i<t} L_i u_1(t_i) \le 0$, $\forall t \in J$. By the randomness of g, we know $p(t) \le p'(t) \le 0$, $\forall t \in J$.

Lemma 4 — Let $\sigma \in PC[J, E]$, $\eta \in PC^1[J, E]$, $M(t), N(t), L(t)$ are bounded integrable non-negative functions on J, $L_i \ge 0, L_i^* \le 0$, $(i = 1, 2, \ldots, m)$ are constants, then the IVP for linear impulsive integro-differential equation

$$\begin{cases} u''(t) = -M(t)u - N(t)u'(t) - L(t)(Tu)(t) + \sigma(t), & \forall t \in J, t \ne t_i, \\ \Delta u|_{t=t_i} = L_i u'(t_i), \\ \Delta u'|_{t=t_i} = I_i(\eta(t_i), \eta'(t_i)) + L_i^*(u(t_i) - \eta(t_i)), & (i = 1, 2, \ldots, m), \\ u(0) = x_0, u'(0) = x_1, \end{cases} \quad \ldots (11)$$

has a solution in $PC^1[J, E] \cap C^2[J', E]$.

PROOF : Let

$$f_*(t, u, u', Tu) = \sigma(t) - M(t)u(t) - N(t)u'(t) - L(t)((Tu)(t)), t \in J,$$

Firstly, we consider the following linear integro-differential equation

$$\begin{cases} u'' = f_*(t, u, u', Tu), & t \in J_0, \\ u(0) = x_0,\ u'(0) = x_1, \end{cases} \qquad \dots (12)$$

it is easy to verify that $u \in C^2[J_0, E]$ is a solution of IVP (12), if and only if $u \in C^1[J_0, E]$ is a solution of the following integral equation

$$u(t) = x_0 + t x_1 + \int_0^t (t-s)[\sigma(s) - M(s)u(s) - N(s)u'(s) - L(s)(Tu)(s)]\,ds.$$

Let $$(A_0 u)(t) = x_0 + tx_1 + \int_0^t (t-s)[\sigma(s) - M(s)u(s) - N(s)u'(s) - L(s)(Tu)(s)]\,ds, \qquad \dots (13)$$

then $$(A_0 u)'(t) = x_1 + \int_0^t [\sigma(s) - M(s)u(s) - N(s)u'(s) - L(s)(Tu)(s)]\,ds. \qquad \dots (14)$$

For any $u, v \in C^1[J_0, E]$ by (13) (14) we have

$$\|(A_0 u)(t) - (A_0 v)(t)\| \le \int_0^t (t-s)[M_* \|u(s) - v(s)\| + N_* \|u'(s) - v'(s)\| + L_* \tau k_0 \|u(s) - v(s)\|]\,ds$$

$$\le \int_0^t \tau[(M_* + L_* k_0 \tau)\|u - v\| + N_* \|u'(s) - v'(s)\|]\,ds$$

$$\le (\tau + 1)\, t\,(M_* + L_* k_0 \tau + N_*)\|u - v\|_{C^1[J_0]}, \quad t \in J_0,$$

$$\|(A_0 u)'(t) - (A_0 v)')(t)\| \le \int_0^t [M_* \|u(s) - v(s)\| + N_* \|u'(s) - v'(s)\| + L_* k_0 \tau \|u(s) - v(s)\|]\,ds$$

$$\le (\tau + 1)(M_* + L_* k_0 \tau + N_*)\|u - v\|_{C^1[J_0]}\, t, \quad t \in J_0,$$

$$\|(A_0^2 u)(t) - (A_0^2 v)(t)\| \le \int_0^t \tau[(M_* + L_* k_0 \tau)\|(A_0 u)(s) - (A_0 v)(s)\| + N_* \|(A_0 u)'(s) - (A_0 v)'(s)\|]\,ds$$

$$\leq (\tau + 1)^2 (M_* + L_* k_0 \tau + N_*)^2 \| u - v \|_{C^1[J_0]} \frac{t^2}{2},$$

$$\| (A_0^2 u)'(t) - (A_0^2 v)'(t) \| \leq \int_0^t [(M_* + L_* k_0 \tau + N_*) \| (A_0 u)(s) - (A_0 v)(s) \|$$

$$+ N_* \| (A_0 u)'(s) - (A_0 v)'(s) \|] \, ds$$

$$\leq (\tau + 1)^2 (M_* + L_* k_0 \tau + N_*)^2 \| u - v \|_{C^1[J_0]} \frac{t^2}{2}.$$

By mathematical induction we can obtain

$$\| (A_0^n u)(t) - (A_0^n v)(t) \| \leq (\tau + 1)^n (M_* + L_* k_0 \tau + N_*)^n \| u - v \|_{C^1[J_0]} \frac{t^n}{n!},$$

$$\| (A_0^n u)'(t) - (A_0^n v)'(t) \| \leq (\tau + 1)^n (M_* + L_* k_0 \tau + N_*)^n \| u - v \|_{C^1[J_0]} \frac{t^n}{n!},$$

i.e.,

$$\| A_0^n u - (A_0^n) \|_{C[J_0]} \leq (\tau + 1)^n (M_* + L_* k_0 \tau + N_*)^n \| u - v \|_{C^1[J_0]} \frac{t^n}{n!}$$

$$\| (A_0^n u)' - (A_0^n v)'(t) \|_{C^1[J_0]} \leq (\tau + 1)^n (M_* + L_* k_0 \tau + N_*)^n \| u - v \|_{C^1[J_0]} \frac{t^n}{n!},$$

hence

$$\| A_0^n u - A_0^n \|_{C^1[J_0]} \leq (\tau + 1)^n (M_* + L_* k_0 \tau + N_*)^n \| u - v \|_{C^1[J_0]} \frac{\tau^n}{n!}, \quad \text{... (15)}$$

and so, we can choose $n_0 \in N$, such that

$$(\tau + 1)^{n_0} (M_* + L_* k_0 \tau + N_*)^{n_0} \frac{\tau^{n_0}}{n_0!} < 1, \quad \text{... (16)}$$

hence, by (15) (16) and the Banach contraction mapping principle, we see that $A_0^{n_0}$ has a unique fixed point ρ_0 in $C^1[J_0]$, i.e., (12) has a unique solution in $C^2[J_0, E]$, then ρ_0 satisfies

$$\begin{cases} \rho_0'' = f_*(t, \rho_0, \rho', T\rho_0), t \in J_0, \\ \rho_0(0) = x_0, \rho_0'(0) = x_1, \end{cases} \quad \text{... (17)}$$

Secondly, we consider the following linear integro-differential equation

$$\begin{cases} u'' = f_*\left(t, u, u', \bar{\rho}_0 + \int_{t_1}^{t} k(t,s)\, u(s)\, ds \right), \quad t \in J_1, \\ u(t_1^+) = L_1\, \rho_0'(t_1) + \rho_0(t_1), \\ u'(t_1^+) = I_1(\eta(t_1), \eta'(t_1)) - L_1^*\, \eta(t_1) + L_1^*\, \rho_0(t_1) + \rho_0'(t_1), \end{cases} \qquad \ldots (18)$$

where $\qquad \rho_{\bar{0}}(t) = \int_{t_0}^{t_1} k(t,s)\, \rho_0(s)\, d.$

It is easy to verify that $u \in PC^1[J_1, E] \cap C^2[(t_1, t_2), E]$ is a solution of (17), if and only if $u \in PC^1[J_1, E]$ is a solution of the following integral equation

$$u(t) = L_1\, \rho_0'(t_1) + \rho_0(t_1) + (t - t_1)\, [I_1(\eta(t_1), \eta'(t_1)) - L_1^*\, \eta(t_1) + L_1^*\, \rho_0(t_1) + \rho_0'(t_1)]$$

$$+ \int_{t_1}^{t} (t-s) \left[\sigma(s) - M(s)\, u(s) - N(s)\, u'(s) - L(s) \left(\bar{\rho}_0(s) + \int_{t_1}^{s} k(s,\xi)\, u(\xi)\, d\xi \right) \right] ds,$$

Let

$$(A_1 u)(t) = L_1\, \rho_0(t_1) + \rho_0(t_1) + (t - t_1)\, [I_1(\eta(t_1), \eta'(t_1))$$

$$- L_1^*\, \eta(t_1) + L_1^*\, \rho_0(t_1) + \rho_0'(t_1)]$$

$$+ \int_{t_1}^{t} (t-s) \left[\sigma(s) - M(s)\, u(s) - N(s)\, u'(s) - L(s) \left(\bar{\rho}_0(s) + \int_{t_1}^{s} k(s,\xi)\, u(\xi)\, d\xi \right) \right] ds,$$

hence, $\qquad (A_1 u)'(t) = I_1(\eta(t_1), \eta'(t_1)) - L_1^*\, \eta(t_1) + L_1^*\, \rho_0(t_1) + \rho_0'(t_1)$

$$+ \int_{t_1}^{t} \left[\sigma(s) - M(s)\, u(s) - N(s)\, u'(s) - L(s) \left(\bar{\rho}_0(s) + \int_{t_1}^{s} k(s,\xi)\, u(\xi)\, d\xi \right) \right] ds,$$

then A_1 is a operator from $PC^1[J_1, E]$ to $PC^1[J_1, E]$. For any $u, v \in PC^1[J_1, E]$, by the same was that we used in obtaining (15), we have

$$\| A_1^n u - A_1^n v \|_{PC^1[J_1]} \le (\tau + 1)^n (M_* + N_* + L_* k_0 \tau)^n \| u - v \|_{PC^1[J_1]} \frac{\tau^n}{n!}, \quad \dots (19)$$

by (16) (19) and the Banach contraction mapping principle, we see that A_1^no has a unique fixed point ρ_1 in $PC^1[J_1]$, therefore (18) has a unique solution ρ_1 in $PC^1[J_1, E] \cap C^2[(t_1, t_2), E]$. Then ρ_1 satisfies

$$\begin{cases} \rho_1'' = f_*\left(t, \rho_1, \rho_1', \overline{\rho}_0 + \int_{t_1}^{t} k(t,s)\, \rho_1(s)\, ds\right), & t \in J_1, \\ \rho_1(t_1^+) = L_1 \rho_0'(t_1) + \rho_0(t_1), \\ \rho_1'(t_1^+) = I_1(\eta(t_1), \eta'(t_1)) - L_1^* \eta(t_1) + L_1^* \rho_0(t_1) + \rho_0'(t_1). \end{cases} \quad \dots (20)$$

Next, by repeating above process, we can further to prove that, for $\forall i$ $(i = 1, 2, \dots, m)$, the following integro-differential equation

$$\begin{cases} u'' = f_*\left(t, u, u', \overline{\rho}_0 + + \overline{\rho}_1 + \dots + \overline{\rho}_{i-1} + \int_{t_1}^{t} k(t,s)\, u(s)\, ds\right), & t \in J_1, \\ u(t_1^+) = L_1 \rho_{i-1}'(t_i) + \rho_{i-1}(t_i), \\ u'(t_1^+) = I_i(\eta(t_i), \eta'(t_i)) - L_i^* \eta(t_i) + L_i^* \rho_{i-1}'(t_i) + \rho_{i-1}'(t_i). \end{cases}$$

has a unique solution $\rho_i \in PC^1[J_i, E] \cap C^2[(t_i, t_{i+1}), E]$ and ρ_i satisfies

$$\begin{cases} \rho_i'' = f_*\left(t, \rho_i, \overline{\rho}_0 + \overline{\rho}_1 + \dots + \overline{\rho}_{i-1} + \int_{t_1}^{t} k(t,s)\, \rho_i(s)\, ds)\right), & t \in J_1, \\ \rho_i(t_i^+) = L_i \rho_{i-1}'(t_i) + \rho_{i-1}(t_1), \\ \rho_i'(t_i^+) = I_i(\eta(t_i), \eta'(t_i)) - L_i^* \eta(t_1) + L_1^* \rho_{i-1}(t_i) + \rho_{i-1}'(t_1). \end{cases} \quad \dots (21)$$

where $\overline{\rho}_i(t) = \int_{t_i}^{t_{i+1}} k(t,s)\, \rho_i(s)\, ds, \; i = 0, 1, 2, \dots, m-1.$

Let $$\rho(t) = \begin{cases} \rho_0(t), & t \in J_0, \\ \rho_1(t), & t \in J_1, \\ \cdots\cdots & \\ \rho_m(t), & t \in J_m. \end{cases} \quad \ldots (22)$$

From (17) (20) (21) (22), it is easy to prove that $\rho \in PC^1[J, E] \cap C^2[J' E]$ is a solution of IVP (11).

Lemma 5 — Let $\sigma \in PC[J, E]$, $\eta \in PC^1[J, E]$, then $u \in PC^1[J, E] \cap C^2[J', E]$ is a solution of (11), if and only if $u \in PC^1[J, E]$ is a solution of the following equation

$$u(t) = x_0 + tx_1 + \int_0^t (t-s)[\sigma(s) - M(s)u(s) - N(s)(u'(s)) - L(s)(Tu)(s)]\,ds$$

$$+ \sum_{0<t_i<t} \left\{ L_i u'(t_i) + (t-t_i)[I_i(\eta(t_i), \eta'(t_i)) + L_i^*(u(t_i) - \eta(t_i))] \right\}, \; \forall t \in J. \quad \ldots (23)$$

Lemma 6[7] — Let $m \in PC[J, R^+]$, $k \in C[D, R^+]$, $b\eta_i$ $(i = 1, 2, \ldots)$ are nonnegative constants, and

$$m(t) \le \int_0^t k(t,s)\,m(s)\,ds + \sum_{0<t_i<t} \beta_i m(t_i), \; \forall t \in J.$$

Then $m(t) \le 0$, $\forall t \in J$.

3. Main Theorem

We shall state and prove our main theorem in this section. For convenience, let us list some conditions for use.

(H_1) There exist $u_0, v_0 \in PC^1[J, E] \cap C^2[J', E]$ those are the lower and upper solution of IVP (1), i.e., (2) (3) holding :

(H_2) There exist bounded, integrable and nonnegative functions $M(t$, $N(t)$, $L(t)$ and constants $L_i^* \le 0$, $(i = 1, 2, \ldots, m)$ such that $\forall t \in J$ and $\forall x, y, \in [u_0, v_0]$ satisfying

$$f(t, x, x', Tx) - f(t, y, y', Ty) \ge -M(t)(x-y) - N(t)(x'-y') - L(t)(Tx - Ty),$$

$$I_i(x, x') - I_i(y, y') \ge L_i^*(x-y),$$

where $$u_0 \le y \le x \le v_0,\; u_0'(t) \le y' \le x' \le v_0'(t),\; Tu_0 \le Ty \le Tx \le Tv_0,$$

(H_3) For $\forall t \in J$, and any bounded equicontinuous on each J_i (i =1, 2, ..., m), monotone sequence $B = \{u_n\} \subset PC^1[J, E] \cap C^2[J', E]$, there exist nonnegative constants c_1, c_2, c_3 and a_i (i = 1, 2, .., m) such that

$$\alpha(f(t, B(t), B'(t), (TB)\, t))) \leq c_1\, \alpha(B(t)) + c_2\, \alpha(B'(t)) + c_3\, \alpha((TB)(t)), \ \forall t \in J,$$

and $$\alpha(I_i(B(t), B'(t))) \leq a_i \max\{\alpha(B(t)), \alpha(B'(t))\}, \ \forall t \in J.$$

In the following, we let

$$[u_0, v_0] = \left\{ u \in PC[J, E] : u_0(t) \leq u(t) \leq v_0(t), u_0'(t) \leq u'(t) \leq v_0'(t) \right\} \ \forall t \in J.$$

***Theorem* 1** — *Let P is normal, suppose that conditions* $(H_1)(H_2)(H_3)$ *are satisfied,* $L_i \geq 0$, *(i =1, 2, .., m), and one inequality of (5) hold. Then, there exist monotone sequence* $\{u_n\}$, $\{v_n\}$ $\in PC^1[J, E] \cap C^2[J', E]$, *which converge to the minimal and maximal solutions* u^*, $v^* \in PC^1[J, E] \cap C^2[J', E]$ *of IVP (1) respectively in* $[u_0, v_0]$ *and* $\{u_n'\}$, $\{v_n'\}$ *converge to* $(u^*)', (v^*)'$ *in* $[u_0, v_0]$. *For any solution* $u \in PC^1[J, E] \cap C^2[J', E]$ *of IVP (1) in* $[u_0, v_0]$, *satisfies*

$$u_0(t) \leq u_1(t) \leq \ldots \leq u_n(t) \leq \ldots \leq u^*(t) \leq u(t)$$

$$\leq v^*(t) \leq \ldots. \leq v_n(t) \leq \ldots \leq v_1(t) \leq v_0(t),$$

$$u_0'(t) \leq u_1'(t) \leq \ldots \leq u_n'(t) \leq \ldots \leq (u^*)'(t) \leq u'(t)$$

$$\leq (v^*)'(t) \leq \ldots \leq v_n'(t) \leq \ldots \leq v_1'(t \leq v_0'(t),$$

PROOF : For any $\eta \in [u_0, v_0]$, let

$$\sigma(t) = f(t, \eta(t), \eta'(t), (T\eta)(t)) + M(t)\,\eta(t) + N(t)\,\eta'(t) + L(t)\,(T\dot{\eta})(t), t \in J, \quad \ldots (25)$$

then $\sigma(t) \in PC[J, E]$. ByLemma 3 and Lemma 4, IVP (1) has a unique solution $u \in P\dot{C}[j, E] \cap C^2[J', E]$. Let $u = A\eta$, then, A is a operator from $[u_0, v_0]$ to $PC^1[J, E] \cap C^2[J', E]$. Now, we show that (a) $u_0 \leq Au_0, u_0' \leq (Au_0)', A\, v_0 \leq v_0, (A\, v_0)' \leq v_0'$ (b) If $\eta_1, \eta_2 \in [u_0, v_0]$ and $\eta_1 \leq \eta_2$, then $A\,\eta_1 \leq A\,\eta_2, (A\,\eta_1)' \leq (A\,\eta_2)'$. To prove (a), let $v_1 = A\, v_0, x = v_1 - v_0$, by (11) and $(H_1)(H_2)$ we have

$$x'' = v_1'' - v_0'' \le f(t, v_0, v_0', T v_0) - M(t)(v_1 - v_0) - N(t)(v_1' - v_0') - L(t) T(v_1 - v_0)$$

$$-f(t, v_0, v_0', T v_0) \le -M(t)x - N(t)x' - L(t)Tx$$

$$\Delta x|_{t=t_i} = \Delta v_1|_{t=t_i} - \Delta v_0|_{t=t_i} = L_i v_1'(t_i) - L_i v_0'(t_i) = L_i x'(t_i),$$

$$\Delta x'|_{t=t_i} = \Delta v_1'|_{t=t_1} - \Delta v_0'|_{t=t_i} = I_i(v_0(t_i)) + L_i^*(v_1(t_i) - v_0(t_i)) - I_i(v_0(t_i), v_0'(t_i))$$

$$\le L_i^* x(t_i), (i = 1, 2, \ldots, m),$$

$$x(0) = v_1(0) - v_0(0) = x_0 - x_0 = 0$$

$$x'(0) = v_1'(0) - v_0(0) \le x_1 - x_1 = 0$$

and by Lemma 2 we can obtain $x(t) \le 0, x'(t) \le 0,\ \forall t \in J$, i.e., $A v_0 = v_1 \le v_0, (A v_0)' = v_1' \le v_0'$.

To prove (b), let $x = u_1 - u_2$, where $u_1 = A\eta_1, u_2 = A\eta_2$, by (11) and (H_2) we can obtain

$$x'' = u_1'' - u_2'' = f(t, \eta_1, \eta_1', T\eta_1) - f(t, \eta_2, \eta_2', T\eta_2) - M(t)(u_1 - \eta_1) - N(t)(u_1' - \eta_1)$$

$$-L(t)(Tu_1 - T\eta_1) - M(t)(u_2 - \eta_2) - N(t)(u_2' - \eta_2') + L(t)(Tu_2 - T\eta_2)$$

$$= -[f(t, \eta_2, \eta_2', T\eta_2) - f(t, \eta_1, \eta_1', T\eta_1)] - M(t)(u_1 - u_2) - N(t)(u_1' - u_2')$$

$$-L(t)(Tu_1 - Tu_2) - M(t)(\eta_2 - \eta_1) - N(t)(\eta_2' - \eta_1)$$

$$-N(t)(\eta_2' - \eta_1') - l(t)(T\eta_2 - t\eta_1)$$

$$\le -M(t)x - N(t)x' - L(t)Tx,$$

$$\Delta x|_{t=t_i} = \Delta u_1|_{t=t_i} = -\Delta u_2|_{t=t_i} = L_i u_1'(t_i) - L_i u_2'(t_i) = L_i x'(t_i),$$

$$\Delta x'|_{t=t_i} = \Delta u_1'|_{t=t_i} - \Delta u_2'|_{t=t_i} = I_i(\eta_1(t_i), \eta_1'(t_i))$$

$$+L_i^*(u_1(t_i) - \eta_1(t_i)) - I_1(\eta_2(t_i), \eta_2'(t_i))$$

$$-L_1^*(u_2(t_i), \eta_2(t_i)) = L_i^* x(t_i),$$

$$x(0) = u_1(0) - u_2(0) = 0,$$

$$x'(0) = u_1'(0) - u_2'(0) = 0,$$

and so, by Lemma 2 we have $x(t) \le 0, x'(t) \le 0,\ \forall t \in J$, i.e.,

$$u_1 = A\,\eta_1 \le u_2 = A\,\eta_2,\ u_1' = (A\,\eta_1)' \le u_2' = (A\,\eta_2)'.$$

Let $u_n = Au_{n-1},\ v_n = A\,v_{n-1}$ $(n = 1, 2, ...)$, by conclusion (a) (b) we have

$$u_0 \le u_1 \le \dots \le u_n \le .... \le v_n \le .... \le v_1 \le v_0, \qquad ...\ (26)$$

and

$$u_0' \le u_1' \le \dots \le u_n' \le .... \le v_n' \le .... \le v_1' \le v_0', \qquad ...\ (27)$$

Let $U = \{u_n\}$, $U' = \{u_n'\}, V = \{v_n\}$, $V' = \{v_n'\}$, by the normality of P and (26) (27) we know U, U', V, V' are bounded set in $PC\,[J, E]$. For $\forall \eta \in [u_0, v_0]$. By $(H_1)\,(H_2)$ we have

$$u_0''(t) + M(t)\,u_0(t) + N(t)\,u_0'(t) + L(t)\,(Tu_0)(t)$$

$$\le f(t, u_0, u_0', Tu_0) + M(t)\,u_0(t) + N(t)\,u_0'(t) + L(t)\,(Tu_0)(t)$$

$$\le f(t, \eta, \eta', T\eta) + M(t)\,\eta(t) + N(t)\,\eta'(t) + L(t)\,(T\eta)(t)$$

$$\le f(t, v_0, v_0', T\,v_0) + M(t)\,v_0(t) + n(t)\,v_0'(t) + L(t)\,(T\,v_0)(t)$$

$$\le v_0''(t) + M(t)\,v_0(t) + N(t)\,v_0'(t) + L(t)\,(T\,v_0)(t),$$

therefore $\{f(t, \eta, \eta', T\eta) + M(t)\,\eta + N(t)\,\eta' + L(t)\,(T\eta) : \eta \in [u_0, v_0]\}$ is a bounded set in $PC\,[J, E]$, and so, there exist a constant $h > 0$, such that

$$\| f(t, u_{n-1}(t), u_{n-1}'(t), (Tu_{n-1}(t)) - M(t)\,(u_n(t) - u_{n-1}(t)) - N(t)\,(u_n'(t) - u_{n-1}'(t))$$
$$- L(t)\,(T(u_n - u_{n-1})(t)) \| \le h, \ (t \in J, n = 1, 2, \ldots). \qquad ...\ (28)$$

and $\{\sigma_n : n = 1, 2, \ldots\}$ is a bounded set in $PC\,[J, E]$, where

$$\sigma_n = f(t, u_{n-1}, u_{n-1}', tu_{n-1}) + M(t)\,u_{n-1} + N(t)\,u_{n-1}' + L(t)\,Tu_{n-1}.$$

By the definition of u_n and Lemma 5 we have

$$u_n(t) = x_0 + tx_1 + \int_0^t (t-s)\,[-M(s)\,u_n(s) - N(s)\,u_n'(s) - L(s)\,(Tu_n)(s) + \sigma_{n-1}(s)]\,ds$$

$$+ \sum_{0 < t_i t} \left\{ L_i\,u_n'(t_i) + (t - t_i)\,[I_i(u_{n-1}(t_i)\,u_{n-1}'(t_i)) + L_i^*(u(t_i) - u_{n-1}(t_i))] \right\}, \qquad ...\ (29)$$

$\forall t \in J, (n = 1, 2, ...),$

and

$$u_n'(t) = x_1 + \int_0^t [-M(s)\,u_n(s) - N(s)\,u_n'(s) - L(s)\,(Tu_n)(s) + \sigma_{n-1}(s)]\,ds$$

$$+ \sum_{0<t_i<t} [I_i(u_{n-1}(t_i)\,u_{n-1}'(t_i)) + L_i^*(u(t_i) - u_{n-1}(t_i))], \qquad \dots (30)$$

$\forall t \in J$, $(n = 1, 2, \dots)$,

by (29) (30) and (28) we know U, U' are equicontinuous on each J_i ($i = 0, 1, \dots$). By Lemma 1 we have

$$\alpha_{PC^1}(U) = \max\left\{ \sup_{t\in J} \alpha(U(t)),\ \sup_{t\in J} \alpha(U'(t)) \right\},\quad t \in J.$$

For $\forall t \in J$, by (29), (H_3) and Lemma 2 we have

$$\alpha(U(t)) \le \alpha\left(\left\{ \int_0^t (t-s)\,[\sigma_{n-1}(s) - M(s)\,u_n(s) - N(s)\,u_n'(s) - L(t)\,(Tu_n)(s)]\,ds \mid n = 1, 2, \dots \right\} \right)$$

$$+ \sum_{0<t_i<t} \alpha\ (\{L_i\,u_n'(t_i) + (t-t_i)\,[I_i(u_n(t_i), u_n'(t_i))$$

$$+ L_i^*(u_n(t_i) - u_{n-1}(t_i))] \mid n = 1, 2, \dots\})$$

$$\le 2\,\alpha \int_0^t \alpha(f(s, U(s), U'(s), (TU)(s))) + 4aM_* \int_0^t \alpha(U(s))\,ds + 4a\,N_* \int_0^t \alpha(U'(s))\,ds$$

$$+ 4aL_* \int_0^t \alpha((TU)(s))\,ds + \sum_{0<t_i<t} [L_i\,\alpha(U'(t_i)) + a\max\left\{ \alpha(U(t_i)), \alpha(U'(t_i)) \right\}$$

$$+ 2a\,L_i^*\,\alpha(U(t_i))]$$

$$\le 2a \int_0^t (c_1\,\alpha(U(s)) + c_2\,\alpha(U'(s)) + c_3\,\alpha((TU)(s)))\,ds$$

$$+ 4aM_* \int_0^t \alpha(U(s))\,ds + 4a\,N_* \int_0^t \alpha(U'(s))\,ds + 4aL_* \int_0^t \alpha((TU)(s))\,ds$$

$$+ \sum_{0<t_i<t} (L_i + 2aL_i^* + aa_i)\ \max\ \{\alpha(U(t_i)), \alpha\,U'(t_i))\}$$

$$\le (2ac_1 + 4aM_*) \int_0^t \alpha(U(s))\,ds + (2ac_2 + 4aN_*) \int_0^t \alpha(U'(s))\,ds + 2k_0 a\,(2ac_3 + 4aL_*)$$

$$\int_0^t \alpha(U(s))\,ds + \sum_{0<t_i<t} (L_i + aa_i + 2aL_i^*) \ \max\ \{\alpha(U(t_i)), \alpha U'(t_i))\}$$

$$\le (2ac_1 + 4aM_* + 4k_0 a^2 c_3 + 8k_0 a^2 L_*) \int_0^t \alpha(U(s))\,ds + (2ac_2 + 4aN_*) \int_0^t \alpha(U'(s))\,ds$$

$$+ \sum_{0<t_i<t} (L_i + aa_i + 2aL_i^*) \max\left\{ \alpha(U(t_i)), \alpha(U'(t_i))\right\}$$

$$\le (2ac_1 + 4aM_* + 4k_0 a^2\ c_3 + 8k_0 a^2 L_* + 2ac_2 + 4aN_*) \int_0^t \max\left\{ \alpha(U(s)), \alpha(U'(s))\right\} ds$$

$$+ \sum_{0<t_i<t} (L_i + aa_i + 2aL_i^*) \max\left\{ \alpha(U(t_i)), \alpha(U'(t_i))\right\}. \quad \ldots (31)$$

Similarly, we have

$$\alpha(V'(t)) \le (2ac_1 + 4aM_* + 4k_0 a^2 c_3 + 8k_0 a^2 L_* + 2ac_2 + 4aN_*) \int_0^t$$

$$\max\ \{\alpha(U(s)), \alpha(U'(s))\}\ ds$$

$$+ \sum_{0<t_i<t} (L_i + aa_i + 2aL_i^*) \max\left\{ \alpha(U(t_i)), \alpha(U'(t_i))\right\}. \quad \ldots (32)$$

Let $m(t) = \max\ \{\alpha(U(t_i)), \alpha(U'(t_i))\}$, by the boundedness on $PC\ [J, E]$ and the equicontinuous on $J_i\ (i = 1, 2, \ldots, m)$ of U, U' we can know $m(t) \in PC\,[J, E], m(t) \ge 0$. By (31) (32) we can obtain

$$m(t) \le K \int_0^t m(s)\,ds + \sum_{0<t_i<t} (L_i + aa_i + 2aL_i^*)\, m(t_i),$$

where

$$K = 2ac_1 + 4aM_* + 4k_0 a^2 + 8k_{0a}^2 L_* + 2ac_2 + 4aN_*,$$

therefore, by Lemma 8 we see $m(t) \leq 0$, i.e., $m(t) \equiv 0, t \in J$ and so $\alpha(U(t)) \equiv 0, \alpha(U'(t)) \equiv 0, t \in J$. Hence $\alpha_{PC^1}(U) = 0$, $\alpha_{PC}(U') = 0$. Then U is a relatively compact set in $PC^1[J, E]$, U' is a compact set in $PC\,[J, E]$. According (26) (27) and the normality of P, we know $\{u_n\}$ $\{u_n'\}$ are convergent sequence respectively in $PC^1[J, E]$ and $PC\,[J, E]$. Hence, there exist $u^* \in PC^1[J, E]$ satisfies $u_n \to u^*, u_n' \to (u^*)'$, i.e.,

$$\| u_n - u^* \|_{PC}^1 \to 0, \| u_n' - (u^*)' \| \to 0. \qquad \text{... (33)}$$

By the continuity of f, the definition of σ_n and (33) we can prove

$$\| \sigma_n - \sigma^* \| \to 0, (n \to \infty), \qquad \text{... (34)}$$

where $\quad \sigma^* = f(t, u^*, (u^*)', Tu^*) + M(t)\, u^* + N(t)\,(u^*)' + L(t)\, Tu^*.$

And by (28), (33) and (34) and Lebesgue control theorem, take limits in (29) (30) as $n \to \infty$ we have

$$u^*(t) = x_0 + tx_1 + \int_0^t (t-s)\,[\sigma^*(s) - M(s)\,u^*(s) - N(s)\,(u^*)'(s) - L(s)\,(Tu^*)(s)]\,ds$$

$$+ \sum_{0 < t_i < t} [L_i\,(u^*)'(t_i) + (t - t_i)\,L_i^*\,u^*(t_i)],$$

and

$$(u^*)'(t) = x_1 + \int_0^t [\sigma^*(s) - M(s)\,u^*(s) - N(s)\,(u^*)'(s) - L(s)\,(Tu^*)(s)]\,ds,$$

$$+ \sum_{0 < t_i < t} L_i^*\,u^*(t_i)$$

according Lemma 5 we know $u^* \in PC^1[J, E] \cap C^2[J', E]$, and $u^*(t)$ s a solution of IVP ( ).

Similarly, we can prove that there exists a $v^*\, P\, C^1[J, E] \cap C^1[J'\, E]$, such that $\| v_n - v^* \| \to 0 \ \| v_n' - (v_n^*)' \| \to 0$ and $v^*(t)$ is a solution of IVP (1). And by (26) (27) we can obtain

$$u_0(t) \leq u_1(t) \leq \ldots \leq u_n(t) \leq \ldots \leq u^*(t)$$

$$\leq v^*(t) \leq \ldots \leq v_n(t) \leq \ldots \leq \ldots \leq v_1(t) \leq v_0(t), \quad \ldots (35)$$

$$u_0'(t) \leq u_1'(t) \leq \ldots \leq u_n'(t) \leq \ldots \leq (u^*)'(t)$$

$$\leq (v^*)'(t) \leq \ldots \leq v_n'(t) \leq \ldots \leq \ldots \leq v_1'(t) \leq v_0'(t), \quad \ldots (36)$$

Let $u \in PC'[J< E] \cap C^2[J', E]$ is a solution of IVP (1) in $[u_0, v_0]$, assume that $u_{n-1}(t) \leq u(t) \leq v_{n-1}(t), u_{n-1}'(t) \leq v_{n-1}'(t), \quad \forall t \in J$. By the increasing of A we can see $u_n(t) \leq u(t) \leq v_n(t), u_n'(t) \leq u'(t) \leq v_n'(t), \quad \forall t \in J$. By the normality of P, take limits in above inequality as $n \to \infty$, we can obtain $u^*(t) \leq u(t), \ \forall t \in J$. Observing (35) (36) we know that (24) hold.

Theorem 2 — *Let P be a regular cone, and* $(H_1)(H_2)$ *be satisfied, then the conclusion of theorem 1 hold.*

PROOF : The proof is almost the same as that of Theorem 1. The only difference is that, instead of using condition (H_3), the conclusion of relatively compact of U, U' is implied directly by (26) (27) and the regularity of P.

Remark 1 : In paper [1-3], when the author discussed the second order integro-differential equation in which f doesn't contain different argument u', and established comparison theorem, the only thing is to improve $p(t) \leq 0$, but $p'(t) \leq 0$. In this paper, we consider the second order impulsive integro-differential equation in which f contain differential argument u', and established a new comparison theorem, therefore, we obtained the existence theorem of extremal solutions and generalized the main result in paper [1-3].

Remark 2 : In paper [4], the author considered the integro-differential equation which doesn't contain impulsive argument, and assume that f is M-increasing about u', f is increasing u, Tu. In this paper, we consider the second order impulsive integro-differential equation, and assume that $f(t, u, u', Tu)$ is $M(t)$-increasing about u, u', Tu. By establishing a new comparison theorem, we obtain the existence theorem of extremal solutions and generalized the main result in paper [5].

Remark 3 : In paper [5], the author considered the second order integro-differential equation in which f contain differential argument u', but doesn't contain impulsive argument in infinite domain. In this paper, we considered the second order impulsive integro-differential equation which contain the differential argument u' in finite domain, therefore, the method used in paper [5] isn't suitable to this paper, and so, this paper improved the main result in paper [5].

4. AN EXAMPLE

Example — Consider the IVP of second impulsive integro-differential equation

$$\begin{cases} u_n'' = \dfrac{1}{6n(1+t)^5}[(t^2-u_n)^2+(2t-u_n')^2+tu_{n+1}^2+t_2(u_{2n}')^3] \\ +\dfrac{1}{n^2(1+t)^8}\left(\dfrac{t^2}{2}-\displaystyle\int_0^t \dfrac{u_n(s)\,ds}{1+t+s}\right)^3, \quad \forall t\in\left[0,\dfrac{1}{2}\right], t\neq\dfrac{1}{4}; \\ \Delta u_n|_{t=\frac{1}{4}} = \dfrac{1}{2}u_n'\left(\dfrac{1}{4}\right); \\ \Delta u_n'|_{t=\frac{1}{4}} = -\dfrac{7}{48}u_n\left(\dfrac{1}{4}\right); \\ u_n(0)=u_n'(0)=0\ (n=1,2,\ldots). \end{cases} \quad \ldots (37)$$

Evidently, $u_n(0)\equiv 0$ isn't a solution of IVP (37).

CONCLUSION

IVP (37) has maximal and minimal solutions $u(t), \upsilon(t)\in C^2\left[1,\frac{1}{2}\right)\cup\left(\frac{1}{4},\frac{1}{2}\right]$ satisfy

$$0\le u_n(t),\ \upsilon_n(t)\le\begin{cases}\dfrac{t^2}{\eta}, & 0\le t<\dfrac{1}{4};\\[2mm] \dfrac{t^2+\frac{1}{4}}{n}, & \dfrac{1}{4}<t\le\dfrac{1}{2},\end{cases}$$

$$0\le u_n'(t),\ \upsilon_n'(t)\le\frac{2t}{n},\quad \forall t\in\left[0,\frac{1}{2}\right](n=1,2,\ldots).$$

PROOF : Let $J=\left[0,\frac{1}{2}\right]$, $E=c_0=\{u=(u_1,u_2,\ldots,u_n,\ldots):u_n\to 0\}$, $\|u\|=\sup_n|u_n|$, $P=\{u=(u_1,\ldots,u_n,\ldots)\in c_0:u_n\ge 0, n=1,2,\ldots\}$, then cone P is a normal cone in E, IVP (37) can be regarded as the form IVP (1). Where, $x_0=x_1=(0,\ldots,0,\ldots)$, $k(t,s)=(1+t+s)^{-1}$, $u=(u_1,\ldots,u_n,\ldots)$, $\upsilon=(\upsilon_1,\ldots,\upsilon_n,\ldots)$, $w=(w_1,\ldots,w_n,\ldots)$, $f=(f_1,\ldots,f_n,\ldots)$, here

$$f_n(t,u,\upsilon,w)=\frac{1}{6n(1+t)^5}[(t^2-u_n)^2+(2t-\upsilon_n)^2+tu_{n+1}^2+t^2(\upsilon_{2n}^2]$$

$$+ \frac{1}{n^2 (1+t)^8} \left(\frac{t^2}{2} - w_n \right)^2, \qquad \text{.... (38)}$$

$$I_1 (u, v) = -\frac{7}{48} u.$$

Then, we have $f \in C[J \times E \times E \times E, E]$, $I_1 \in C[E, E]$. Let

$$u_0 (t) = (0, \ldots., 0, \ldots),$$

$$v_0 (t) = \begin{cases} \frac{t^2}{n}, & 0 \le t \le \frac{1}{4}; \\ \frac{t^2 + \frac{1}{4}}{n}, & \frac{1}{4} < t \le \frac{1}{2} \end{cases}$$

we have

$$u_0 \, v_0 \in PC^1 \left[1, \frac{1}{2} \right] \cap C^2 \left[0, \frac{1}{4} \right) \cup \left(\frac{1}{4}, \frac{1}{2} \right], u_0 (t) \le v_0 (t) \; (t \in J)$$

and $$u_0' (t) = (0, \ldots, 0, \ldots) \le \left(2t, \ldots, \frac{2t}{n}, \ldots \right) = v_0' (t) \quad \forall t \in J,$$

$$u_0 (0) = v_0 (u) = (0, \ldots, 0, \ldots) = x_0,$$

$$u_0' (0) = v_0' (0) = (0, \ldots, 0, \ldots) = x_0 = x_1,$$

$$u_0'' (t) = (0, \ldots, 0, \ldots), \; v_0'' (t) = \left(2, \ldots, \frac{2}{n}, \ldots \right), \quad \forall t \in J,$$

$$f_n (t, u_0 (t), u_0' (t), (Tu_0) (t)) = \frac{1}{6n (1+t)^5} [t^4 + 4t^2] + \frac{t^6}{8n^2 (1+t)^8} \ge 0, \quad \forall t \in J,$$

$$f_n (t, v_0 (t), (T v_0 (t))$$

$$< \frac{1}{6n} \left(1 + 4 + \frac{5}{4} \right) + \frac{1}{8n} < \frac{2}{n}, \quad \forall t \in J \; (n = 1, 2, 3, \ldots).$$

Therefore u_0, v_0 satisfy (H_1). On the other hand, for $\forall t \in J, u_0 (t) \le \bar{u} \le u \le v_0 (t), u_0' (t) \le \bar{v} \le v \le v_0'$, $(Tu_0) (t) \le \bar{w} \le w \le (T v_0) (t)$ by (38) we can know

$$f_n (t, u, v, w) - f_n (t, \bar{u}, \bar{v}, \bar{w}) \ge -\frac{t^2}{3 (1+t)^5} (u_n - \bar{u}_n)$$

$$-\frac{2t}{3\,(1+t)^5}\,(v_n-\overline{v}_n)-\frac{3t^4}{4\,(1+t)^8}\,(w_n-\overline{w}_n)\;(n=1,2,\ldots),$$

$$I_1\,(u,\overline{u})-I_1\,(v,\overline{v})=-\frac{7}{48}\,(u-v).$$

Consequently, (H_2) is satisfied for $M(t)=t^2/[3\,(1+t)^5]$, $N(t)=2t/[3\,(1+t)^5]$, $L(t)=3t^4/[4\,(1+t)^8]$. At this time, =1, $t_1=1/4$, $M_*=1$? 3, $N_*=2/3$, $L_*=3/4$, a = 1/2, $L_1=1/2$, $L_1^*=-7/48$, k_0 = 1, so, we have

$$(a+L_1)=1\le\frac{1-aN_*}{a\,(M_*+k_0\,L_1*a)-L_1^*}=\frac{4}{3},$$

For $\forall t\in J$, any monotone equicontinuous sequence $B\subset[u_0,v_0]$, it is easy to verify that $f(t,B(t),(TB)(t))$ is relatively compact in c_0, so $c_1=c_2=c_3=0$, and

$$\alpha\,(I_1\,(B\,(t),B'\,(t)))=\frac{7}{48}\,\alpha\,(B\,(t))\le\max\left\{\,\alpha\,(B\,(t)),\,\alpha\,(B'\,(t))\,\right\},$$

therefore (H_3) hold. Hence, the conclusion hold by Theorem 1.

Remark 4 : This example is a second order impulsive integro-differential equation in which f contain differential argument u' in finite domain, but the equations discussed in paper [1-4] are not according to this form, so, this problem can't be resolved by the main theorem obtained in paper [1-4]. Therefore, this paper improved these paper.

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