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FIXED POINT THEOREMS ON UNIFORM SPACES

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Two general fixed point theorems for $F:(X, \mathcal{U}) \to (2^X, \mathcal{U}^*)$ are proved. Examples are given to illustrate the generality of the theorems.

Key Words : Fixed Point; Set-Valued Mapping; Uniform Space.

1. INTRODUCTION AND PRELIMINARIES

Let $(X, \mathcal{U})$ be a uniform space. A family $\{d_i : i \in I\}$ of pseudometrics on X with indexing set I, is called an associated family for the uniformity $\mathcal{U}$ if the family

$$\beta = \{ V(i, r) : i \in I, r > 0 \},$$

where $\quad V(i, r) = \{(x, y) : x, y \in X, d_i(x, y) < r\}$

is a subbase for the uniformity $\mathcal{U}$. We may assume that itself is a base by adjoining finite intersection of members of β, if necessary. The corresponding family of pseudometrics is called an augmented associated family for $\mathcal{U}$. An associated family for $\mathcal{U}$ will be denoted by P^*. For details the reader is referred to Ganguly[1], Turkoglu *et al.*[5], Pai and Veeramani[6], Taraftar[7], Qureshi and Upadhyay[8], Mishra[9], Mishra and Singh[10], Acharya[11], Thorn[14].

Let A be a nonempty subset of a uniform space X. Define

$$2^X = \{A : A \text{ is a nonempty } P^* - \text{bounded subset of } X\}.$$

For any nonempty subset A and B of 2^X, define

$$d_i(x, A) = \inf \{d_i(x, a) : a \in A, i \in I\},$$

$$\delta_i(A, B) = \sup \{d_i(a, b) : a \in A, b \in B, i \in I\}.$$

The function δ_i satisfies the following conditions :

(*i*) $\delta_i(A,B)=\delta_i(B,A)\geq 0$ and $\delta_i(A,B)=0$ implies that $A = B$ and this set consist of only one point.

(*ii*) $\delta_i(A,B)\leq \delta_i(A,C)+\delta_i(A,C)$ for $A,B,C\in 2^X$.

Also, if $A = \{a\}$ we write $\delta_i(A,B)=\delta_i(a,B)$ and furthermore, if $B = \{b\}$, we write $\delta_i(A,B)=\delta_i(a,b)=d_i(a,b)$.

A sequence $\{A_n\}$ of sets in 2^X is said to converge to the subset A of X if the following two conditions are satisfied :

(*a*) for each point a in A, there is a sequence $\{a_n\}$ such that $a_n\in A_n$ for all n and $a_n\to a$.

(*b*) for every $\varepsilon>0$ there is an integer N such that $A_n\subseteq A^{\varepsilon}$ for all $n\geq N$, where

$$A^{\varepsilon}=\bigcup_{x\in A} U(x) = \{y\in X : d_i(x,y)<\varepsilon \text{ for some } x \text{ in } A,\ i\in I\}.$$

In such a case, A is said to be the limit of the sequence $\{A_n\}$ and we write $\lim_{n\to\infty} A_n=A$ or $A_n\to A$.

The mapping $F:X\to 2^X$ is said to be continuous at $x_0\in X$ if whenever $\{x_n\}$ is a sequence of points in X converging to x, the sequence $\{Fx_n\}$ in 2^X converges to Fx in 2^X. We say that F is a continuous mapping of X into 2^X if F is continuous at each point x in X.

The usual definition of a fixed point x of a set valued mapping F is that $x\in Fx$.

A good reference, for theorems in this setting, is the paper by Rhoades and Watson[2], Fisher[3].

Let $(X, \mathcal{U})$ be a uniform space and let $U\in \mathcal{U}$ be an arbitrary entourage. For each subset A of X, define

$$U\,[A] = \{y\in X : (x,y)\in U \text{ for some } x\in A\}.$$

The uniformity $2^{\mathcal{U}}$ on 2^X is defined by the base

$$2^{\beta}=\left\{\tilde{U} : U\in \mathcal{U}\right\},$$

where
$$\tilde{U}=\left\{(A,B)\in 2^X\times 2^X : A\times B\subset U\right\}\cup\Delta.$$

(Here Δ denotes the diagonal of $X\times X$).

The augmented associated family P^* also induces a uniformly $\mathcal{U}^*$ on 2^X defined by the base

$$\beta^* = \{V^*(i,r) : i\in I.\ r>0\},$$

where $V^*(i,\ r) = \left\{(A,B)\in 2^X\times 2^X : \delta_i(A,B)<\varepsilon\right\}\cup\Delta.$

The uniformities $2^{\mathcal{U}}$ and $\mathcal{U}^*$ on 2^X are uniformly isomorphic. The space $(2^X, \mathcal{U}^*)$ is thus a uniform space called the hypersurface of $(X, \mathcal{U})$.

The following theorem was proved in [12].

Theorem 1.1 — *If (Y, d) is complete metric space and $F: Y \to CL(Y)$ is a multi valued function which fulfills the inequality $D(Fx, Fy) \le \Psi(d(x, y))$ for all x, y in X and some strictly increasing function Ψ such that $\lim_{k \to \infty} \Psi^k(t) = 0$ for every t, then*

(a_1) for every $y_0 \in Y$ and for every fixed point $y \in Y$ of F there exists a sequence of iterates of F at y_0 which converges to y,

and (a_2) if $\Sigma\ \Psi^k(t) < \infty$, for $t > 0$, then the set of fixed points of F is nonempty.

In this theorem $\Psi: [0, \infty) \to [0, \infty)$, D is the Hausdorff metric and $CL(Y) = \{A : A$ is closed in $Y\}$.

2. Main Results

Theorem 2.1 — *Let $(X, \mathcal{U})$ be a complete Hausdorff uniform space defined by $\{d_i : i \in I\} = P^*$ and $(2^X, \mathcal{U}^*)$ a hyperspace, let $F: X \to 2^X$ be a continuous mapping and Fx compact for each x in X. Assume that*

$$\delta_i(Fx, Fy) \le K(d_i(x, y)) \qquad \dots (1)$$

for all $i \in I$ and $x, y \in X$, where $K: [0, \infty) \to [0, \infty)$, $K(0) = 0$ and K is nondecreasing. Then there exists z in X with $z \in Fz$ if and only if there exists x_0 in X with $\sum_{n=1}^{\infty} K^n(d_i(x_0, Fx_0)) < \infty$.

Note that in this theorem K is not assumed to be continuous and $K^n(t) = K(K^{n-1}(t))$.

PROOF : If $z \in \overline{F_Z}$ then $d_i(z, \overline{F_z}) = 0$, $0 = K(0) = K^2(0) = \dots = K^n(0) = \dots$ for each $i \in I$ and $\Sigma K^n(d_i(z, \overline{F_Z})) = 0$. Let $x_0 \in X$ and $x_1 \in \overline{Fx_0}$ be arbitrary point. Suppose that there exists x_0 such that $\Sigma K^n(d_i(x_0, Fx_0)) < \infty$ for each $i \in I$. Let $U \in \mathcal{U}$ be an arbitrary entourage. Since β is a base for $\mathcal{U}$, there exists $V(i, r) \in \beta$ such that $V(i, r) \subseteq U$. Now $y \to d_i(x_0, y)$ is continuous on the compact set $\overline{Fx_0}$ and this implies that there exists $x_1 \in \overline{Fx_0}$ such that $d_i(x_0, x_1) = d_i(x_0, \overline{Fx_0})$. Similarly, Fx_1 is compct so there exists $x_2 \in \overline{Fx_1}$ such that $d_i(x_1, x_2) = d_i(x_1, \overline{Fx_1})$. Continuing, we obtain a sequence $\{x_n\}$ such that $x_{n+1} \in \overline{Fx_n}$ and $d_i(x_n, x_{n+1}) = d_i(x_n, \overline{Fx_n})$. Noting that K is nondecreasing and using inequality (1), we have

$$d_i(x_n, x_{n+1}) = d_i(x_n, \overline{Fx_n}) \leq \delta_i(\overline{Fx_{n-1}}, \overline{Fx_n})$$

$$= \delta_i(Fx_{n-1}, Fx_n) \leq K(d_i(x_{n-1}, x_n)) = K(d_i(x_{n-1}, \overline{Fx_{n-1}}))$$

$$\leq \delta_i(\overline{Fx_{n-2}}, \overline{Fx_{n-1}}) = \delta_i(Fx_{n-2}, Fx_{n-1}) \leq K^2(d_i(x_{n-2}, x_{n-1}))$$

$$\leq \ldots.$$

$$\leq K^n(d_i(x_0, x_1)) = K^n(d_i(x_0, Fx_0)).$$

Hence we obtain

$$d_i(x_n, x_{n+m}) \leq d_i(x_n, x_{n+1}) + d_i(x_{n+1}, x_{n+2}) + \ldots + d_i(x_{n+m-1}, x_{n+m})$$

$$\leq K^n(d_i(x_0, Fx_0)) + K^{n+1}(d_i(x_0, Fx_0)) + .. + K^{n+m-1}(d_i(x_0, Fx_0))$$

$$= \sum_{k=n}^{n+m-1} K^k(d_i(x_0, Fx_0)).$$

Since $\sum_{n=1}^{\infty} K^n(d_i(x_0, Fx_0)) < \infty$, it follows that there exists p such that $d_i(x_n, x_m) < r$ and hence $(x_n, x_m) \in U$ for all $n, m \geq p$. Therefore, the sequence $\{x_n\}$ is a Cauchy sequence in the d_i-uniformity on X.

Let $S_p = \{x_n ; n \geq p\}$ for all positive integer p and let β be the filter basis $\{S_p : p = 1, 2, 3, \ldots\}$. Then since $\{x_n\}$ is a d_i Cauchy sequence for each $i \in I$, it is easy to see that the filter basis β is Cauchy filter in the uniform space $(X, \mathcal{U})$. To see this we first note that family $\{V(i, r) : i \in I, r > 0\}$ is a base $\mathcal{U}$ as $P^* = \{d_i : i \in I\}$. Now, since $\{x_n\}$ is a d_i Cauchy sequence in X, there exists a positive integer p such that $d_i(x_n, x_m) < r$ for $m \geq p, n \geq p$. This implies that $S_p \, x \, S_p \subset U$. Hence, β is a Cauchy filter in $(X, \mathcal{U})$. Since $(X, \mathcal{U})$. is complete Hausdorff space, the Cauchy filter $\beta = \{S_p\}$ converges to a unique point $z \in X$.

We will show that $d_i(z, Fz) = 0$, that is $z \in \overline{Fz}$.

Now let $W \in \mathcal{U}$ be arbitrary and let $V(j, t) \in \beta, j \in I$ and $t > 0$ be such that $V(j, t) \subseteq W$. For a given $\varepsilon > 0, S_p \to z$ there exists a positive integer N_1 such that

$$d_j(z, x_n) < \frac{\varepsilon}{3} \qquad \ldots (2)$$

for all $n \geq N_1$. On the other hand, since F is continuous, for the same ε, we can find an N_2 such that

$$Fx_n \subseteq A^{\frac{\varepsilon}{3}} = \bigcup_{a \in Fz} U(a)$$

for all $n \geq N_2$. Further, since $x_{n+1} \in \overline{Fx_n}$, then there exists a $y \in Fx_n$ such that $d_j(x_n, y) < \frac{\varepsilon}{3}$ and

$$y \in \overline{Fx_n} \subseteq \bigcup_{a \in Fz} U(a)$$

implies that there exists an $a \in Fz$ such that $d_j(a, y) < \frac{\varepsilon}{3}$. Thus

$$d_j(x_{n+1}, Fz) \leq d_j(x_n, a) \leq d_j(x_n, y) + d_j(y, a) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3} \qquad \ldots (3)$$

for all $n \geq N_2$. Now let $N = \max\{N_1, N_2\}$, from (2) and (3), we have

$$d_j(z, Fz) \leq d_j(z, x_{n+1}) + d_j(x_{n+1}, Fz) < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon$$

for all $n > N$. Since $\varepsilon > 0$ is arbitrary, we have $d_j(z, Fz) = 0 < t$. Hence, $(z, Fz) \in V(j, t) \subseteq W$. it follows that $z \in \overline{Fz}$.

***Theorem* 2.2** — *Change Theorem 2.1 by removing the colosure symbol from the three places it appears.*

PROOF : The same proof works. In the last line of the proof one has $z \in \overline{Fz} = Fz$.

Remark 1 : To apply Theorem 2.1 or Theorem 2.2 one needs a nondecreasing function K and x in X with

$$\sum_{n=1}^{\infty} K^n(d_i(x, Fx)) < \infty$$

The following examples satisfy these conditions and therefore illustrate the generality of the Theorems; let X denote complete Hausdorff uniform space defined by $\{d_i : i \in I\} = P^*$

Example 2.3 — Suppose $0 < \lambda_i < 1$. Let $K(t) = \lambda_i t$ for $t \geq 0$. Then $\delta_i(Fx, Fy) \leq K(d_i(x, y)) = \lambda_i d_i(x, y)$ and $K^n(d_i(x, Fx)) = \lambda_i^n d_i(x, Fx)$ for any x in X. Clearly

$$\sum_{n=1}^{\infty} \lambda_i^n d_i(x, Fx) < \infty.$$

Example 2.4 — Suppose that F satisfies $\delta_i(Fx, Fy) \leq \Phi(d_i(x, y))\, d_i(x, y)$ for all $x, y \in X$, where Φ $[0, \infty)$ $[0, 1)$ and Φ is nondecreasing. Then $K(t) = t\,\Phi(t), K$ is nondecreasing and $K : [0, \infty) \to [0, \infty)$. It follows that by induction that $K^n(t) \leq t\,[\Phi(t)]^n$, since $\Phi(t) < 1$ and $\sum_{n=1}^{\infty} K^n(t) < \infty$.

Example 2.5 — Consider $K(t) = t\,\Phi(t)$, where $\Phi : [0, \infty) \to [0, \infty)$, $\Phi(t) \leq 1$ for $t \leq 1$. If $t < 1$ it follows that $K^n(t) \leq t\,[\Phi(t)]^n$. If K is nondecreasing, then Theorems can be applied.

Example 2.6 — $K(t) = t\,\Phi(t)$, where $\Phi : [0, \infty) \to [0, \infty)$, $\Phi(\alpha_i t) \leq \alpha_i \Phi(t)$ for $\alpha_i \in (0, 1)$. If $\Phi(t) \leq 1$, then $K^n(t) \leq K(t)\,[\Phi(t)]^n$ for all $n \geq 2$.

Example 2.7 — Assume K is nondecreasing, K is convex on $[0, 1)$, and $K(t) < t$ for all $0 < t < 1$. If $t < 1, K(t) < t$, then $K(t) = \alpha_i t$ for some $0 < \alpha_i < 1$. It can be shown that $K^n(t) \leq \alpha_i^n t$ for all n and thus $\sum_{n=1}^{\infty} K^n(t) < \infty$.

Theorem 2.8 — *Let* $(X, \mathcal{U})$ *be a complete Hausdorff uniform space defined by* $\{d_i : i \in I\} = P^*$, *let* $F : X \to 2^X$ *be a multi valued mapping and Fx compact for each x in X. Assume that* $\delta_i(Fx, Fy) \leq [d_i(x, y)]^q$, *where* $q > 1$, *then F has a fixed point in X.*

PROOF : Let $K(t) = t^q$ for $t \geq 0$. Then $K(0) = 0$ and K is increasing, $K(t) < t$ if $t < 1$ and K is convex. If $t = d_i(x, Fx) < 1$, then $\sum_{n=1}^{\infty} K^n(t) < \infty$ from the previous example. Also F is continuous, so Theorem 2.2 applies.

Remark 2 : If we replace the uniform space $(X, \mathcal{U})$ in Theorem 2.2 and 2.8 and Examples 1-5 by a metric space (i.e. a metrizable uniform space), then the results of Hicks [13] will follow as special cases of our results.

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