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ON THE CONVERGENCE OF MOVING AVERAGE PROCESSES UNDER NEGATIVELY ASSOCIATED RANDOM VARIABLES

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Let $\{Y_i, -\infty < i < \infty\}$ be a sequence of identically distributed negatively associated (NA) random variables with $EY_1 = 0$ and $\{a_i - \infty < i < \infty\}$ an absolutely summable sequence of real numbers. In this paper, we discuss complete convergence of $\left\{ \sum_{k=1}^{n} \sum_{i=-\infty}^{\infty} a_{i+k} Y_i / H(n), n \geq 1 \right\}$ under some suitable conditions, where $H(n) = n^{1/t}$ $(1 \leq t < 2)$ and $H(n) = (nL(n))^{1/2}$ $(L(x) = \max\{1, \log x\})$. These results extend and generalize Li *et al.*[8] result and complement the result for $t = 2$, which had not been discussed by Li *et al.*[7]. Moreover, a conjecture for i.i.d random variables of Gut[4] is answered and generalized in NA setting.

Key Words : Convergence; Negatively Associated Random Variable; Moving Average; Law of Logarithm

1. INTRODUCTION

Let $\{Y_i, -\infty < i < \infty\}$ be a sequence of identically distributed random variables and $\{a_i, -\infty < i < \infty\}$ a sequence of real numbers with $\sum_{i=-\infty}^{\infty} |a_i| < \infty$. Write

$$X_k = \sum_{i=-\infty}^{\infty} a_{i+k} Y_i, \quad k \geq 1. \qquad \text{... (1.1)}$$

When $\{Y_i, -\infty < i < \infty\}$ is a sequence of independent random variables, there have been some authors who studied limit properties for the moving average process $\{x_k, k \geq 1\}$. In particular, Ibragimov[5] had established the Central Limit Theorem for $x_k, k \geq 1$, Burton and Dehling[3] had obtained large deviation principle for $\{X_k, k \geq 1\}$ assuming $E \exp(tY_1) < \infty$ for all t, and Li *et al.*[7] had obtained the following result on complete convergence.

Theorem A — *Suppose* $\{Y_i, -\infty < i < \infty\}$ *is a sequence of independent and identically distributed (i.i.d) random variables. Let* $\{X_k, k \geq 1\}$ *be defined as above and* $1 \leq t < 2$. *Then* $EY_1 = 0$ *and* $E|Y_1|^{2t} < \infty$ *imply*

$$\sum_{n=1}^{\infty} P\left(\left|\sum_{k=1}^{n} X_k\right| \geq \varepsilon n^{1/t}\right) < \infty, \ \forall \varepsilon > 0. \quad \ldots (1.2)$$

Recently, Zhang[13] *gave a general version of Theorem A under identically distributed* ϕ*-mixing assumptions. Clearly, Theorem A implies*

$$\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} Y_i\right| \geq \varepsilon n^{1?t}\right) < \infty, \ \forall \varepsilon > 0. \quad \ldots (1.3)$$

While, by Kolmogorov's law of iterated logarithim, we know

$$\limsup_{n \to \infty} \frac{\left|\sum_{i=1}^{n} Y_i\right|}{n^{1/2}} = \infty \text{ a.s..}$$

Therefore, (1.3) is not true for $t = 2$, further (1.2) does not hold for $t = 2$.

The main purpose of this paper is to extend and generalize Theorem A to NA random variables; discuss the result for $t = 2$ in NA setting, which had not been settled by Li *et al.*[7] in i.i.d. setting

A finite family of random variables $\left\{X_i, 1 \leq i \leq n\right\}$ is said to be negatively associated (NA) if for every pair of disjoint subsets A and B of $\{1, 2, \ldots, n\}$,

$$\text{Cov}\ (f_1(X_i, i \in A), f_2(X_j, j \in B)) \leq 0,$$

whenever f_1 and f_2 are coordinatewise increasing and such that the covariance exists. An infinite family of random variables is *NA* if every finite subfamily is *NA*. This definition is introduced by Alam and Saxena[1] and carefully studied by Joag-Dev and Proschan[6], a number of well-known multivariate distributions possess the NA property, such as (*a*) multinomial, (*b*) convolution of unlike multinomials, (*c*) multivariate hypergeometric (*d*) Dirichlet, (*e*) Dirichlet compound multinomial, (*f*) negatively correlated normal distribution, (*g*) permutation distribution, (*h*) random sampling without replacement, and (*i*) joint distribution of ranks. Because of its wide applications in multivariate statistical analysis and reliability, the notion of NA have received considerable attention recently.

We refer to Joag-Dev and Proschan[6] for fundamental properties, Newman[9] for the central limit theorem, Matula[8] for the three series thorem, Su *et al.*[12] for a moment inequality, a weak invariance principle and an example to show that there exists finite family of non-degenerate non-independent strictly stationary NA random variables, Shao and Su[17] for the law of the iterated logarithm, Roussas[10] for the central limit theorem of random fields, some examples and applications.

2. Main Results

Here, let $\left\{ Y_i, -\infty, < i < \infty \right\}$ be a sequence of identically distributed NA random variables (r.v.'s) with $EY_1 = 0$ and $\left\{ X_k, k \geq 1 \right\}$ be defined as in (1.1) of section 1. Denote by $L(x) = \max(1, \log x)$.

Theorem 2.1 — *Let $h(x) > 0$ be a slowly varying function as $x \to \infty$ and $r \geq 1, 1 \leq t < 2$, $h(x)$ is non-decreasing when $r = 1$. If $E[|Y_1|^{rt} h(|Y_1|^t)] < \infty$, then $\forall\ \varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{r-2} h(n) P\left(\left| \sum_{k=1}^{n} X_k \right| \geq \varepsilon n^{1/t} \right) < \infty.$$

Theorem 2.2 — *Let $r > 1$. If $E[|Y_1|^{rt} h(|Y_1|^t)] < \infty$, then there exists some $\varepsilon_0 > 0$ such that $\varepsilon > \varepsilon_0$,*

$$\sum_{n=1}^{\infty} n^{r-2} P\left(\left| \sum_{k=1}^{n} X_k \right| \geq \varepsilon (nL(n))^{1/2} \right) < \infty.$$

Theorem 2.3 — *For $\eta > 0$, if $E[|Y_1^2/L(|Y_1|^{1-\eta})] < \infty$, then $\forall \varepsilon > 0$,*

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\left| \sum_{k=1}^{n} X_k \right| \geq \varepsilon (nL(n))^{1/2} \right) < \infty.$$

Remark 2.1 : Since i.i.d. random variables are a special case of *NA* random variables, Theorem 2.1 generalizes and extends Theorem A. Theorems 2.2-2.3 complement the results for $t = 2$ in *NA* setting, which had not been discussed by Li *et al.*[7] in i.i.d. setting.

Remark 2.2 : Gut[4] conjectured that under $\left\{ Y_i \right\}$ is a sequence of i.i.d symmetric random variables, for $\eta > 0$, if $E[Y_1^2/(L(|Y_1|))^{1-\eta}] < \infty$, then $\forall \varepsilon > 0$,

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\left| \sum_{i=1}^{n} Y_i \right| \geq \varepsilon (nL(n))^{1/2} \right) < \infty.$$

Clearly, Theorem 2.3 extends and generalizes (taking $a_0 = 1, a_j = 0, j \neq 0$) Gut's[4] above conjecture.

3. Proof of Main Result

In this section, $a << b$ means $a = O(b)$. C and $C_q\,(q \geq 1)$ will represent positive constants which their value may change from one place to another.

Lemma 1[3] — Let $\sum_{i=-\infty}^{\infty} a_i$ be an absolutely convergent series of real numbers with $\sum_{i=-\infty}^{\infty} a_i$, $b = \sum_{i=-\infty}^{\infty} |a_i|$. Suppose $\Phi : [-b, b] \to R$ is a function satisfying the following conditions :-

(*i*) Φ is bounded and continuous at a.

(*ii*) There exist $\delta > 0$ and $C > 0$ such that for all $|x| \leq \delta$, $\Phi(x) | \leq C |x|$.

Then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=-\infty}^{\infty} \Phi\left(\sum_{j=i+1}^{i+n} a_j\right) = \Phi(a).$$

Remark 3.1 : Taking $\Phi(x) = |x|^q, q \geq 1$, from Lemma 1 we have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=-\infty}^{\infty} \left|\sum_{j=i+1}^{i+n} a_j\right|^q = |a|^q. \qquad \text{... (3.1)}$$

Lemma 2[11 & 12] — Let $p \geq 2$ and let $\{X_i, i \geq 1\}$ be a sequence of *NA* random variables with $EX_i = 0$ and $E|X_i|^p < \infty$. Then, there exist constant $A_p > 0$ and $B_p > 0$ such that

$$E\left|\sum_{i=1}^{n} X_i\right|^p \leq A_p \left\{\left(\sum_{i=1}^{n} EX_i^2\right)^{p/2} + \sum_{i=1}^{n} E|X_i|^p\right\},$$

$$E \max_{1 \leq k \leq n} \left|\sum_{i=1}^{k} X_i\right|^p \leq B_p \left\{\left(\sum_{i=1}^{n} EX_i^2\right)^{p/2} + \sum_{i=1}^{n} E|X_i|^p\right\}.$$

Lemma 3[11] — Let $\{X_j, 1 \leq j \leq n\}$ be mean zero NA random variables with finite variance. Denote by $B_n = \sum_{j=1}^{n} EX_j^2$. Then for any $x > 0$, $\alpha > 0$ and $0 < b\eta < 1$,

$$P\left(\max_{1\le k\le n} |S_k| \ge x\right) \le 2P\left(\max_{1\le k\le n} |X_k| > \alpha\right) + \frac{2}{1-\beta}$$

$$\exp\left\{-\frac{x^2\beta}{2(\alpha x + B_n)}\left(1+\frac{2}{3}\ln\left(1+\frac{\alpha x}{B_n}\right)\right)\right\}.$$

Remark 3.2 : Suppose $\left\{Z_i; -\infty < i < \infty\right\}$ is a sequence of identically distributed mean zero *NA* random variables with $E|Z_1| < \infty$, finite variance and $\left\{a_i, -\infty < i < \infty\right\}$ a sequence of real numbers with $\sum_{i=-\infty}^{\infty} |a_i| < \infty$. Put $\sum_{i=-\infty}^{\infty} E|a_i Z_i|^2 < \infty$. From Lemma 3, we have

$$P\left(\left|\sum_{i=-\infty}^{\infty} a_i Z_i\right| \ge 2x\right) \le P\left(\left|\sum_{i=-m}^{m} a_i Z_i\right| \ge x\right) + P\left(\left|\sum_{|i|>m} a_i Z_i\right| \ge x\right)$$

$$\le 2P\left(\sup_i |a_i Z_i| > \alpha\right) + \frac{2}{1-\beta}\exp\left\{-\frac{x^2\beta}{2(\alpha x + B)}\right\} + \frac{E|Z_1|}{x}\sum_{|i|>m} |a_i|.$$

Since $\sum_{i=-\infty}^{\infty} |a_i| < \infty$, $\forall\ \varepsilon > 0$, choose m such that $\frac{E|Z_1|}{x}\ \Sigma_{|i|>m}\ |a_i| < \varepsilon$. Thus, we get

$$P\left(\left|\sum_{i=-\infty}^{\infty} a_i Z_i\right| \ge 2x\right) \le 2P\left(\sup_i |a_i Z_i| > \alpha\right) + \frac{2}{1-\beta}\exp\left\{-\frac{x^2\beta}{2(\alpha x + B)}\right\}. \quad \text{... (3.2)}$$

PROOF OF THEOREM 2.1

Note that $$\sum_{k=1}^{n} X_k = \sum_{i=-\infty}^{\infty}\left(\sum_{k=1}^{n}(a_{i+k}\right)Y_i = \sum_{i=-\infty}^{\infty} a_{mi} Y_i. \quad \text{... (3.3)}$$

It suffices to show that for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{r-2} h(n)\, P\left(\left|\sum_{i=-\infty}^{\infty} a_{ni}^{+} Y_i\right| > \varepsilon n^{1/t}\right) < \infty \quad \text{... (3.4)}$$

and $$\sum_{n=1}^{\infty} n^{r-2} h(n)\, P\left(\left|\sum_{i=-\infty}^{\infty} a_{ni}^{-} Y_i\right| > \varepsilon n^{1/t}\right) < \infty, \quad \text{... (3.5)}$$

where $a_{ni}^{+} = a_{ni} \vee 0$, $a_{ni}^{-} (-a_{ni}) \vee 0$. We prove only (3.4), the proof of (3.5) is analogous. Let

$$Y_{ni} = -n^{1/t} I(a_{ni}^{+} Y_i < -n^{1/t}) + a_{ni}^{+} Y_i I\left(|a_{ni}^{+} Y_i| \le n^{1/t} \right) + n^{1/t} I(a_{ni}^{+} Y_i > n^{1/t}).$$

Then

$$\sum_{n=1}^{\infty} n^{r-2} h(n) P\left(\left| \sum_{i=-\infty}^{\infty} a_{ni}^{+} \; Y_i \right| > \varepsilon n^{1/t} \right)$$

$$\le \sum_{n=1}^{\infty} n^{r-2} h(n) P\left(\left| \sum_{i=-\infty}^{\infty} Y_{ni} \right| \ge \frac{\varepsilon}{2} n^{1/t} \right) + \sum_{n=1}^{\infty} n^{r-2} h(n)$$

$$\sum_{i=-\infty}^{\infty} P\left(|a_{ni}^{+} Y_i| > n^{1/t} \right)$$

$$= : I_1 + I_2.$$

From (3.1), we can assume, without loss of generality, that $\sum_{i=-\infty}^{\infty} a_{ni}^{+} \le n, a_{ni}^{+} \le 1$ and denote by $I_{nj} = \left\{ i \in Z : (j=1)^{-1/t} < a_{ni}^{+} \le j^{-1/t} \right\}$. It is easy to verify from Lemma 1 that

$$\sum_{j=1}^{k} \# I_{nj} \le Cn(k+1)^{1/t}. \qquad \text{... (3.6)}$$

For I_2, using (3.6) we have

$$I_2 \le \sum_{n=1}^{\infty} n^{r-2} h(n) \sum_{j=1}^{\infty} (\# In_j) \sum_{k=nj}^{\infty} P(k \le |Y_1|^t < k+1)$$

$$\le \sum_{n=1}^{\infty} n^{r-2} h(n) \sum_{k=n}^{\infty} \sum_{j=1}^{[k/n]} (\# In_j) P(k \le |Y_1|^t < k+1)$$

$$<< \sum_{n=1}^{\infty} n^{r-1} h(n) n^{-1/t} \sum_{k=n}^{\infty} k^{1/t} P(k \le |Y_1|^t < k+1)$$

$$<< E[|Y_1|^{rt} h(|Y_1|^t)] < \infty.$$

By $EY_1 = 0$, we get

$$\left| \sum_{i=-\infty}^{\infty} EY_{ni} \right| / n^{1/t} \leq 2E \mid Y_1 \mid^t I(\mid Y_1 \mid > n^{1/t}) \to 0, \text{ as } n \to \infty.$$

Thus, to prove $I_1 < \infty$, we need only to show that

$$I_1^* =: \sum_{n=1}^{\infty} n^{r-2} h(n) P\left(\left| \sum_{i=-\infty}^{\infty} (Y_{ni} - EY_{ni}) \right| \geq \varepsilon n^{1/t} \right) < \infty, \ \forall \ \varepsilon > 0.$$

In fact, we use the Markov's inequality for a suitably large M, which will be determined later, Lemma 2 and note that for each $n \geq 1$, $\left\{ Y_{ni}, -\infty < i < \infty \right\}$ is still a sequence of NA random variables from the definition, we have

$$I_1^* << \sum_{n=1}^{\infty} n^{r-2} h(n) n^{-M/t} \left\{ \left(\sum_{i=-\infty}^{\infty} E \mid Y_{ni} \mid^2 \right)^2 + \sum_{i=-\infty}^{\infty} E \mid Y_{ni} \mid^M \right\}$$

$$=: I_3 + I_4.$$

If $r > 2$, note that $E \mid Y_1 \mid^2 < \infty$ and $\sum_{i=-\infty}^{\infty} \mid a_{ni} \mid^q \leq Cn$ for $q \geq 1$, taking $M > 2t(r-1)/(2-t)$, we get

$$I_3 \leq \sum_{n=1}^{\infty} n^{r-2} h(n) n^{-M/t}$$

$$\left\{ \cdot \sum_{i=-\infty}^{\infty} [n^{2/t} P(\mid a_{ni}^{+} Y_i \mid > n^{1/t}) + E \mid a_{ni}^{+} Y_i \mid^2 I(a_{ni}^{+} Y_i \mid \leq n^{1/t})] \right\}^{M/2}$$

$$<< \sum_{n=1}^{\infty} n^{r-2(1/t-1/2)M} h(n) < \infty.$$

If $1 < r \leq 2$ and $1 < rt \leq 2$, then there exists some s such that $r > s > 1$, taking $M >$ $2(r-1)/(s-1)$, we have

$$I_3 \leq \sum_{n=1}^{\infty} n^{r-2} h(n) n^{-M/t}$$

$$\left\{ \sum_{i=-\infty}^{\infty} [n^{2/t} P(|a_{ni}^{+} Y_i| > n^{1/t}) + E|a_{ni}^{+} Y_i|^2 I(a_{ni}^{+} Y_i| \le n^{1/t}) \right\}^{M/2}$$

$$= \sum_{n=1}^{\infty} {}^{r-2-M/t} h(n) \left[\sum_{i=-\infty}^{\infty} \int_0^{2/t} P(|a_{ni}^{+} Y_i|^2 > x)\, dx \right]^{M/2}$$

$$<< \sum_{n=1}^{\infty} n^{r-2-(s-1)M/2} h(n) < \infty.$$

If $r = 1$. Choose $M = 2$. Similarly to the below proof of $I_4 < \infty$, we get $I_3 < \infty$. As to I_4, we have

$$I_4 << \sum_{n=1}^{\infty} n^{r-2} h(n) \sum_{i=-\infty}^{\infty} P(|a_{ni}^{+} Y_i| > n^{1/t})$$

$$+ \sum_{n=1}^{\infty} n^{r-2-M/t} h(n) \sum_{i=-\infty}^{\infty} E|a_{ni}^{+} Y_i|^M I(|a_{ni}^{+} Y_i| \le n^{1/t})$$

$$=: I_5 + I_6.$$

From the proof of $I_2 < \infty$ we know $I_5 < \infty$.

$$I_6 \le \sum_{n=1}^{\infty} n^{r-2-M/t} h(n) \sum_{j=1}^{\infty} (\# I_{nj}) j^{-M/t} \sum_{k=1}^{2n} E|Y_1|^M I(k_1 < |Y_1|^t \le k)$$

$$+ \sum_{n=1}^{\infty} n^{r-2-M/t} h(n) \sum_{j=1}^{\infty} (\# I_{nj}) j^{-M/t} \sum_{k=2n+1}^{n(j+1)}$$

$$E|Y_1|^M I(k_1 < |Y_1|^t \le k)$$

$$=: I_7 + I_8.$$

Note that for $M \ge 1$ and $k \ge 1$, we have

$$\sum_{j=k}^{\infty} (\# I_{nj}) j^{-M/t} \le C n k^{-(M-1)/t}.$$

Hence, taking $M > rt$, we get

$$I_7 << \sum_{k=1}^{\infty} \sum_{n=[k/2]}^{\infty} n^{r-1-M/t} h(n) E|Y_1|^M I(k_1 < |Y_1|^t \leq k)$$

$$<< \sum_{k=1}^{\infty} k^{r-M/t} h(k) E|Y_1|^M I(k_1 < |Y_1|^t \leq k)$$

$$<< E[|Y_1|^{rt} h(|Y_1|^t)] < \infty.$$

$$I_8 << \sum_{n=1}^{\infty} n^{r-2-M/t} h(n) \sum_{k=2n+1}^{\infty} n\left(\frac{k}{n}\right)^{-(M-1)/t}$$

$$E|Y_1|^M I(k_1 < |Y_1|^{ty} \leq k)$$

$$<< \sum_{k=2}^{\infty} \sum_{n=1}^{[k/2]} n^{r-1-M/t} h(n) k^{-(M-1)/t} E|Y_1|^M I(k_1 < |Y_1|^t \leq k)$$

$$<< E[|Y_1|^{rt} h(|Y_1|^t)] < \infty.$$

PROOF OF THEOREM 2.2 : We need only to prove that for $\varepsilon > \varepsilon_0/2$,

$$\sum_{n=1}^{\infty} n^{r-2} P\left(\left|\sum_{i=-\infty}^{\infty} a_{ni}^{+} Y_i\right| > \varepsilon (n L(n))^{1/2}\right) < \infty, \quad \text{... (3.7)}$$

and

$$\sum_{n=1}^{\infty} n^{r-2} P\left(\left|\sum_{i=-\infty}^{\infty} a_{ni}^{-} Y_i\right| > \varepsilon (n L(n))^{1/2}\right) < \infty. \quad \text{... (3.8)}$$

We give proof of (3.7), the proof of (3.8) is analogous. Let

$$\lambda_n = \frac{10}{2\varepsilon} \sqrt{n/L(n)}, \rho_n = \frac{\varepsilon}{4N} \sqrt{n L(n)},$$

$$Y_{ni}^{(1)} = -\lambda_n I(a_{ni}^{+} Y_i < -\lambda_n) + a_{ni}^{+} Y_i I(|a_{ni}^{+} Y_i| \leq \lambda_n) + \lambda_n I(a_{ni}^{+} Y_i > \lambda_n),$$

$$Y_{ni}^{(2)} = (a_{ni}^{+} Y_i - \lambda_n) I(\lambda_n < a_{ni}^{+} Y^i < \rho_n),$$

$$Y_{ni}^{(3)} = (a_{ni}^{+} Y_i + \lambda_n) I(-\lambda_n > a_{ni}^{+} Y_i > -\rho_n)$$

and

$$Y_{ni}^{(4)} = (a_{ni}^{+} Y_i + \lambda_n) I(a_{ni}^{+} Y_i \leq -\rho_n) + (a_{ni}^{+} Y_i - \lambda_n) I(a_{ni}^{+} Y_i \geq \rho_n),$$

where N is some large positive integer, which will be specified later on. Then

$$\sum_{n=1}^{\infty} n^{r-2} P\left(\left|\sum_{i=-\infty}^{\infty} a_{ni}^{+} Y_i\right| > \varepsilon (n L(n))^{1/2}\right)$$

$$\leq \sum_{n=1}^{\infty} n^{r-2} P\left(\left|\sum_{i=-\infty}^{\infty} Y_{ni}^{(1)}\right| \geq \frac{\varepsilon}{4} (n L(n))^{1/2}\right)$$

$$+ \sum_{n=1}^{\infty} n^{r-2} P\left(\left|\sum_{i=-\infty}^{\infty} Y_{ni}^{(2)}\right| \geq \frac{\varepsilon}{4} (nL(n))^{1/2}\right)$$

$$+ \sum_{n=1}^{\infty} n^{r-2} P\left(\left|\sum_{i=-\infty}^{\infty} Y_{ni}^{(3)}\right| \geq \frac{\varepsilon}{4} (n L(n))^{1/2}\right)$$

$$+ \sum_{n=1}^{\infty} n^{r-2} P\left(\left|\sum_{i=-\infty}^{\infty} Y_{ni}^{(4)}\right| \geq \frac{\varepsilon}{4} (nL(n))^{1/2}\right) =: J_1 + J_2 + J_3 + J_4.$$

From (3.1), we can assume $a_{ni}^{+} \leq (2L(2)^{-1/2})$, denote by $I_{nj} = \left\{ i \in Z : ((j+2) L(j+2))^{-1/2} < a_{ni}^{+} \leq ((j+1) L(j+1))^{-1/2} \right\}$. Note that $\sum_{j=1}^{k} \# I_{nj} \leq C n ((k+2) L(k+2))^{1/2}$. Similarly, to the proof of $I_2 < \infty$, we get

$$J_4 \leq \sum_{n=1}^{\infty} n^{r-2} \sum_{i=-\infty}^{\infty} P\left(|a_{ni}^{+} Y_i| \geq \frac{\varepsilon}{4N} (nL(n))^{1/2}\right)$$

$$\leq \sum_{n=1}^{\infty} n^{r-2} \sum_{j=1}^{\infty} (\# I_{nj}) \sum_{k=nj}^{\infty} P(|Y_1|^2/L(|Y_1|) \geq C nj)$$

$$<< \sum_{k=1}^{\infty} k^{1/2} \sum_{n=1}^{k} n^{r-3/2} (L(2k/n))^{1/2} P\left(k \leq \frac{|Y_1|^2}{CL(|Y_1|)} < k+1 \right).$$

Choose $\beta > 0$ such that $r - 1/2 > \beta$, $L(x) \leq Cx^{2\beta}$ when $x \geq 2k_0$ for some $k_0 > 0$. Hence,

$$\sum_{n=1}^{k} n^{r-3/2} (L(2k/n))^{1/2} << \int_1^k x^{r-3/2} (L(2k/x))^{1/2} dx$$

$$<< \int_1^{k/k_0} x^{r-3/2}(k/x)^{\beta}\,dx + \int_{k/k_0}^{k} x^{r-3/2}(L(2k_0))^{1/2}\,dx << k^{r-1/2}.$$

Therefore, $$J_4 << \sum_{k=1}^{\infty} k^r P\left(k \le \frac{|Y_1|^2}{CL(|Y_1|)} < k+1 \right) << E[|Y_1|^2/L(|Y_1|)]^r < \infty.$$

Choose r_0 such that $r > r_0 > 1$, hence $E|Y_1|^{2r_0} < \infty$. From the deifnition of $Y_{ni}^{(2)}$, we know that $Y_{ni}^{(2)} > 0$, taking $N > (r-1)/(r_0-1)$, by the property of NA, we have

$$J_2 = \sum_{n=1}^{\infty} n^{r-2} P\left(\sum_{i=-\infty}^{\infty} Y_{ni}^{(2)} \ge \frac{\varepsilon}{4}(nL(n))^{1/2} \right)$$

$$\le \sum_{n=1}^{\infty} n^{r-2} P\,(\text{there are at least } N\ i\text{'s such that } Y_{ni}^{(2)} \ne 0)$$

$$\le \sum_{n=1}^{\infty} n^{r-2} \left[\sum_{i=-\infty}^{\infty} P(a_{ni}^{+} Y_i > \lambda_n) \right]^N$$

$$<< \sum_{n=1}^{\infty} n^{r-2-(r_0-1)N} (L(n))^{Nr_0} < \infty.$$

Similarly, $Y_{ni}^{(3)} < 0$ and $J_3 < \infty$. By $EY_1 = 0$ and $E|Y_1|^{2r_0} < \infty$, $\sum_{i=-\infty}^{\infty} (a_{ni}^{+})^{2r_0} \le Cn$, we have

$$\left| \sum_{i=-\infty}^{\infty} EY_{ni}^{(1)} \right| / (nL(n))^{1/2} \le \sum_{i=-\infty}^{} [\lambda_n P(|a_{ni}^{+} Y_i| > \lambda_n)$$

$$+ E|a_{ni}^{+} Y_i| I(|a_{ni}^{+} Y_i| > \lambda_n)]/(nL(n))^{1/2}$$

$$<< 1/n^{-(r_0-1)} (L(n))^{-r_0(1)} \to 0 \text{ as } n \to \infty.$$

Therefore, to prove $J_1 < \infty$, it suffices to show that

$$J_1^* =: \sum_{n=1}^{\infty} n^{r-2} P\left(\left| \sum_{i=-\infty}^{\infty} (Y_{ni}^{(1)} - EY_{ni}^{(1)}) \right| \ge \frac{\varepsilon}{5}(nL(n))^{1/2} \right) < \infty.$$

Note that, for each $n \geq 1, \left\{ Y_{ni}^{(1)}, -\infty, < i < \infty \right\}$ is still a sequence of NA random variables. Taking $\alpha = \frac{10}{\varepsilon} \sqrt{n/L(n)}, x = \frac{\varepsilon}{10} \sqrt{nL(n)}, \beta = \frac{1}{2}$ and it is easy to verify that

$$\sup_i \left| Y_{ni}^{(1)} - EY_{ni}^{(1)} \right| \leq 2\lambda_n = \alpha,\ B = \sum_{i=-\infty}^{\infty} E(Y_{ni}^{(1)} - EY_{ni}^{(1)})^2 \leq C_0\, n\, EY_1^2,$$

where C_0 satisfies $\sum_{i=-\infty}^{\infty} a_{ni}^2 \leq n\, C_0/2$. Hence, by using (3.2) we get

$$J_1^* \leq 4 \sum_{n=1}^{\infty} n^{r-2} \exp\left(- \frac{1/2 \cdot \varepsilon^2/100 \cdot nL(n)}{2(n + C_0\, nEY_1^2)} \right)$$

$$= 4 \sum_{n=1}^{\infty} n^{r-2-\varepsilon^2/400(1+C_0 EY_1^2)} < \infty$$

here, $\varepsilon_0 = 40\sqrt{(r-1)(1+c_0 EY_1^2)}$.

Proof of Theorem 2.3

Similarly to the proof of Theorem 2.2, we prove only that

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\left| \sum_{i=-\infty}^{\infty} a_{ni}^+ Y_i \right| > \varepsilon (nL(n))^{1/2} \right) < \infty,\ \forall\ \varepsilon > 0.$$

We may assume $\eta < 1$ and choose $\alpha > 0$ such that $\alpha < \eta$. Denote by

$$\lambda_n = n^{1/2} (L(n))^{(1-\alpha)/2}.$$

$$Y_{ni}^{(1)} = -\lambda_n I(a_{ni}^+ Y_i - \lambda_n) + a_{ni}^+ Y_i I(|a_{ni}^+ Y_i| \leq \lambda_n) + \lambda_n I(a_{ni}^+ Y_i > \lambda_n),$$

$$Y_{ni}^{(2)} = (a_{ni}^+ Y_i - \lambda_n) I(\lambda_n < a_{ni}^+ Y_i \leq \frac{\varepsilon}{4N} (nL(n))^{1/2}),$$

$$Y_{ni}^{(3)} = (a_{ni}^+ Y_i + \lambda_n) I(-\lambda_n > a_n^+ Y_i \geq -\frac{\varepsilon}{4N} (nL(n))^{1/2})$$

and

$$Y_{ni}^{(4)} = (a_{ni}^+ Y_i + \lambda_n) I(a_{ni}^+ Y_i < -\frac{\varepsilon}{4N} (nL(n))^{1/2})$$

$$+ (a_{ni}^+ Y_i - \lambda_n) I(a_{ni}^+ Y_i > \frac{\varepsilon}{4N} (nL(n))^{1/2}),$$

where N is some large positive integer, which will be specified later on. Then

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\left|\sum_{i=-\infty}^{\infty} a_{ni}^{+} Y_i\right| > \varepsilon (nL(n))^{1/2}\right)$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{n} P\left(\left|\sum_{i=-\infty}^{\infty} Y_{ni}^{(1)}\right| \geq \frac{\varepsilon}{4} (nL(n))^{1/2}\right)$$

$$+ \sum_{n=1}^{\infty} \frac{1}{n} P\left(\left|\sum_{i=-\infty}^{\infty} Y_{ni}^{(2)}\right| \frac{\varepsilon}{4} (nL(n))^{1/2}\right)$$

$$+ \sum_{n=1}^{\infty} \frac{1}{n} P\left(\left|\sum_{i=-\infty}^{\infty} Y_{ni}^{(3)}\right| \geq \frac{\varepsilon}{4} (nL(n))^{1/2}\right)$$

$$+ \sum_{n=1}^{\infty} \frac{1}{n} P\left(\left|\sum_{i=-\infty}^{\infty} Y_{ni}^{(4)}\right| \geq \frac{\varepsilon}{4} (nL(n))^{1/2}\right) =: Q_1 + Q_2 + Q_3 + Q_4.$$

From (3.1), similarly to the proof of $J_4 < \infty$ we can get $Q_4 < \infty$. From the definition of $Y_{ni}^{(2)}$ we know that $Y_{ni}^{(2)} > 0$, hence, taking $n > 1/(\eta - \alpha)$ and noticing that $\sum_{j=1}^{\infty} (\# I_{nj}) j^{-\delta} << n$ for $\delta > 0$ (the definition of I_{nj} is as in the proof of Theorem 2.2).

$$Q_2 = \sum_{n=1}^{\infty} \frac{1}{n} P\left(\sum_{i=-\infty}^{\infty} Y_{ni}^{(2)} \geq \frac{\varepsilon}{4} (nL(n))^{1/2}\right)$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{n} \left[\sum_{i=-\infty}^{\infty} P(a_{ni}^{+} Y_i > \lambda_n)\right]^N$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{n} \left[\sum_{j=1}^{\infty} (\# I_{nj}) P(Y_1^2 > nL^{1-\alpha}(n)(j+1)L(j+1))\right]^N$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{n} [(\sum_{j=1}^{\infty} (\# I_{nk}) P\left(\frac{Y_1^2}{(L(|Y_1|))^{1-\eta}} \geq \frac{CnL^{1-\alpha(n)}(j+1)L(j+1)}{(L(nL^{1-\alpha}(n)) + L((j+1)L(j+1)))^{1-\eta}}\right)^N$$

$$<< \sum_{n=1}^{\infty} \frac{1}{n}$$

$$\left\{ \sum_{J=1}^{\infty} (\# I_{nj}) \left[(njL(j+1))^{-1} (L(n))^{-(\eta-\alpha)} + (njL^{\eta}(j+1))^{-1} (L(n))^{-(1-\alpha)} \right] \right\}^{N}$$

$$\ll \sum_{n=1}^{\infty} \frac{1}{n} \left[(L(n))^{-N(\eta-\alpha)} + (L(n))^{-n(1-\alpha)} \right] < \infty.$$

Similarly, $Y_{ni}^{(3)} < 0$ and $Q_3 < \infty$. By $EY_1 = 0$ and $E[Y_1^2/(L(|Y_1|))^{1-\eta}] < \infty$, we get

$$\left| \sum_{i=-\infty}^{\infty} EY_{ni}^{(1)} \right| /(nL(n))^{1/2} \leq \frac{1}{(nL(n))^{1/2}}$$

$$\sum_{i=-\infty}^{\infty} [n^{1/2} L^{(1-\alpha)/2}(n) P(|a_{ni}^{+} Y_i| > n^{1/2} L^{(1-\alpha)/2}(n))$$

$$+ E|a_{ni}^{+} Y_i| I(|a_{ni}^{+} Y_i| > n^{1/2} L^{(1-\alpha)/2}(n))]$$

$$\leq \frac{1}{nL(n))^{1/2}} \sum_{j=1}^{\infty} (\# I_{nj}) [n^{1/2} L^{(1-\alpha)/2}(n) P(Y_1^2 > nL^{1-\alpha}(n)(j+1)L(j+1)$$

$$+ ((j+1)L(j+1))^{-1/2} E|Y_1| I(Y_1^2 > nL^{1-\alpha}(n)(j+1)L(j+1))$$

$$\ll (L(n))^{-(\eta-\alpha/2)} + (L(n))^{-(1-\alpha/2)} \rightarrow \text{ as } n \rightarrow \infty.$$

Thus, to prove $Q_1 < \infty$, it suffices to show that

$$Q_1^* =: \sum_{n=1}^{\infty} \frac{1}{n} P\left(\left| \sum_{i=-\infty}^{\infty} (Y_{ni}^{(1)} - EY_{ni}^{(1)}) \right| \geq \varepsilon \cdot (nL(n))^{1/2} \right) < \infty, \ \forall \ \varepsilon > 0.$$

Since, for each $n \geq 1, \left\{ Y_{ni}, -\infty < i < \infty \right\}$ remains a sequence of NA random variables, by using Lemma 2, choose $p > \max\{2/\eta, 2(1-\eta)/\alpha + 2\}$ we have

$$Q_1^* \ll \sum_{n=1}^{\infty} \frac{1}{n} \cdot (nL(n))^{-p/2} \left\{ \left(\sum_{i=-\infty}^{\infty} E|Y_{ni}^{(1)}|^2 \right)^{p/2} + \sum_{i=-\infty}^{\infty} E|Y_{ni}^{(1)}|^p \right\}$$

$$=: Q_5 + Q_6.$$

While

$$Q_5 \le \sum_{n=1}^{\infty} \frac{1}{n} \cdot (nL(n))^{p-2} \left[\sum_{i-\infty}^{\infty} [nL^{1-\alpha}(n)\, P(|a_{ni}^{(1)}|\, Y_i| > n^{1/2} L^{(1-\alpha)/2}(n)) \right.$$

$$\left\{ + E|a_{ni}^{+} Y_i|^2 I(|a_{ni}^{+} Y_i| \le n^{1/2} L^{(1-\alpha)/2}(n))\,]^{p/2} \right\}$$

$$\ll \sum_{n=1}^{\infty} \frac{1}{n} \cdot (nL(n))^{-p/2} \left\{ \sum_{j=1}^{\infty} (\# I_{nj})\, [nL^{1-\alpha}(n)\, P(Y_1^2 > C\, nL^{1-\alpha}(n)\, jL(j)) \right.$$

$$+ (jL(j))^{-1} E(Y_1^2/(L(|Y_1|))^{1-\eta}) \cdot (L(|Y_1|))^{1-\eta} I(Y_1^2 \le C\, nL^{1-\alpha}(n)\, j\, L(j))]\}^{p/2}$$

$$\ll \sum_{n=1}^{\infty} \frac{1}{n} [(L(n))^{-p\,\eta/2} + (L(n))^{-p/2}] < \infty.$$

$$Q_6 \le \sum_{n=1}^{\infty} \frac{1}{n} \cdot (nL(n))^{-p/2} \sum_{i=-\infty}^{\infty} [E|a_{ni}^{+} Y_i|^p I(|a_{ni}^{+} Y_i| \le n^{1/2} L^{(1-\alpha)/2}(n))$$

$$+ n^{p/2} L^{(1-\alpha)p/2}(n)\, P(|a_{ni}^{+} Y_i| > n^{1/2} L^{(1-\alpha)/2}(n))]$$

$$\ll \sum_{n=1}^{\infty} \frac{1}{n} \cdot (nL(n))^{-p/2} \sum_{j=1}^{\infty} (\# I_{nj})\, [(jL(j))^{-p/2}$$

$$E|Y_1|^p E|Y_1|^p I(Y_1^2 \le C\, nL^{1-\alpha}(n)\, jL(j))$$

$$+ n^{p/2} L^{(1-\alpha)p/2}(n)\, P(Y_1^2 > C\, nL^{1-\alpha}(n)\, jL(j))]$$

$$\le \sum_{n=1}^{\infty} \frac{1}{n} \cdot (l(n))^{-p/2} \sum_{j=1}^{\infty} (\# I_{nj})\, [(jL(j))^{-p/2}$$

$$\cdot E\left(\frac{Y_1^2}{(L(|Y_1|))^{1-\eta}} \right) \cdot |Y_1|^{p-2} (L(|Y_1|))^{1-\eta} I(Y_1^2 \le C\, nL^{1-\alpha}(n)\, jL(j))$$

$$\left. + n^{p/2} L^{(1-\alpha)p/2}(n)\, P\left(\frac{Y_1^2}{(L(|Y_1|))^{1-\eta}} \ge \frac{C\, nL^{1-\alpha}(n)\, jL(j)}{(L(nL^{1-\alpha}(n)) + L(jL(j)))^{1-\eta}} \right) \right]$$

$$\ll \sum_{n=1}^{\infty} \frac{1}{n} \left\{ (L(n))^{-\left[\frac{p\alpha}{2} + (\eta - \alpha)\right]} + (L(n))^{-\left[\frac{p\alpha}{2} + (1-\alpha)\right]} \right\} < \infty.$$

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